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COHOMOLOGICAL PROPERTIES OF THE RESTRICTION OF SCHUR FUNCTORS OF $\Omega_{\mathbf{P}^3}$ TO SPACE CURVES

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Abstract: We consider cohomological properties of the restriction of $\Omega_{\mathbf{P}^3}$ and iits related sheaf (by any Schur functor) to many hugh general space curves.

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1. Introduction

Let Y be any connected projective scheme algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ and F any vector bundle on Y. For any integer $m \geq 1$ let $\Pi(m)'$ denote the set of all partitions $a_1 \geq \cdots \geq a_k > 0$ of m. For any $\tau \in \Pi(m)'$ let $\tau(F)$ denote the corresponding direct factor of $F^{\otimes m}$. Thus $F^{\otimes m} = \oplus_{\tau \in \Pi(m)'} \tau(F)$. Notice however that $\tau(F) = 0$ if τ is associated to a partition $a_1 \geq \cdots \geq a_k > 0$ of m with $k > 0$. Let $\Gamma(m)'$ denote the set of all pairs (u, v), where u is any unordered decompostion $m_1 + \cdots + m_s$ of m with $s > 0$ and $m_i > 0$ for all i, while $v = \tau_1 \cdots \tau_s$ with $\tau_i \in \Pi(m_i)'$ for all i. For any $\gamma \in \Gamma(m)$ let $\gamma(F)$ denote the associated vector bundle on Y. Set $r_\gamma(F) := \mathrm{rank}(\gamma(F))$. The integer $r_\gamma(F)$ depends only from γ and the integer $\mathrm{rank}(F)$. If $\mathrm{rank}(F) = 3$, then we will write r_γ instead of $r_\gamma(F)$. We write

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$\Pi(m)$ (resp. $\Gamma(m)$) to denote the set of all $\tau \in \Pi(m)'$ (resp. $\Gamma(m)'$) such that $\tau(F) \neq 0$ when $\mathrm{rank}(F) = 3$. For instance $\Pi(m)$ is the set of all partitions $a_1 \geq \cdots \geq a_k > 0$ of m with $k \leq 3$. Set $\Omega := \Omega_{\mathbf{P}^3}$.

Here we descibe the space curves we are interested in.

Remark 1. For any integer $t > 0$ let C_t denote any arithmetically Cohen-Macaulay space curve whose minimal free resolution is of the form

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-t-1)^{\oplus t} \to \mathcal{O}_{\mathbf{P}^3}(-t)^{\oplus t+1} \to \mathcal{I}_{C_t} \to 0\,. \tag{1}$$

Thus $\deg(C_t) = t(t+1)/2$ and $p_a(C_t) = 1 + t(t+1)(2t-5)/6$ and the homogeneous ideal of C_t is generated by $t+1$ forms of degree t. The family Φ_t of all such curves C_t's is irreducible (as for all arithmetically Cohen-Macaulay space curves with fixed degree and arithmetic genus ([4], Theorem 2)) and the general $C_t \in \Phi_t$ is smooth ([8], or [5], or [2], Section 1). For instance, every C_1 is a line, the smooth C_2's are the smooth linearly normal rational curves, while the smooth C_3's are exactly the smooth genus 3 and degree 6 curves not contained in a quadric, i.e. the non hyperelliptic ones (see [7]) By Castelnuovo-Mumford's Lemma every space curve D such that $h^i(\mathbf{P}^3, \mathcal{I}_D(t-1)) = 0$ for $i = 0, 1, 2$, is an element of Φ_t. If $t \geq 2$ (resp. $t \geq 4$), then $\Omega|C_t$ is semistable (resp. stable) for a general $C_t \in \Phi_t$ ([3], Lemma 2.7). Fix a smooth quadric surface $Q \subset \mathbf{P}^3$. For each integer $t > 0$ there are curves $C_t \in \Phi_t$ and $C_{t+2} \in \Phi_{t+2}$ such that C_t is smooth, C_{t+2} is nodal, $C_{t+2} = C_t \cup A$ with A general in $|\mathcal{O}_Q(t+1, t+2)|$ and $\sharp(C_t \cap A) = t(t+1)$, i.e such that $C_t \cap Q \subseteq A$ ([3], Lemma 2.2).

For a general $C_t \in \Phi_t$ we will prove the following result.

