International Journal of Pure and Applied Mathematics
Volume 35 No. 1 2007, 31-45

ON A B.V.P. IN A QUADRANT WITH NON-LOCAL BOUNDARY CONDITIONS

A. Merzon[1 §], M.A. Taneco-Hernández[2]
[1,2]Institute of Physics and Mathematics
University of Michoacán of San Nicolás de Hidalgo, ed. C-3
Ciudad Universitaria, 58090, Morelia, Michoacán, MÉXICO
[1]e-mail: anatoli@ifm.umich.mx
[2]e-mail: math23@itzel.ifm.umich.mx

Abstract: We consider a boundary value problem (b.v.p.) in a quadrant with absorbing boundary value conditions introduced by Engquist and Majda [3]. We prove the uniqueness of the solution in the Schwartz space of distributions.

AMS Subject Classification: 35J05, 35J25, 35J67
Key Words: Helmholtz equation, method of characteristics complex, absorbing boundary value conditions

1. Introduction

The aim of this paper is to prove that the non-local boundary value conditions, introduced by Engquist and Majda [3], [4] lead to the well-posed boundary value problem for the Helmholtz equation in a quadrant.

These conditions appeared as some pseudo-differential operators that turn the boundary of the half-plane into transparent one for some traveling waves. After the Fourier transform with respect to t, this operator has the form

Received: November 25, 2006

[§]Correspondence author

$$Bu(x,y)=\left(\frac{d}{dx}-i\sqrt{\omega^2+\frac{d^2}{dy^2}}\right)u(x,y),\quad (x,y)\in\mathbb{R}^2,\ \ \omega>0, \tag{1}$$

where for $v\in\mathbb{S}(\mathbb{R})$ (the Schwartz space of the smooth rapidly decreasing functions)

$$\sqrt{\omega^2+\frac{d^2}{dy^2}}\,v(y):=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-iy\eta}\sqrt{\omega^2-\eta^2}\,\hat{v}(\eta)d\eta, \tag{2}$$

with

$$\hat{v}(\eta)=\int_{\mathbb{R}}e^{i\eta y}v(y)dy. \tag{3}$$

Let $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$ be the Schwartz spaces of the the smooth rapidly decreasing functions and the tempered distributions respectively. Then B is well-defined operator from $\mathbb{S}(\mathbb{R}^2)$ to $\mathbb{S}'(\mathbb{R}^2)$. By continuity it admits the extension to the operator from $\mathbb{S}'(\mathbb{R}^2)$ to $\mathbb{S}'(\mathbb{R}^2)$. This extension has the following remarkable property: it annihilates the travelling to the left plane waves on the line $x=0$. These waves have the form $u=e^{i\omega t}\,e^{-i\omega\vec{n}\cdot(x,y)}$, with $\vec{n}=(\cos\phi,\sin\phi)$, $\phi\in(\pi/2,3\pi/2)$.

The approximations of this operator by the differential linear operators are used as approximate absorbing boundary conditions in numerical computations with artificial boundaries, introduced to limit a computational domain, (see [5] for an overview). These approximations may lead the boundary value problem in a half-plane to either well-posed or ill-posed problem according to the criterion introduced in [3] and [10].

There are several papers where the well-posed problems are studied for these boundary conditions, for example [10] for the half-plane, [1] for a non-stationary problem in a quadrant. We prove the uniqueness of the stationary problem for the *exact* non-local boundary value conditions in a quadrant.

Namely, we prove that in a quadrant the boundary value problem for the Helmholtz equation is well-posed in the following sense: there exist no non-trivial solutions of the Helmholtz equation for $x\in K:=\{(x_1,x_2)\in\mathbb{R}^2\mid x_1>0,\ x_2>0\}$

$$(\Delta+\omega^2)u(x)=0,\ \ \omega>0, \tag{4}$$

with the boundary conditions of the type (1) in the Schwartz space of *distributions*. In this case we need to give an appropriate definition of the operator (1) because this operator is defined a *priory* on the functions given on $\mathbb{R}^2$. We do it by means of the Paley-Wiener theory using [9].

Note that the function

$$u(x_1,x_2):=\sin\Big(\frac{\omega}{\sqrt{2}}\,x_1\Big)\sin\Big(\frac{\omega}{\sqrt{2}}\,x_2\Big),$$

is a solution to the boundary value problem for the Helmholtz equation (4) with the Dirichlet boundary condition in the quadrant K. This function belongs to the space $\mathbb{C}^\infty(\overline{K})$ of the smooth functions in $\overline{K}$ that are bounded with all their derivatives. Moreover, $u(x)$ is equal to 0 on the lattice $L := \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = \frac{k\pi\sqrt{2}}{\omega},\ k \in \mathbb{Z},\ x_2 \in \mathbb{R}\Big\} \cup \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \frac{k\pi\sqrt{2}}{\omega},\ k \in \mathbb{Z},\ x_1 \in \mathbb{R}\Big\}$. In particular, this is the reason why the Dirichlet boundary conditions are not suitable for computing simulations of the wave propagation problems (see [6], [2]). Here we prove that the same is not possible in a quadrant if the boundary operators are the modified operator of the type (1).

