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GOODNESS-OF-FIT TESTS VIA PARTITIONS OF CHI-SQUARED TEST STATISTICS DERIVED FROM CHARACTERIZING CONDITIONS

Dominik Szynal

Institute of Mathematics
Maria Curie-Skłodowska University
Pl. M. Curie-Skłodowskiej 1
Lublin, PL-20-031, POLAND
e-mail: szynal@golem.umcs.lublin.pl

Abstract: There are presented goodness-of-fit tests via conditions characterizing continuous distributions in terms of moments of record values. We study partitions of the chi-squared test statistics for testing goodness-of-fit into two components, each asymptotically distributed as a chi-squared variate with one degree of freedom. The formulae of the tests obtained in this way are remarkably simple and they are expressed in original sample values or in terms of extremal statistics.

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1. Introduction and Preliminaries

Let $\{X_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F and pdf f. For a fixed integer $k \geq 1$ we define a sequence $U_k(1), U_k(2), \ldots$ of k-th upper record times of $\{X_n,\, n \geq 1\}$ as follows:

$$U_k(1) = 1,$$
$$U_k(n+1) = \min\{j > U_k(n) : X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geq 1.$$

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$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1},\ n \geq 1.$$

The sequence $\{Y_n^{(k)},\ n \geq 1\}$ is called the sequence of k-th (upper) record values of the above sequence. Note that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$.

The k-th lower record times $L_k(n)$, $n \geq 1$, are defined as

$$L_k(1) = 1,$$
$$L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, \quad n \geq 1,$$

and the k-th lower record values as

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \quad n \geq 1.$$

Note that $Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k)$.

The marginal and joint cdf and pdf of the k-th upper and lower record values are given in Dziubdziela and Kopociński [4], Nevzorov [14] and Pawlas and Szynal [15].

We shall use the following recurrence relations for cdf of $Y_n^{(k)}$ and $Z_n^{(k)}$:

$$F_{Y_n^{(k)}}(x) = F_{Y_{n-1}^{(k)}}(x) - \frac{k^{n-1}}{(n-1)!}[1 - F(x)]^k[-\log(1 - F(x))]^{n-1} \tag{1.1}$$

and

$$F_{Z_n^{(k)}}(x) = F_{Z_{n-1}^{(k)}}(x) + \frac{k^{n-1}}{(n-1)!}[F(x)]^k[-\log F(x)]^{n-1} \tag{1.2}$$

for $n \geq 2$ and $k \geq 1$ (cf. Bieniek and Szynal [3], Grudzień and Szynal [5]).

Conditions characterizing continuous distributions by expected values of two functions of record values, were studied in Lin [8], Grudzień and Szynal [5] and Malinowska et al [9]. Tests derived from those conditions were given in Morris and Szynal [10]-[13]. The aim of this paper is to discuss partitions of the chi-squared test statistics for testing goodness-of-fit (cf. Morris and Szynal [13]) into two components, each asymptotically distributed as a chi-squared variate with one degree of freedom. The tests obtained in this way depend on three parameters: $k \in \mathbb{N}$, $r > -1/2$, $a \in \mathbb{R}$ or $b \in \mathbb{R}$ which appear in characterizing conditions (cf. Malinowska et al [9]). It turns out that the formulae of tests are remarkably simple and they are expressed in original sample values or in terms of extremal statistics. Tests in Morris and Szynal [13] are elements of the family of tests given in this contribution.

2. Characterization Conditions Via Moments of k-th Record Values

Let $I(F)$ stand for the minimal interval containing the support of F. Recall that a subdistribution G on $\mathbb{R}$ is nondecreasing and right-continuous function from $\mathbb{R}$ to the interval [0,1].

In what follows we write

$$\begin{aligned} &H_t(x) = (-\log(1-G(x)))^t, \quad h_t(x) = (-\log(1-F(x)))^t, \\ &H_t^*(x) = (-\log G(x))^t, \quad h_t^*(x) = (-\log F(x))^t, \quad x \in \mathbb{R},\, t \neq 0. \end{aligned} \tag{2.1}$$

Theorem 1. *Let $\{X_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with a continuous distribution function F and G subdistribution on $\mathbb{R}$. Futher, let $\boldsymbol{B}$ be a non-singular matrix*