Theorem 1. *Fix positive integers t, k, m such that $t \geq 2$ and $\gamma \in \Gamma(m)$. Set $\eta_m = 1$ if $m \equiv 1, 2 \pmod 3$ and $\eta_m = 2$ if $m \equiv 0 \pmod 3$. Set $\epsilon_t = \lfloor (t+1)/2 \rfloor$. Fix a general $C \in \Phi_t$ and general finite sets $A \subset C$, $B \subset C$ such that $\sharp(A) \leq \lfloor (-2mt(t+1) + kmt(t+1)/2)/3 \rfloor - t(t+1)(2t-5)/6 - \eta_t \cdot \epsilon_t$ and $\sharp(B) \geq \lceil (-2mt(t+1) + kmt(t+1)/2)/3 \rceil - t(t+1)(2t-5)/6 + \eta_t \cdot \epsilon_t$. Then $h^1(C, (\gamma(\Omega)(k)|C)(-A)) = 0$ and $h^0(C, (\gamma(\Omega)(k)|C)(-B)) = 0$.*

Lemma 1. *Let Y a reduced equidimensional projective curve such that $Y = C \cup D$ with $C \neq \emptyset$ and $D \neq \emptyset$. Set $Z := C \cap D$ (scheme-theoretic intersection). Let F be a vector bundle on Y and $A \subset C \backslash Z_{red}$, $B \subset D \backslash Z_{red}$ closed subschemes such that $h^1(C, \mathcal{I}_{A,C} \otimes (F|C)) = h^1(D, \mathcal{I}_{B \cup Z, D} \otimes (F|D)) = 0$. Then $h^1(Y, \mathcal{I}_{A \cup B, Y} \otimes F) = 0$.*

Proof. Since $A_{red} \cap Z_{red} = B_{red} \cap Z_{red} = \emptyset$ and F is locally free, we have a Mayer-Vietoris exact sequence on Y:

$$0 \to \mathcal{I}_{A \cup B, Y} \otimes F \to \mathcal{I}_{A,C} \otimes (F|C) \oplus \mathcal{I}_{B,D} \otimes (F|D) \to F|Z \to 0\,. \tag{2}$$

Since $h^1(D, \mathcal{I}_{B\cup Z,D} \otimes (F|D)) = 0$, the restriction map $H^0(D, \mathcal{I}_{B,D} \otimes (F|D) \to H^0(Z, F|Z)$ is surjective. Apply the cohomology exact sequence associated to (2). □

Remark 2. In the set-up of the statement of Lemma 1 we have $h^1(D, \mathcal{I}_{B\cup Z,D} \otimes (F|D)) = 0$ if $D \cong \mathbf{P}^1$ and $F|D$ has splitting type $a_1 \geq \cdots \geq a_n$ with $a_n \geq \text{length}(Z \cup B) - 1$.

Taking $A = B = \emptyset$ in the proof of Lemma 1 we get the following result.

Lemma 2. *Let Y a reduced equidimensional projective curve such that $Y = C \cup D$ with $C \neq \emptyset$ and $D \neq \emptyset$. Set $Z := C \cap D$ (scheme-theoretic intersection). Let F be a vector bundle on Y such that the restriction map $H^0(D, F|D) \to H^0(Z, F|Z)$ is surjective. Then:*

$$h^0(Y, F) = h^0(C, F|C) + h^0(D, F|D) - \text{legth}(Z) \cdot \text{rank}(F) \tag{3}$$

$$h^1(Y, F) = h^1(C, F|C) + h^1(D, F|D)\,. \tag{4}$$

Remark 3. Let $D \subset \mathbf{P}^3$ be a rational normal curve. Then $T\mathbf{P}^n|D$ is isomorphic to the direct sum of 3 line bundles of degree 4 (see e.g. [3], Lemma 1.3).