2. Fourier-Laplace Transform

Let us introduce some notations. Let $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$; $K_+^n := \{x \in \mathbb{R}^n \mid x_i > 0,\ i = 1, \ldots, n\}$, $n = 1, 2$; $K_+^1 := \mathbb{R}^+$, $K_+^2 := K$. We use the Fourier transform $\mathbb{F} : \mathbb{S}(\mathbb{R}^n) \longrightarrow \mathbb{S}(\mathbb{R}^n)$ in the form

$$\hat{f}(\xi) \equiv \mathbb{F}_{x\to\xi}[f](\xi) := \int_{\mathbb{R}^n} e^{i\langle\xi\cdot x\rangle} f(x)dx, \quad \xi \in \mathbb{R}^n. \tag{5}$$

This transform is extended by the continuity to the Fourier transform of the space $\mathbb{S}'(\mathbb{R}^n)$. By $\mathbb{S}'(\overline{K_+^n})$ we denote the space of the tempered distributions with the support in $\overline{K_+^n}$, and by $\mathbb{S}'(K_+^n)$ we denote the space of its restrictions to the open region K_+^n: $\mathbb{S}'(K_+^n) := \mathbb{S}'(\mathbb{R}^2)|_{K_+^n}$. The inverse transform reads as

$$\mathbb{F}^{-1}_{\xi\to x}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle\xi\cdot x\rangle} \hat{f}(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{6}$$

For an open region A in $\mathbb{C}^n$ the symbol $\mathcal{H}(A)$ denotes the set of all analytic functions in A, and $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. We will use the complex Fourier-Laplace transform and a version of the Paley-Wiener Theorem.

Theorem 2.1. (Paley-Wiener, see [8]) *i) Let $f \in \mathbb{S}'(\overline{K_+^n})$. Then $\hat{f}(\xi)$ has an unique analytic extension, $\tilde{f}(z)$, to the domain $\mathbb{C}K_+^n := \mathbb{R}^n \oplus iK_+^n$, $z := (z_1, \ldots, z_n) = (\xi_1 + i\tau_1, \ldots, \xi_n + i\tau_n)$, $\xi_i,\ \tau_i \in \mathbb{R}^+$, $i = 1, \ldots, n$, such that*

$$\left|\tilde{f}(z)\right| \leq C(1 + |z|)^\mu \operatorname{dist}\{z, \mathbb{R}^n\}, \quad z \in \mathbb{C}K_+^n, \tag{a}$$

for some constants $\mu \in \mathbb{R}$, $C > 0$ with $\operatorname{dist}\{z, \mathbb{R}^n\} := \min_{i=1,\ldots,n}\{\operatorname{Im} z_i\}$, *for $z \in \mathbb{C}K_+^n$ and*

$$\tilde{f}(\xi_1 + i\tau_1, \ldots, \xi_n + i\tau_n) \xrightarrow[\tau_{1,\ldots,n}\to 0+]{} \hat{f}(\xi_1, \ldots, \xi_n) \tag{b}$$

in $\mathbb{S}'(\mathbb{R}^n)$.

ii) Conversely, if $\tilde{f}(z)$ *is analytic function on* $\mathbb{C}K_+^n$ *and (a), (b) hold, then there exists*

$$\hat{f}(\xi) = \lim_{\operatorname{Im} z \to 0+} \tilde{f}(z),$$

in $\mathbb{S}'(\mathbb{R}^n)$ *and it is the Fourier transform of a distribution* $f(x)$ *belonging to* $\mathbb{S}'(\overline{K_+^n})$.

If $\hat{f}(\xi) = \lim_{\operatorname{Im} z \to 0+} \tilde{f}(z)$, then we write

$$\tilde{f}(z) := \mathbb{F}_{x \to z}[f(x)], \qquad f(x) = \mathbb{F}^{-1}_{z \to x}[\tilde{f}(z)]$$

and $\mathbb{F}$ is the complex Fourier-Laplace transform for $f \in \mathbb{S}'(\overline{K_+^n})$. In the next section we introduce the absorbing boundary value conditions.

3. Absorbing Boundary Value Conditions and Main Result

We modify the definition of the absorbing boundary value conditions proposed in [3]. For this we use the method [9] of the representation of boundary value conditions via Cauchy data.

We are going to define the absorbing boundary value conditions for functions defined on K, using Theorem 2.1. Namely, we define the absorbing operator as follows.

Let $u(x) \in \mathbb{S}'(K)$ be a solution of Helmholtz equation (4). Then $u \in \mathbb{C}^\infty(K)$ by ellipticity, and its traces on the rays $x_i = \epsilon$, $\epsilon > 0$, $i = 1, 2$, parallel to the side of the quadrant K are well defined. It is easy to demonstrate that these traces belong to $\mathbb{S}'(\mathbb{R}^+)$. The same is valid for all derivatives of u on these rays. We use the following theorem proved in [9].

Theorem 3.1. (see Proposition 2.3, Proposition 2.4 [9]) *i) There exist the Cauchy data of the solution u:*

$$\begin{cases} u(x_1, 0+) := u(x_1, x_2)|_{x_2=0+} = \lim\limits_{\epsilon \to 0+} u(x_1, \epsilon), \ x_1 > 0, \\ u(0+, x_2) := u(x_1, x_2)|_{x_1=0+} = \lim\limits_{\epsilon \to 0+} u(\epsilon, x_2), \ x_2 > 0, \\ \partial_{x_2} u(x_1, 0+) := \partial_{x_2} u(x_1, x_2)\big|_{x_2=0+} = \lim\limits_{\epsilon \to 0+} \left(\partial_{x_2} u(x_1, x_2)\big|_{x_2=\epsilon}\right), \ x_1 > 0, \\ \partial_{x_1} u(0+, x_2) := \partial_{x_1} u(x_1, x_2)\big|_{x_1=0+} = \lim\limits_{\epsilon \to 0+} \left(\partial_{x_1} u(x_1, x_2)\big|_{x_1=\epsilon}\right), \ x_2 > 0, \end{cases} \tag{7}$$

where the limits are understood in the sense $\mathbb{S}'(\mathbb{R}^+)$.