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{2.2}$$

and assume that $n \geq 1$, $k \geq 1$ and $s \geq 0$ are given integers and $r \neq 0$ a given real number such that $n + r + 1 > 0$. Then $F(x) = G(x)$ on $\mathbb{R}$ iff

$$\left\{ \begin{aligned} &b_{11} k^r EH_r\left(Y_{n+1}^{(k)}\right) + b_{12} k^{r+n-s} EH_{r+n-s}\left(Y_{s+1}^{(k)}\right) \\ &\qquad = \left(\tfrac{1}{n!} b_{11} + \tfrac{1}{s!} b_{12}\right) \Gamma(n+r+1), \\ &b_{21} k^r EH_r\left(Y_{n+1}^{(k)}\right) + b_{22} k^{r+n-s} EH_{r+n-s}\left(Y_{s+1}^{(k)}\right) \\ &\qquad = \left(\tfrac{1}{n!} b_{21} + \tfrac{1}{s!} b_{22}\right) \Gamma(n+r+1), \end{aligned} \right. \tag{2.3}$$

or iff

$$\left\{ \begin{aligned} &b_{11} k^r EH_r^*\left(Z_{n+1}^{(k)}\right) + b_{12} k^{r+n-s} EH_{r+n-s}^*\left(Z_{s+1}^{(k)}\right) \\ &\qquad = \left(\tfrac{1}{n!} b_{11} + \tfrac{1}{s!} b_{12}\right) \Gamma(n+r+1), \\ &b_{21} k^r EH_r^*\left(Z_{n+1}^{(k)}\right) + b_{22} k^{r+n-s} EH_{r+n-s}^*\left(Z_{s+1}^{(k)}\right) \\ &\qquad = \left(\tfrac{1}{n!} b_{21} + \tfrac{1}{s!} b_{22}\right) \Gamma(n+r+1) \end{aligned} \right. \tag{2.3*}$$

(cf. Malinowska et al [9]).

Letting in (2.2), $b_{11} = b_{22} = 1$, $b_{12} = a \in \mathbb{R}$, $b_{21} = 0$, i.e. taking

$$\boldsymbol{B} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix},$$

and applying (2.3) and (2.3*), we can use

$$\left\{ \begin{aligned} &k^r Eh_r\left(Y_{n+1}^{(k)}\right) + a k^{r+n-s} Eh_{r+n-s}\left(Y_{s+1}^{(k)}\right) = \left(\tfrac{1}{n!} + \tfrac{a}{s!}\right) \Gamma(n+r+1), \\ &k^{r+n-s} Eh_{r+n-s}\left(Y_{s+1}^{(k)}\right) = \tfrac{1}{n!} \Gamma(n+r+1), \end{aligned} \right. \tag{2.4}$$

or

$$\begin{cases} k^r Eh_r^* \left(Z_{n+1}^{(k)}\right) + ak^{r+n-s} Eh_{r+n-s}^* \left(Z_{s+1}^{(k)}\right) = \left(\frac{1}{n!} + \frac{a}{s!}\right) \Gamma(n+r+1)\,, \\ k^{r+n-s} Eh_{r+n-s}^* \left(Z_{s+1}^{(k)}\right) = \frac{1}{n!} \Gamma(n+r+1)\,. \end{cases} \tag{2.4*}$$

If $n = 1$, $s = 0$ in (2.4) and (2.4*) then we can apply

$$\begin{cases} k^r Eh_r \left(Y_2^{(k)}\right) + ak^{r+1} Eh_{r+1} \left(Y_1^{(k)}\right) = (a+1)\Gamma(r+2)\,, \\ k^{r+1} Eh_{r+1} \left(Y_1^{(k)}\right) = \Gamma(r+2)\,, \end{cases} \tag{2.5}$$

or

$$\begin{cases} k^r Eh_r^* \left(Z_2^{(k)}\right) + ak^{r+1} Eh_{r+1}^* \left(Z_1^{(k)}\right) = (a+1)\Gamma(r+2)\,, \\ k^{r+1} Eh_{r+1}^* \left(Z_1^{(k)}\right) = \Gamma(r+2). \end{cases} \tag{2.5*}$$

Taking into account that

$$E\left[h_r\left(Y_2^{(k)}\right)\right] = E\left[h_r(X_{1:k})\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r}\,,$$
$$E\left[h_r^*\left(Z_2^{(k)}\right)\right] = E\left[h_r^*(X_{k:k})\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r}\,,$$

which follow from (1.1) and (1.2), we lead to the following corollary.