Remark 4. Fix an integer $m \geq 1$, a reduced and connected projective curve Y, $M \in \text{Pic}(Y)$ and a rank 3 vector bundle F on Y. Set $x := \deg(F)$; here deg means the total degree. For any vector bundle A on Y let $\mu(A) := \deg(A)/\text{rank}(A)$ denote the slope of A. We have $\mu(A \otimes B) = \mu(A) + \mu(B)$ for all vector bundles A, B. Furthermore, all the factors $\tau(A)$, $\tau \in \Pi(m)$, have the same slope. Hence we get $\mu(\gamma(F)) = m \cdot \mu(F) = mx/3$ for all $\gamma \in \Gamma(m)$. Thus $\deg(\tau(A)) = mx \cdot r_\gamma/3$ for all $\gamma \in \Gamma(m)$. We have $\deg(\gamma(A) \otimes M) = \deg(\gamma(A)) + r_\gamma \cdot \deg(M)$ ([6], Lemma 2.1). By Riemann-Roch we have $\chi(\gamma(A) \otimes M) = \deg(\gamma(A) \otimes M) + r_\gamma(1 - p_a(Y))$.

Lemma 3. *Let $C \subset \mathbf{P}^3$ be a smooth linearly normal elliptic curve and $R \in Pic^t(C)$, $t \geq 3$. Fix an integer $m \geq 1$, any $\gamma \in \Gamma(m)$, and any $A, B \subset C$, $A \subset C$, such that $\sharp(A) = \lfloor(-16m)/3\rfloor + t$ and $\sharp(B) = \lceil(-16m)/3\rceil + t$. If $m \equiv 0 \pmod 3$, then fix a general $A' \in C$ such that $\sharp(A') = t - (16m)/3$. Then:*

(i) *$\Omega|C$ is stable;*

(ii) *$h^1(C, (\Omega|C) \otimes (R(-A)) = h^0(C, (\Omega|C) \otimes (R(-B)) = 0$;*

(iii) *$\gamma(\Omega|C)$ is semistable;*

(iv) *assume* $3t > 4m$ *and* $m \equiv 1, 2 \pmod 3$*; then* $h^1(C, (\gamma(\Omega|C)) \otimes (R(-A')) = h^0(C, (\gamma(\Omega|C)) \otimes (R(-B')) = 0$*;*

(v) *assume* $3t > 4m$ *and* $m \equiv 0 \pmod 3$*; then* $h^1(C, (\gamma(\Omega|C)) \otimes (R(-A')) = h^0(C, (\gamma(\Omega|C)) \otimes (R(-A')) = 0$.

Proof. Taking duals, to prove (i) it is sufficient to prove that $T\mathbf{P}^3(-1)|C$ is semistable. $T\mathbf{P}^3(-1)|C$ is a spanned vector bundle with rank 3 and degree 4. Since C is not planar, it is easy to check (use the Euler's sequence) that $\mathcal{O}_C$ is not a factor of $T\mathbf{P}^3(-1)|C$. No degree 1 line bundle on C is spanned. No degree 1 or degree 2 rank two indecomposable vector bundle on C is spanned. Hence $T\mathbf{P}^3(-1)|C$ is indecomposable. Thus $T\mathbf{P}^3(-1)|C$ is semistable by Atiyah's classification of vector bundles on an elliptic curve ([1], Part (ii)). Part (ii) follows from part (i), because $(\Omega|C) \otimes R$ has slope $t - (4/3)$. Furthermore $\gamma(\Omega|C)$ is semistable for all $\gamma \in \Gamma(m)$. Hence parts (iii), (iv) and (v) follow from the cohomology of semistable vector bundles on C ([1], Part II). □

Lemma 4. *Fix positive integers* x, y, a, b, m*, a smooth quadric surface* $Q \subset \mathbf{P}^3$*, a general* $C \in |\mathcal{O}_Q(4,4)|$*, and general* $A, B \subset C$ *such that* $\sharp(A) = a$ *and* $\sharp(B) = b$*. If* $m \equiv 1, 2 \pmod 3$*, then assume* $a \leq \lfloor(-16m/3)\rfloor + 4x + 4y - 4$ *and* $b \geq \lceil(-16m/3)\rceil + 4x + 4y - 4$*. If* $m \equiv 0 \pmod 3$*, then assume* $a \leq \lfloor(-16m/3)\rfloor + 4x + 4y - 5$ *and* $b \geq \lceil(-16m/3)\rceil + 4x + 4y - 3$*. Then*