ii) There exists the canonical extension, $u_0(x) := u_0(x_1, x_2) \in \mathbb{S}'(\overline{K})$ *such that*

$$\gamma_0(x) := \big(\Delta + \omega^2\big) u_0(x), \quad x \in \mathbb{R}^2, \tag{8}$$

and has the form for $x \in \mathbb{R}^2$

$$\gamma_0(x) = v_1^1(x_1)\delta(x_2) + v_2^1(x_2)\delta(x_1) + v_1^0(x_1)\delta'(x_2) + v_2^1(x_2)\delta'(x_2), \tag{9}$$

where $v_l^\beta(x_l) \in \mathbb{S}'(\overline{\mathbb{R}^+})$, $l = 1, 2$, $\beta = 0, 1$ *are the extensions by zero of* $u_l^{(\beta)}(x)|_{x_l}$, *i.e.*

$$\begin{aligned} &v_1^0(x_1) = u(x_1, 0+),\ v_1^1(x_1) = \partial_{x_2} u(x_1, 0+), \quad x_1 > 0, \\ &v_2^0(x_2) = u(0+, x_2),\ v_2^1(x_2) = \partial_{x_1} u(0+, x_2), \quad x_2 > 0. \end{aligned} \tag{10}$$

Note that the Cauchy data (7) are defined only on $\mathbb{R}^+$, and its extensions by 0, $v_l^\beta(x_l)$, $l = 1, 2$, $\beta = 0, 1$, are defined on $\mathbb{R}$.

Let $\omega > 0$. We denote $\Gamma_1 := \{(x_1, 0) \mid x_1 \ge 0\}$, $\Gamma_2 := \{(0, x_2) \mid x_2 \ge 0\}$. Let $\mathbb{L} := (-\infty, -\omega] \cup [\omega, \infty)$.

Definition 3.2. We define $\sqrt{\omega^2 - z^2} : \mathbb{C}\backslash\mathbb{L} \longrightarrow \mathbb{C}$ as the analytic branch of $\sqrt{\cdot}$, such that $\operatorname{Re}\sqrt{\omega^2 - z^2} > 0$, for all $z \notin \mathbb{L}$.

Now we are able to define $Bu|_{\Gamma_l}$, $l = 1, 2$.

Definition 3.3. Let $v(x) \in \mathbb{S}'(\overline{\mathbb{R}^+})$. We define the pseudo-differential operator $\sqrt{\omega^2 + \dfrac{\partial^2}{\partial x^2}} : \mathbb{S}'(\overline{\mathbb{R}^+}) \longrightarrow \mathbb{S}'(\overline{\mathbb{R}^+})$ as

$$\sqrt{\omega^2 + \frac{\partial^2}{\partial x^2}}\,[v(x)] := \mathbb{F}^{-1}_{z\to x}[\sqrt{\omega^2 - z^2}\,\tilde{v}(z)], \quad \operatorname{Im} z > 0.$$

This definition is correct by Theorem 2.1. Now we define the non-local boundary value operators.

Definition 3.4. Let $u(x_1, x_2)$ be a solution of the equation (4) and v_l^β, $(l = 1, 2,\ \beta = 1, 2)$ be its extended Cauchy data. We define

$$B_l\, u(x_1, x_2)|_{\Gamma_l} := v_l^1(x_l) - i\sqrt{\omega^2 + \frac{\partial^2}{\partial x_l^2}}\, v_l^0(x_l), \quad l = 1, 2.$$

Note that for $u \in \mathbb{C}^\infty(\overline{K})$ this definition coincides with $B\,u(x_1, x_2)|_{\Gamma_l}$, where B is defined by (1).

In such a way we define the boundary operator (B_1, B_2) on the generalized solutions of the Helmholtz equation in quadrant, with the values in the space $\mathbb{S}'(\overline{\mathbb{R}^+}) \times \mathbb{S}'(\overline{\mathbb{R}^+})$.

Now let us consider the following homogeneous boundary value problem for the Helmholtz equation for $u \in \mathbb{S}'(K)$:

$$\begin{cases} (\Delta + \omega^2)u(x) = 0, & x \in K, \\ B_l u(x)|_{\Gamma_l} = 0, & l = 1, 2, \end{cases} \tag{11}$$

where B_l, $l = 1, 2$, are defined by Definition 3.4. The goal is to prove the following main theorem.

Theorem 3.5. *Let $\omega > 0$. The problem (11) has only a trivial solution in $\mathbb{S}'(K)$.*

In the rest of the paper we prove this theorem.

4. Reducing to an Algebraic Equation on the Riemann Surface

We are going to apply the complex Fourier transform $\mathbb{F}_{x\to z}$ to (9). Let $u \in \mathbb{S}'(K)$ be a solution to the problem (11). Then by Theorem 3.1 ii) there exists its extension $u_0 \in \mathbb{S}'(\overline{K})$ such that (8) holds, moreover $\operatorname{supp}\gamma_0 \subset \partial K$. Therefore, by Theorem 2.1, $\tilde{u}_0(z)$ and $\tilde{\gamma}_0(z)$ are holomorphic functions of two complex variables in the tube domain $\mathbb{C}K$. Hence, (8) implies

$$\begin{aligned} \mathsf{H}(z)u_0(z) := (-z_1^2 - z_2^2 + \omega^2)\tilde{u}_0(z_1, z_2) &= \tilde{\gamma}_0(z) \\ &= \tilde{v}_1^1(z_1) + \tilde{v}_2^1(z_2) - iz_2\tilde{v}_1^0(z_1) - iz_1\tilde{v}_2^0(z_2), \\ &\qquad (z_1, z_2) \in \mathbb{C}K. \end{aligned} \tag{12}$$

Here $\tilde{v}_l^\beta(z_l) \in H(\mathbb{C}^+),\ l = 0, 1; \beta = 0, 1$ also by Theorem 2.1.