Corollary 1. *If $X \sim F$ and F is continuous then*

$$\begin{cases} k^r E\left[h_r\left(X_{1:k}\right)\right] + ak^{r+1} E\left[h_{r+1}\left(X_{1:k}\right)\right] = \Gamma(r+1) + a\Gamma(r+2)\,, \\ k^{r+1} E\left[h_{r+1}(X_{1:k})\right] = \Gamma(r+2)\,, \end{cases} \tag{2.6}$$

$$\begin{cases} k^r E\left[h_r^*\left(X_{k:k}\right)\right] + ak^{r+1} E\left[h_{r+1}^*\left(X_{k:k}\right)\right] = \Gamma(r+1) + a\Gamma(r+2)\,, \\ k^{r+1} E\left[h_{r+1}^*(X_{k:k})\right] = \Gamma(r+2). \end{cases} \tag{2.6*}$$

Now letting in (2.2), $b_{11} = b_{22} = 1$, $b_{12} = 0$, $b_{21} = b \in \mathbb{R}$, i.e. taking

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \qquad b \in \mathbb{R},$$

we lead to the following result.

Corollary 2. *If $X \sim F$ and F is continuous then*

$$\begin{cases} k^r E\left[h_r\left(X_{1:k}\right)\right] = \Gamma(r+1)\,, \\ bk^r E\left[h_r(X_{1:k})\right] + k^{r+1} E\left[h_{r+1}(X_{1:k})\right] = b\Gamma(r+1) + \Gamma(r+2)\,, \end{cases} \tag{2.7}$$

$$\begin{cases} k^r E\left[h_r^*\left(X_{k:k}\right)\right] = \Gamma(r+1)\,, \\ bk^r E\left[h_r^*(X_{k:k})\right] + k^{r+1} E\left[h_{r+1}^*(X_{k:k})\right] = b\Gamma(r+1) + \Gamma(r+2). \end{cases} \tag{2.7*}$$

3. Goodness-of-Fit Tests

3.1. Parameters of F are Specified

We construct here tests of fit for continuous distributions using the conditions (2.6), (2.6*) and (2.7), (2.7*).

We write

$$U_k := X_{1:k} = \min(X_1, \dots, X_k), \qquad V_k := X_{k:k} = \max(X_1, \dots, X_k)\,,$$

and we define

$$R_k^{(r)} = k^r[-\log(1 - F(U_k))]^r, \qquad R_k^{*(r)} = k^r[-\log F(V_k)]^r.$$

Put

$$Y = R_k^{(r+1)}, \quad Z = R_k^{(r)}, \quad Y^* = R_k^{*(r+1)}, \quad Z^* = R_k^{*(r)}.$$

Note that the conditions (2.6) and (2.6*) can be written as follows

$$E\,[Z + aY] = \Gamma(r+1) + a\Gamma(r+2), \qquad E[Y] = \Gamma(r+2)\,,$$
$$E\,[Z^* + aY^*] = \Gamma(r+1) + a\Gamma(r+2), \qquad E[Y^*] = \Gamma(r+2).$$

Similarly the conditions (2.7) and (2.7*) are as follows

$$E\,[Z] = \Gamma(r+1), \qquad E[bZ + Y] = b\Gamma(r+1) + \Gamma(r+2)\,,$$
$$E[Z^*] = \Gamma(r+1), \qquad E[bZ^* + Y^*] = b\Gamma(r+1) + \Gamma(r+2).$$

Write

$$Z_1 = Z + aY, \qquad Z_1^* = Z^* + aY^*$$

and

$$Y_1 = bZ + Y, \qquad Y_1^* = bZ^* + Y^*.$$

To discuss the above variates we need the following lemma.

Lemma 1. *Suppose that $X \sim \mathrm{Exp}(1)$. Then if $r > -\frac{1}{2}$ the joint distribution of (Y, Z) is a singular distribution concentrated on the line $C : z = y^{r/(r+1)}$ with cdf*

$$F(y, z) = P[Y < y] = 1 - \exp\left(-y^{1/(r+1)}\right), \qquad (y, z) \in C.$$

Moreover,

$$EY = \Gamma(r+2), \qquad EZ = \Gamma(r+1)\,,$$

$$\operatorname{Var} Y = \Gamma(2r+3) - \Gamma^2(r+2), \qquad \operatorname{Var} Z = \Gamma(2r+1) - \Gamma^2(r+1)$$
$$E[YZ] = \Gamma(2r+2), \qquad \operatorname{Cov}(Y,Z) = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)$$

(cf. Morris and Szynal [13]).

Hence we have the following result.