$$h^1(C, (\Omega(x,y)|C) \otimes (-A)) = h^0(C, (\Omega(x,y)|C)(-B)) = 0.$$

Proof. Notice that C has genus 5. Let D, D' be general elements of $|\mathcal{O}_Q(2,2)|$. Hence $T := D \cup D'$ is a nodal member of $|\mathcal{O}_Q(4,4)$. First assume $m \equiv 1, 2 \pmod 3$. To prove the statement for A it is sufficient to prove the case $a = \lfloor(-16m/3)\rfloor + 4x + 4y - 4$. By Lemma 4 applied to D' and the line bundle $\mathcal{O}_{D'}(x,y)|$ there is $E \subset D' \backslash (D \cap D')$ such that $\sharp(E) = \lfloor(-8m)/3\rfloor + 2x + 2y$ and $h^1(D', (\gamma(\Omega|D')) \otimes \mathcal{O}_{D'}(x,y)(-E)) = 0$. By Lemma 3 applied to D and the line bundle $\mathcal{O}_D(x-2, x-2)$ there is $E' \subset D$ such that $h^1(D, (\gamma(\Omega|D')) \otimes \mathcal{O}_{D'}(x-2, y-2)(-E')) = 0$. Notice that $D \cap D' \in |\mathcal{O}_D(2,2)|$. Hence $h^1(D, (\gamma(\Omega|D')) \otimes \mathcal{O}_{D'}(x,y)(-E' - (D \cap D'))) = 0$. hence a Mayer-Vietoris excat sequence gives $h^1(T, (\Omega(x,y)|E) \otimes (-E - E')) = 0$. Notice that $\sharp(E \cup E') = \lfloor(-16m/3)\rfloor + 4x + 4y - 4$. Then by semicontinuity we get the statement for a general $C \in |\mathcal{O}_Q(4,4)|$ and a general $A \subset C$. The same proof works for the set B and in the case $m \equiv 0 \pmod 3$. □

Lemma 5. *Fix positive integers* $s, x, , y, a, b, m$ *such that* $s \geq 2$*, a smooth quadric surface* $Q \subset \mathbf{P}^3$*, a general* $C \in |\mathcal{O}_Q(s,s)|$*, and general* $A, B \subset C$

such that $\sharp(A) = a$ *and* $\sharp(B) = b$.*If* $m \equiv 1, 2 \pmod 3$, *then assume* $a \le \lfloor(-8sm/3)\rfloor + sx + sy - s^2 + 2s$ *and* $b \ge \lceil(-8sm/3)\rceil + sx + sy - s^2 + 2s$. *If* $m \equiv 0 \pmod 3$, *then assume* $a \le \lfloor(-8sm/3)\rfloor + sx + sy - s^2 + 2s - 1$ *and* $b \ge \lceil(-8sm/3)\rceil + sx + sy - s^2 + 2s + 1$. *Then*

$$h^1(C, (\Omega(x, y)|C) \otimes (-A)) = h^0(C, (\Omega(x, y)|C)(-B)) = 0.$$

Proof. We will do the proof for the set A, the other case being similar. It is sufficient to do the case $a = \lfloor(-8sm)/3\rfloor + s(x + y) - s^2 + 2s$. Notice that $p_a(C) = s^2 - 2s + 1$ (use the adjunction formula). First assume $s \equiv 0 \pmod 3$. Take general $D_i \in |\mathcal{O}_Q(2, 1)|$ and $D'_i \in |\mathcal{O}_Q(1, 2)|$, $1 \le i \le s/3$. Hence $T := \bigcup_{i=1}^{s/3}(D_i \cup D'_i)$ is a nodal element of $|\mathcal{O}_Q(s, s)|$. Take general $E'_i \in D'_i$ and $E_i \subset D_i$ such that $\sharp(E'_i) = -4 + x + 2y + 9(1 - i)$ and $\sharp(E_i) = -4 + 2x + y - 4 + 9(1 - i)$ for all $1 \le i \le s/3$. Set $E := \bigcup_{i=1}^{s/3}(E_i \cup E'_i)$. Notice that $\sharp(E) = \lfloor(-8sm)/3\rfloor + s(x + y) - s^2 + 2s$. Apply Remark 3 with respect to D'_1, the line bundle $\mathcal{O}_{D'_1}(x, y)$ and the set E'_1. Then apply Remark 3 with respect to D_1, the line bundle $\mathcal{O}_{D'_1}(x - 1, y - 2)$ and the set E_1 (as in the proof of Lemma 4). Then apply Remark 3 with respect to D'_2, the line bundle $\mathcal{O}_{D'_1}(x-3, y-3)$ and the set E'_2. And so on. Now assume $s \equiv 2 \pmod 3$. Take general $D_i \in |\mathcal{O}_Q(2, 1)|$ and $D'_i \in |\mathcal{O}_Q(1, 2)|$, $1 \le i \le (s - 2)/3$ and a general $D \in |\mathcal{O}_Q(2, 2)|$. Hence $T := \bigcup_{i=1}^{s/3}(D_i \cup D'_i) \cup D$ is a nodal element of $|\mathcal{O}_Q(s, s)|$. Apply the proof of the case $s \equiv 0 \pmod 3$, except that you must do another step (the last one) using the curve D and applying Lemma 3 to this curve. Now assume $s \equiv 1 \pmod 3$. Take general $D_i \in |\mathcal{O}_Q(2, 1)|$ and $D'_i \in |\mathcal{O}_Q(1, 2)|$, $1 \le i \le (s - 4)/3$ and a general $D \in |\mathcal{O}_Q(4, 4)|$. Hence $T := \bigcup_{i=1}^{s/3}(D_i \cup D'_i) \cup D$ is a nodal element of $|\mathcal{O}_Q(s, s)|$. Apply the proof of the case $s \equiv 0 \pmod 3$, except that you must do another step (the last one) using the curve D and applying Lemma 4 to this curve. □