Definition 4.1. i) $\mathcal{V}$ denotes the Riemann surface of the complex characteristics of the operator $\Delta + \omega^2$

$$\mathcal{V} = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid -z_1^2 - z_2^2 + \omega^2 = 0\}$$

ii) $\mathcal{V}^* = \mathcal{V} \cap \mathbb{C}K$.

Lemma 4.1. (Connection Equation) *Let u be a solution of (11). Then for the $\gamma_0(x)$ from (9) the following identity holds*

$$\tilde{\gamma}_0(z) \equiv \tilde{v}_1^1(z_1) + \tilde{v}_2^1(z_2) - iz_2\tilde{v}_1^0(z_1) - iz_1\tilde{v}_2^0 = 0, \quad z \in \mathcal{V}^*. \tag{13}$$

Proof. The expression (12) holds for $z \in \mathbb{C}K$, in particular for $z \in \mathcal{V}^*$. But the left-hand side of (12) is zero for $z \in \mathcal{V}^*$, hence the right-hand side is also zero. □

We sum up our considerations in the following statement.

Theorem 4.2. *Let $u(x)$ be a solution of (11). Then:*

i) There exists $u_0 \in \mathbb{S}'(\overline{K})$ such that u_0 satisfies (12).

ii) The function $\tilde{\gamma}_0(z)$ fits (13). The solution $u(x) = u_0|_K$ can be expressed in terms of $\tilde{\gamma}_0(z)$ by the formula

$$u_0(x) = \mathbb{F}^{-1}_{z\to x}\left[\frac{\tilde{\gamma}_0(z)}{-z_1^2 - z_2^2 + \omega^2}\right], \quad x \in K, \tag{14}$$

where $\mathbb{F}^{-1}$ is the inverse complex Fourier transformation.

Finally, let us apply the complex Fourier transform to the boundary conditions (11). By Definition 3.4 we obtain

$$\widetilde{B_l u_0(z)}|_{\Gamma_l} = \tilde{v}_l^1(z_1) - i\sqrt{\omega^2 - z_l^2}\,\tilde{v}_l^0(z_l) = 0, \quad z_l \in \mathbb{C}^+. \tag{15}$$

Now, eliminating $\tilde{v}_1^1(z_1)$ and $\tilde{v}_2^1(z_2)$ from (15) and substituting them to the connection equation (13), we get:

$$\Big(\sqrt{\omega^2 - z_1^2} - z_2\Big)\tilde{v}_1^0(z_1) + \Big(\sqrt{\omega^2 - z_2^2} - z_1\Big)\tilde{v}_2^0(z_2) = 0, \;\; z \in \mathcal{V}^*. \tag{16}$$

We are going to prove that (16) has only trivial solution $(\tilde{v}_1^0, \tilde{v}_2^0) \in \mathcal{H}(\mathbb{C}^+) \times \mathcal{H}(\mathbb{C}^+)$. It means by (15) that $\tilde{v}_l^1 \equiv 0$, $l = 1, 2$. Therefore $\tilde{\gamma}_0 \equiv 0$ by (13) and $u_0(x) \equiv 0$ by (14).

The idea is as follows. For $z \in \mathcal{V}^*$ $z_1^2 + z_2^2 = \omega^2$. It implies that $\sqrt{\omega^2 - z_1^2} - z_2 = 0$ or $\sqrt{\omega^2 - z_1^2} - z_2 = -2z_2$ depending on the sign of $\sqrt{\cdot}$. If $\sqrt{\omega^2 - z_1^2} - z_2 = 0$ and $\sqrt{\omega^2 - z_2^2} - z_1 \neq 0$ for (z_1, z_2) belonging to some open part of $\mathcal{V}^*$, then by analytic continuation $\tilde{v}_2^0(z_2) \equiv 0$ in $\mathbb{C}^+$. Similarly $\tilde{v}_1^0(z_1) \equiv 0$, $\operatorname{Im} z_1 > 0$.

In the next sections we justify these arguments. Namely, we complement the equation (16) by two equations of automorphisms, which express that $\tilde{v}_l^0(z_l)$ depends only on one variable. Then we determine the regions where the coefficients in (16) are zero. For this, it is convenient to introduce the uniformization parameter (see Section 6).

5. Automorphisms

Let

$$\mathcal{V}_l^+ := \Big\{z \in \mathcal{V} \mid z_l \in \mathbb{C}^+\Big\}, \quad l = 1, 2. \tag{17}$$

We define the natural lifting $\mathrm{v}_l^0(z)$ of $\tilde{v}_l^0(z_l)$ on the Riemann surface $\mathcal{V}$:

$$\mathrm{v}_l^0(z) := \tilde{v}_l^0(z_l), \quad z \in \mathcal{V}_l^+, \quad l = 1, 2. \tag{18}$$

Note that the functions $\mathrm{v}_l^0(z)$, $l = 1, 2$, are holomorphic in the region $\mathcal{V}_l^+$, $l = 1, 2$, since the functions $\tilde{v}_l^0(z_l)$, $l = 1, 2$ are holomorphic in $\mathbb{C}^+$.

Equation (16) is an undetermined problem of a functional equation with two unknown functions. Nevertheless, it is a well-posed problem because of its structure: the function $\tilde{v}_l^0$ depends only on z_l, $l = 1, 2$. The latter can be expressed algebraically by the method [9]. Namely, the analytic functions v_l^0 in $\mathcal{V}_l^+$ are invariant with respect to the monodromy group of the covering p_l : $\mathcal{V}_l^+ \longrightarrow \mathbb{C}^+$, where $p_l z = z_l$, $l = 1, 2$. The monodromy group is the group of transformations $\mathcal{V} \longrightarrow \mathcal{V}$ that transpose the points $z \in \mathcal{V}$ with the same coordinate z_l. These transformations transpose the "sheets of the covering" p_l, which are the roots of the characteristic equation $\mathsf{H}(z) = 0$. Hence by Vieta theorem, we obtain the following formulas for the generators $h_l : \mathcal{V} \to \mathcal{V}$ of the monodromy groups (automorphisms) of the covering p_l

$$h_1(z_1, z_2) = (z_1, -z_2), \qquad h_2(z_1, z_2) = (-z_1, z_2). \tag{19}$$

Therefore the invariance of v_l^0 with respect to the h_l is written as

$$\mathrm{v}_l^0(z) = \mathrm{v}_l^0(h_l(z)), \quad z \in \mathcal{V}_l^+, \ \ l = 1, 2. \tag{20}$$

We lift the equation (16) to $\mathcal{V}^* = \mathcal{V}_1^+ \cap \mathcal{V}_2^+$. Together with (20) we obtain the following system:

$$\begin{cases} \left(\sqrt{\omega^2 - z_1^2} - z_2\right)\mathrm{v}_1^0(z) + \left(\sqrt{\omega^2 - z_2^2} - z_1\right)\mathrm{v}_2^0(z) = 0, & z \in \mathcal{V}^*, \\ \mathrm{v}_1^0(z) - \mathrm{v}_1^0(h_1 z) = 0, & z \in \mathcal{V}_1^+, \\ \mathrm{v}_2^0(z) - \mathrm{v}_2^0(h_2 z) = 0, & z \in \mathcal{V}_2^+. \end{cases} \tag{21}$$

We have proved the following statement.

Lemma 5.1. *Let $\tilde{v}_l^0(z_l) \in \mathcal{H}(\mathbb{C}^+)$, $l = 1, 2$, be a solution to (16). Then $\mathrm{v}_l^0(z)$, $z \in \mathcal{V}_l^+$, $l = 1, 2$, is a solution to the system (21).*

We are going to prove that the system (21) admits only trivial solutions $\mathrm{v}_l^0 \in \mathcal{H}(\mathcal{V}_l^+)$.

6. Universal Covering Surface

It is more convenient to use the uniformization parameter μ for the Riemann surface $\mathcal{V}$. It is known that there exists a universal covering $\check{\mathcal{V}}$ of the surface $\mathcal{V}$ which is isomorphic to $\mathbb{C}$ (see, e.g. [7], [11]). Let us denote $\Pi : \check{\mathcal{V}} \longrightarrow \mathcal{V}$ the natural projection. We introduce a parameter $\mu \in \check{\mathcal{V}}$ such that

$$\Pi(\mu) = (z_1, z_2) = (-i\omega \sinh \mu, \omega \cosh \mu) \quad \mu \in \check{\mathcal{V}}. \tag{22}$$

Obviously, $\Pi(\mu) \in \mathcal{V}$ for all $\mu \in \check{\mathcal{V}}$. Note that this transformation is periodic with the period $2\pi i$, hence for all $z \in \mathcal{V}$, $\Pi^{-1}(z) = \{\mu + 2i\pi k,\ k \in \mathbb{Z}\}$.

Definition 6.1. Let

$$\check{\mathcal{V}}_l^+ := \Big\{\mu \in \check{\mathcal{V}} \mid \Pi(\mu) \in \mathcal{V}_l^+\Big\}, \quad l = 1, 2. \tag{23}$$

In the following lemma we give the explicit representation of $\check{\mathcal{V}}_l^+$.

Lemma 6.1. *We have:*

$$\check{\mathcal{V}}_1^+ := \check{\mathcal{V}}_{1R}^+ \cup \check{\mathcal{V}}_{1L}^+,$$

where

$$\begin{aligned}
\check{\mathcal{V}}_{1R}^+ &= \Big\{\mu \in \check{\mathcal{V}} \mid \operatorname{Re}\mu > 0,\ \frac{\pi}{2} + 2k\pi < \operatorname{Im}\mu < \frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\Big\} \\
\check{\mathcal{V}}_{1L}^+ &= \Big\{\mu \in \check{\mathcal{V}} \mid \operatorname{Re}\mu < 0,\ -\frac{\pi}{2} + 2k\pi < \operatorname{Im}\mu < \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\Big\}.
\end{aligned}$$

$$\begin{aligned}
\check{\mathcal{V}}_2^+ &= \check{\mathcal{V}}_{2R}^+ \cup \check{\mathcal{V}}_{2L}^+, \\
\check{\mathcal{V}}_{2R}^+ &= \Big\{\mu \in \check{\mathcal{V}} \mid \operatorname{Re}\mu > 0,\ 2k\pi < \operatorname{Im}\mu < \pi + 2k\pi,\ k \in \mathbb{Z}\Big\} \\
\check{\mathcal{V}}_{2L}^+ &= \Big\{\mu \in \check{\mathcal{V}} \mid \operatorname{Re}\mu < 0,\ -\pi + 2k\pi < \operatorname{Im}\mu < 2k\pi,\ k \in \mathbb{Z}\Big\}
\end{aligned}$$

(see Figure 1). and

$$\begin{aligned}
\check{\mathcal{V}}^* &= \Big\{\mu \in \mathbb{C} \mid \operatorname{Re}\mu > 0,\ \frac{\pi}{2} + 2k\pi < \operatorname{Im}\mu < \pi + 2k\pi,\ k \in \mathbb{Z}\Big\} \\
&\cup \Big\{\mu \in \mathbb{C} \mid \operatorname{Re}\mu < 0,\ -\frac{\pi}{2} + 2k\pi < \operatorname{Im}\mu < 2k\pi,\ k \in \mathbb{Z}\Big\}
\end{aligned}$$

(see Figure 2).