Lemma 6. *Fix positive integers* $s, x, , y, a, b, m$, *a smooth quadric surface* $Q \subset \mathbf{P}^3$, *a general* $C \in |\mathcal{O}_Q(s, s+1)|$, *and general* $A, B \subset C$ *such that* $\sharp(A) = a$ *and* $\sharp(B) = b$. *If* $s \equiv 2$ *mod* 3 *assume* $s \ge 5$, *If* $m \equiv 1, 2 \pmod 3$, *then assume* $a \le \lfloor(-4(2s+1)m/3)\rfloor + (s+1)x + sy - s^2 + s + 1$ *and* $b \ge \lceil(-4(2s+1)m/3)\rceil + (s+1)x + sy - s^2 + s + 1$. *If* $m \equiv 0 \pmod 3$, *then assume* $a \le \lfloor(-4(2s+1)m/3)\rfloor + (s+1)x + sy - s^2 + s$ *and* $b \ge \lceil(-4(2s+1)m/3)\rceil + (s+1)x + sy - s^2 + s + 2$. *Then* $h^1(C, (\Omega(x, y)|C) \otimes (-A)) = h^0(C, (\Omega(x, y)|C)(-B)) = 0$.

Proof. We will do the proof for the set A, the other case being similar. It is sufficient to do the case $a = \lfloor(-4(2s + 1)m)/3\rfloor + (s + 1)x + sy - s^2 + s + 1$. Notice that $p_a(C) = s^2 - s$ (use the adjunction formula). First assume $s \equiv 1$