Proof. It follows from (22), $\omega > 0$ and inequalities $\operatorname{Im} z_l(\mu) > 0$, $l = 1, 2$. □

Note that the regions $\check{\mathcal{V}}_l^+$, $\check{\mathcal{V}}_{lR}^+$, $\check{\mathcal{V}}_{lL}^+$, $l = 1, 2$ are invariants with respect to the translation to $2\pi i$, because of the periodicity of the transformation $\Pi(\mu)$.

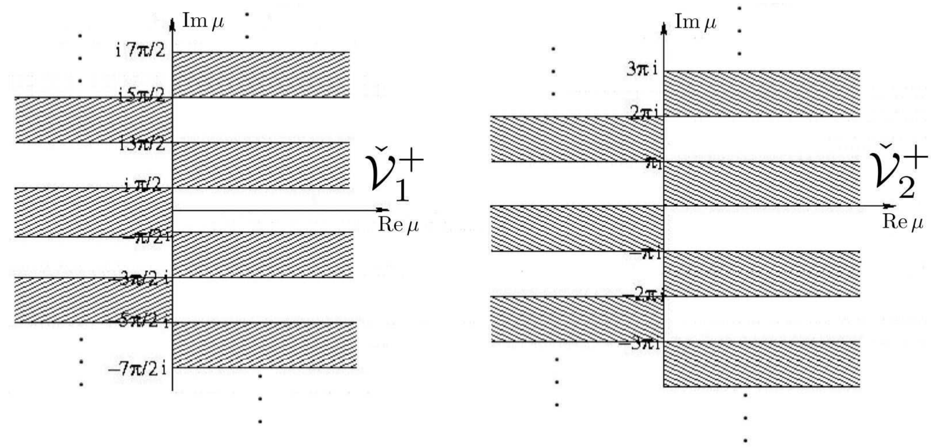

Figure 1: Regions $\check{\mathcal{V}}_1^+$ and $\check{\mathcal{V}}_2^+$

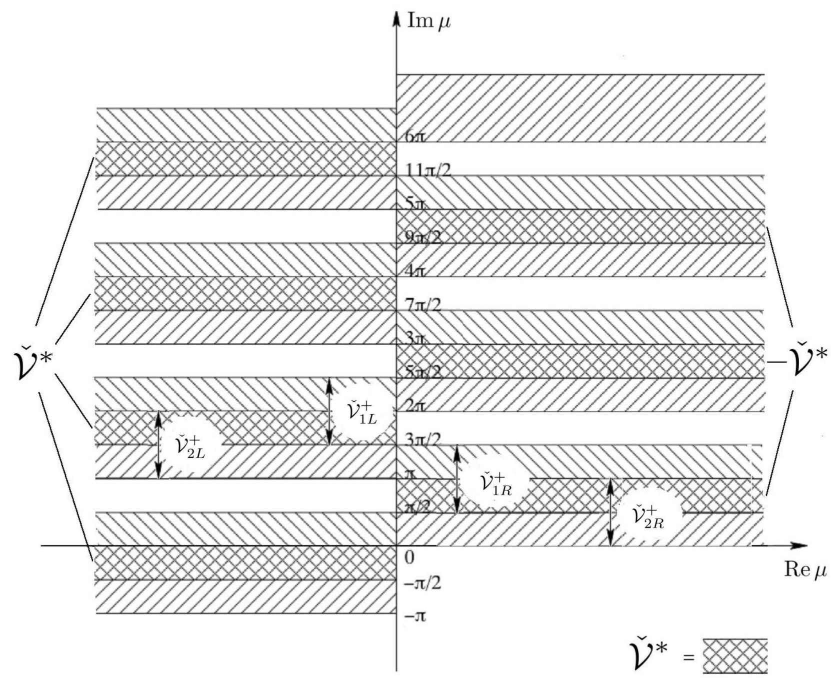

Figure 2: Description of the region $\check{\mathcal{V}}^*$

7. Lifting of the System (21) to the Universal Covering

Lemma 7.1. *There exist the automorphisms* $\check{h}_l : \check{\mathcal{V}} \to \check{\mathcal{V}}$, $l = 1, 2$, *which are the lifting of the automorphisms* h_l, *i.e.*

$$\Pi \check{h}_l = h_l \Pi. \tag{24}$$

They act by formulas:

$$\check{h}_1(\mu) = -\mu + i\pi, \qquad \check{h}_2(\mu) = -\mu, \tag{25}$$

i.e. the automorphism $\check{h}_1$ is the symmetry of $\check{\mathcal{V}}$ with respect to the point $i\pi$, and $\check{h}_2$ is the symmetry with respect to the zero point.

Proof. It suffices to verify that the following diagram is commutative

$$\begin{array}{ccc} \check{\mathcal{V}} & \xrightarrow{\check{h}_l} & \check{\mathcal{V}} \\ \Big\downarrow \Pi & & \Big\downarrow \Pi \\ \mathcal{V} & \xrightarrow{h_l} & \mathcal{V} \end{array}$$

for h_l given by (25). This follows from (22) and (19). □

Further we lift the functions $\mathrm{v}_l^0(z)$ from (18) to $\check{\mathcal{V}}$. Let Π as before, then we define

$$\check{\mathrm{v}}_l^0(\mu) = \mathrm{v}_l^0(\Pi(\mu)), \qquad \mu \in \check{\mathcal{V}}_l^+, \ \ l = 1, 2. \tag{26}$$

Invariance (20), (24) and definition (26) imply that the functions $\check{\mathrm{v}}_l^0(\mu)$ are automorphic with respect to $\check{h}_l$, i.e.,

$$\check{\mathrm{v}}_l^0(\mu) = \check{\mathrm{v}}_l^0(\check{h}_l\mu), \quad \mu \in \check{\mathcal{V}}_l^+, \quad l = 1, 2, \tag{27}$$

where $\check{h}_l$ are given by (25).

In the following we rewrite system (21) in terms of the variable μ. Note that by the periodicity of all lifting it is sufficient to lift (21) to the principal region choosen arbitrarily. We denote subregion of $\check{\mathcal{V}}$

$$\check{V}_{1R}^+ := \Big\{ \operatorname{Re}\mu > 0, \operatorname{Im}\mu \in \Big(\frac{\pi}{2}, \frac{3\pi}{2}\Big) \Big\}, \ \check{V}_{2R}^+ := \Big\{ \operatorname{Re}\mu > 0, \operatorname{Im}\mu \in (0, \pi) \Big\},$$
$$\check{V}_{1L}^+ := \Big\{ \operatorname{Re}\mu < 0, \operatorname{Im}\mu \in \Big(\frac{-\pi}{2}, \frac{\pi}{2}\Big) \Big\}, \check{V}_{2L}^+ := \Big\{ \operatorname{Re}\mu < 0, \operatorname{Im}\mu \in (-\pi, 0) \Big\}.$$

Also we denote

$$\begin{array}{rcl} \check{V}_1^+ & := & \check{V}_{1R}^+ \cup \check{V}_{1L}^+, \ \ \check{V}_2^+ := \check{V}_{2R}^+ \cup \check{V}_{2L}^+, \\ \check{V}_R^* & := & \check{V}_{1R}^+ \cap \check{V}_{2R}^+, \ \ \check{V}_L^* := \check{V}_{1L}^+ \cap \check{V}_{2L}^+, \ \ \check{V}^* = \check{V}_R^* \cup \check{V}_L^* \,, \end{array} \tag{28}$$

see Figure 3.

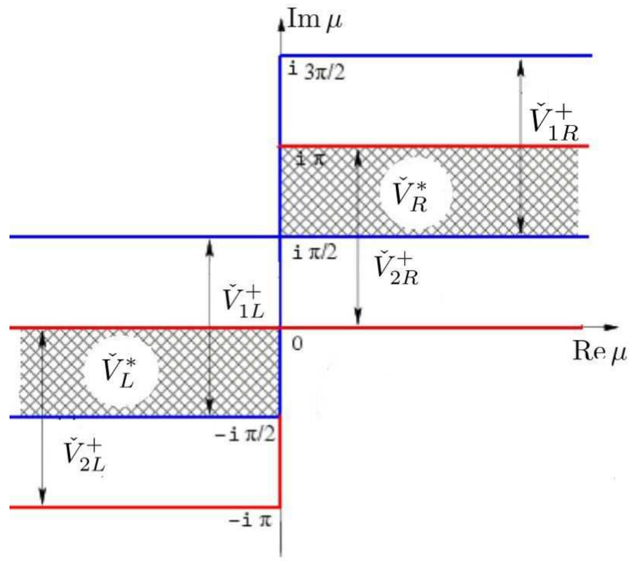

Figure 3: Principal region

It suffices to lift the system (21) to $\check{V}_1^+\cup\check{V}_2^+$, where $\check{V}_l^+$, $l=1,2$ are defined by (28). Using (22), (26), (27)and (25) we obtain the equivalent system:

$$\begin{cases} \left(\omega\sqrt{1+\sinh^2\mu}-\omega\cosh\mu\right)\check{\mathrm{v}}_1^0(\mu) \\ \qquad +\left(\omega\sqrt{1-\cosh^2\mu}+i\omega\sinh\mu\right)\check{\mathrm{v}}_2^0(\mu)=0, \qquad \mu\in\check{V}^*, \\ \check{\mathrm{v}}_1^0(-\mu+i\pi)=\check{\mathrm{v}}_1^0(\mu),\ \mu\in\check{V}_1^+, \\ \check{\mathrm{v}}_2^0(-\mu)=\check{\mathrm{v}}_2^0(\mu),\ \mu\in\check{V}_2^+, \end{cases} \tag{29}$$

where $\sqrt{\cdot}$ corresponds to the branch defined by Definition 3.2.

We prove that (29) has only trivial solution in the class $\mathcal{H}(\check{V}_1^+)\times\mathcal{H}(\check{V}_2^+)$. Dividing the first equation of (29) by ω we obtain

$$\begin{aligned} &\left(\sqrt{1+\sinh^2\mu}-\cosh\mu\right)\check{\mathrm{v}}_1^0(\mu) \\ &\qquad +\left(\sqrt{1-\cosh^2\mu}+i\sinh\mu\right)\check{\mathrm{v}}_2^0(\mu) \quad = \quad 0,\ \ \mu\in\check{V}^*. \end{aligned} \tag{30}$$

Now we analyze $\sqrt{1+\sinh^2\mu}$, $\mu \in \check{V}_1^+$ and $\sqrt{1-\cosh^2\mu}$, $\mu \in \check{V}_2^+$.