(mod 3). Take general $D_i \in |\mathcal{O}_Q(2,1)|$ and $D_i' \in |\mathcal{O}_Q(1,2)|$, $1 \le i \le (s-1)/3$ and a general $D \in |\mathcal{O}_Q(1,2)|$. Hence $T := D \cup \bigcup_{i=1}^{(s-1)/3}(D_i \cup D_i')$ is a nodal element of $|\mathcal{O}_Q(s,s+1)|$. Take general $F \subset D$, $E_i' \in D_i'$ and $E_i \subset D_i$ such that $\sharp(F) = -4m + x + 2y - 1$, $\sharp(E_i') = -4m + x - 1 + 2y - 2 + 9(1-i)$ and $\sharp(E_i) = -4m + 2x - 2 + y - 1 - 4 + 9(1-i)$ for all $1 \le i \le s/3$. Set $E := F \cup \bigcup_{i=1}^{s/3}(E_i \cup E_i')$. Notice that $\sharp(E) = \lfloor(-4s(s+1)m)/3\rfloor + s(x+y) - s^2 + 2s$. Apply Remark 3 first with respect to D, $\mathcal{O}_D(x,y)$ and the set F. Then apply Remark 3 with respect to D_1', the line bundle $\mathcal{O}_{D_1'}(x-1,y-2)$ and the set E_1'. Then apply Remark 3 with respect to D_1, the line bundle $\mathcal{O}_{D_1'}(x-2,y-4)$ and the set E_1 (as in the proof of Lemma 4). Then apply Remark 3 with respect to D_2', the line bundle $\mathcal{O}_{D_1'}(x-4,y-5)$ and the set E_2'. And so on. Now assume $s \equiv 0 \pmod 3$. Take a general $D \in |\mathcal{O}_Q(1,2)|$, a general $D' \in |\mathcal{O}_Q(1,2)|$, general $D_i \in |\mathcal{O}_Q(2,1)|$ and $D_i' \in |\mathcal{O}_Q(2,2)|$, $1 \le i \le (s-3)/3$ and a general $D \in |\mathcal{O}_Q(2,2)|$. Hence $T := D' \cup D \cup \bigcup_{i=1}^{s/3}(D_i \cup D_i')$ is a nodal element of $|\mathcal{O}_Q(s,s+1)|$. Apply the proof of the case $s \equiv 1 \pmod 3$, except that you must do another step (the last one) using the curve D' and applying Lemma 3 to this curve. Now assume $s \equiv 2 \pmod 3$. Take general $D \in |\mathcal{O}_Q(1,2)|$, a general $D' \in |\mathcal{O}_Q(1,2)|$, general $D_i \in |\mathcal{O}_Q(2,1)|$ and $D_i' \in |\mathcal{O}_Q(1,2)|$, $1 \le i \le (s-5)/3$ and a general $D' \in |\mathcal{O}_Q(4,4)|$. Hence $T := D \cup D' \cup \bigcup_{i=1}^{s/3}(D_i \cup D_i')$ is a nodal element of $|\mathcal{O}_Q(s,s)|$. Apply the proof of the case $s \equiv 0 \pmod 3$, except for the curve D': you must apply Lemma 4 to this curve. □

Lemma 7. *Fix positive integers x, y, m, a smooth quadric Q, a general $C \in |\mathcal{O}_Q(2,4)|$ and general $A \subset C$, $B \subset C$ Then*

$$h^1(C, (\Omega(k)|C) \otimes (-A)) = h^0(C, (\Omega(k)|C)(-B)) = 0.$$

Proof. Notice that $p_a(C) = 3$. Fix two general $D, D' \in |\mathcal{O}_Q(1,2)|$. Take a general $F \subset D$ and a general $F' \subset D'$ such that $\sharp(F) = -4 + 2x + y$, $\sharp(F) = -8 + 2x + y$. Set $E := F \cup F'$. Notice that $T := D \cup D'$ is a nodal element of $|\mathcal{O}_Q(2,4)|$, $\sharp(D \cap D') = 4$. Then make the usual proof (first use D and then D'), each time quoting Remark 3. □

Lemma 8. *Let C be a general element of C_3. Fix positive integers m, k and general $A \subset C$, $B \subset C$ such that $\sharp(A) \le \lfloor(-24m+8mk)/3\rfloor - 2$ and $\sharp(B) \ge \lceil(-24m+8mk)/3\rceil - 2$. Then $h^1(C, (\Omega(k)|C) \otimes (-A)) = h^0(C, (\Omega(k)|C)(-B)) = 0$.*

Proof. Fix any smooth quadric Q. The closure of Φ_t in $\mathrm{Hilb}(\mathbf{P}^3)$ contains all degree 6 embeddings of all genus 3 hyperelliptic curves and hence all smooth elements of $|\mathcal{O}_Q(2,4)|$. Apply Lemma 7 and the semicontinuity of cohomology. □

Proof of Theorem 1. Let $Q \subset \mathbf{P}^3$ be a smooth quadric. We start with a general $C_3 \in \Phi_t$ if t is odd and with a general $C_2 \in \Phi_2$ (i.e. a rational normal curve). We apply Lemma 8 to C_3 and Remark 3 to C_2. Now assume $t \geq 4$. We use induction on the integer t passing from $t-2$ to t. By Remark 1 we may find $C_t \in \Phi_t$ such that $C_t = C_{t-2} \cup A$ with C_{t-2} general in Φ_{t-2} and A general in $|\mathcal{O}_Q(t-1,t)|$. We apply Lemma 6 to A and then make the usual proof for a reducible curve. □

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