Lemma 7.2. *For $\mu \in \check{V}^*$ we obtain*

$$\sqrt{1+\sinh^2\mu} = \begin{cases} -\cosh\mu, & \mu \in \check{V}_R^*, \\ \cosh\mu, & \mu \in \check{V}_L^*. \end{cases} \tag{31}$$

$$\sqrt{1-\cosh^2\mu} = \begin{cases} -i\sinh\mu, & \mu \in \check{V}_R^*, \\ i\sinh\mu, & \mu \in \check{V}_L^*. \end{cases} \tag{32}$$

Proof. We consider the function

$$\sqrt{\omega^2 - z_1^2} : \mathbb{C}_{z_1}^+ \to \mathbb{C}.$$

This function is holomorphic in $\mathbb{C}_{z_1}^+$, hence its lifting by the formulas (22) is given by $\sqrt{\omega^2+\omega^2\sinh^2\mu}$ and is holomorphic in $\check{V}_1^+$. In general,

$$\sqrt{\omega^2+\omega^2\sinh^2\mu} = \pm\omega\cosh\mu, \quad \mu \in \check{V}_1^+, \tag{33}$$

and we must choose the sign such that (33) corresponds to the branch fixed by Definition 3.2. It is easy to see that for $\mu \in \check{V}_R^*$ (see Figure 3) we get $\mathrm{Re}(\cosh\mu) < 0$. Therefore we must choose the sign (-) in (33). So we obtain the first formula of (31).

For $\mu \in \check{V}_L^*$ (see Figure 3) we obtain $\mathrm{Re}(\cosh\mu) > 0$. Hence we get the second formula of (31). The formula (32) is proved similarly. □

Now we prove the following theorem, which completes the proof of Theorem 3.5.

Theorem 7.1. *The algebraic equation (16) has no non-trivial solutions in the class $\mathcal{H}(\mathbb{C}^+) \times \mathcal{H}(\mathbb{C}^+)$.*

Proof. Lemma 5.1, the equivalence of the systems (21) and (29) imply that it suffices to prove that the system (29) has no non-trivial solutions in $\mathcal{H}(\check{V}_1^+) \times \mathcal{H}(\check{V}_2^+)$.

Let us consider the equation (30) in the right hand side $\check{V}_R^*$ and in the left hand side $\check{V}_L^*$ of the $\check{V}^*$.

i) We consider (30) in $\check{V}_R^*$ (see Figure 3). By (31) and (32) the equation (30) is equivalent to $-2\cosh\mu\check{v}_1^0(\mu) = 0$, $\mu \in \check{V}_R^*$, which satisfies if and only if $\check{v}_1^0(\mu) = 0$, $\mu \in \check{V}_R^*$. By analyticity $\check{v}_1^0(\mu)$ in $\check{V}_{1R}^+$ it follows that $\check{v}_1^0(\mu) \equiv 0$ in $\check{V}_{1R}^+$. Thus by the automorphism $\check{h}_1$ we conclude $\check{v}_1^0(\mu) \equiv 0, \mu \in \check{V}_1^+$.

ii) Now we consider (30) in $\check{V}_L^*$ (see Figure 3). By (31) and (32) the equation (30) is equivalent to $2i\sinh\mu\check{v}_2^0(\mu)=0$, $\mu\in\check{V}_L^*$, which is satisfied only by $\check{v}_2^0(\mu)=0$, for $\mu\in\check{V}_L^*$. By analyticity $\check{v}_2^0(\mu)$ on $\check{V}_{2L}^+$ it follows that $\check{v}_2^0(\mu)\equiv 0$ on $\check{V}_{2L}^+$. So, by the automorphism $\check{h}_2$ we conclude $\check{v}_2^0(\mu)\equiv 0$, $\mu\in\check{V}_2^+$. The theorem is proved. □

8. Acknowledgments

This paper was inspired by discussions of certain diffraction problems by angles of Professor A. Merzon with A. Komech and P. Joly. The first author's work is supported by CONACyT, México under Project SEP-2004-C01-46769, CIC of UMICH, México and SNI-CONACyT, México. The second autor's work is supported by Proyect SEP-2004-C01-46769, CONACyT, México.

References

[1] A. Bamberger, P. Joly, J. E. Roberts, Second order absorbing boundary conditions for the wave equation: a solution for the corner problem, *SIAM J. Numer. Anal.*, **27**, No. 2 (1990), 323-352.

[2] F. Collino, *Analyse Numérique de Modèles de Propagation D'ondes. Application à la Migration et à L'invesion des Données Sismiques*, Ph.D. Thesis, Université Paris IX (1987).

[3] B. Engquist, E. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.*, **31**, No. 139 (1977), 629-651.

[4] B. Engquist, E. Majda E, Radiation boundary conditions for acoustic and elastic wave calculations, *Comm. Pure Appl. Math.*, **32** (1979), 313-357.

[5] D. Givoli, *Numerical Methods for Problems in Infinite Domains*, Elsevier Scientific Publishing Co., Amsterdam (1992).

[6] P. Joly, *Analyse Numérique et Mathématique de Problèmes Liés a la Propagation D'ondes Acoustiques, Élastiques et Électromagnétiques*, PhD Thesis, Université Paris IX (1987).

[7] Jürgen Jost, *Compact Riemann Surfaces. An introduction to contemporany Mathematics*, Universitext Springer (1997).

[8] A.I. Komech, *Linear Partial Differential Equations with Constant Coefficients*, Partial Differential Equations II, Encyclopaedia of Mathematical Science, Volume **31**, Springer-Verlang, Berlin (1994), 127-260.

[9] A.I. Komech, A. Merzon, P. Zhevandrov, A method of complex characteristics for elliptic problems in angles, and its applications, *Amer. Math. Soc. Transl., Ser. 2*, No. 206, Amer. Math. Soc., Providence, RI (2002), 125-159.

[10] H.O. Kreiss, Inicial-boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.* No. 23 (1970), 277-298.

[11] Zaldivar, Felipe, *Funciones Algebraicas de una Variable Compleja*, Universidad Autónoma Metropolitana, Unidad Iztapalapa, México (1995).