Corollary 3. *For Z_1 and Y_1 we have*

$$\begin{aligned}
EZ_1 &= \Gamma(r+1) + a\Gamma(r+2)\,,\\
\operatorname{Var} Z_1 &= \Gamma(2r+1) - \Gamma^2(r+1) + 2a\left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right)\\
&\quad + a^2\left(\Gamma(2r+3) - \Gamma^2(r+2)\right)\,,\\
\operatorname{Cov}(Y,Z_1) &= \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) + a\left(\Gamma(2r+3) - \Gamma^2(r+2)\right)\,,\\
EY_1 &= b\Gamma(r+1) + \Gamma(r+2)\,,\\
\operatorname{Var}(Y_1) &= \Gamma(2r+3) - \Gamma^2(r+2) + 2b\left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right)\\
&\quad + b^2\left(\Gamma(2r+1) - \Gamma^2(r+1)\right)\,,\\
\operatorname{Cov}(Y_1,Z) &= \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) + b\left(\Gamma(2r+1) - \Gamma^2(r+1)\right).
\end{aligned}$$

Write

$$\begin{aligned}
&a^{(r)} := \operatorname{Var} Z = \Gamma(2r+1) - \Gamma^2(r+1)\,,\\
&b^{(r)} := \operatorname{Cov}(Y,Z) = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\,,\\
&c^{(r)} := \operatorname{Var} Y = \Gamma(2r+3) - \Gamma^2(r+2)\,,\\
&a^{(r,a)} := \operatorname{Var} Z_1 = a^{(r)} + 2ab^{(r)} + a^2c^{(r)}\,,\\
&b^{(r,a)} := \operatorname{Cov}(Y,Z_1) = b^{(r)} + ac^{(r)}\,,\\
&b^{(r,b)} := \operatorname{Cov}(Y_1,Z) = b^{(r)} + ba^{(r)}\,,\\
&c^{(r,b)} := \operatorname{Var}(Y_1) = c^{(r)} + 2bb^{(r)} + b^2a^{(r)}.
\end{aligned}$$

In what follows we write a pair $(r,0)$ as (r).

Suppose now we have a sample $X_1,\dots,X_n$ of size $n = kN$. This provides the samples $R_{k1}^{(r)},\dots,R_{kN}^{(r)}$ and $R_{k1}^{*(r)},\dots,R_{kN}^{*(r)}$, where

$$R_{kj}^{(r)} = k^r h_r(U_{kj}), \quad R_{kj}^{*(r)} = k^r h_r^*(V_{kj}), \quad j = 1,\dots,N,$$

with

$$U_{kj} = \min\left(X_{k(j-1)+1},\dots,X_{kj}\right), \quad V_{kj} = \max\left(X_{k(j-1)+1},\dots,X_{kj}\right).$$

Now we define

$$\boldsymbol{W}_{kj}^{(r)} = \begin{bmatrix} R_{kj}^{(r)} \\ R_{kj}^{(r+1)} \end{bmatrix}, \qquad \boldsymbol{W}_{kj}^{*(r)} = \begin{bmatrix} R^{*}{}_{kj}^{(r)} \\ R^{*}{}_{kj}^{(r+1)} \end{bmatrix},$$

$$\boldsymbol{W}_{kj}^{(r,a)} = \begin{bmatrix} R_{kj}^{(r)} + aR_{kj}^{(r+1)} \\ R_{kj}^{(r+1)} \end{bmatrix}, \qquad \boldsymbol{W}_{kj}^{*(r,a)} = \begin{bmatrix} R^{*}{}_{kj}^{(r)} + aR_{kj}^{*(r+1)} \\ R^{*}{}_{kj}^{(r+1)} \end{bmatrix},$$

$$\boldsymbol{W}_{kj}^{(r,b)} = \begin{bmatrix} R_{kj}^{(r)} \\ bR_{kj}^{(r)} + R_{kj}^{(r+1)} \end{bmatrix}, \qquad \boldsymbol{W}_{kj}^{*(r,b)} = \begin{bmatrix} R^{*}{}_{kj}^{(r)} \\ bR^{*}{}_{kj}^{(r)} + R^{*}{}_{kj}^{(r+1)} \end{bmatrix}.$$

We see that

$$\boldsymbol{\mu}_k^{(r)} := E\boldsymbol{W}_{k1}^{(r)} = \Gamma(r+1) \begin{bmatrix} 1 \\ r+1 \end{bmatrix} = \boldsymbol{\mu}_k^{*(r)} := E\boldsymbol{W}_k^{*(r)},$$

$$\Sigma^{(r)} := \Sigma_k^{(r)} := \mathrm{Var}\left(\boldsymbol{W}_{k1}^{(r)}\right) = \begin{bmatrix} a^{(r)} & b^{(r)} \\ b^{(r)} & c^{(r)} \end{bmatrix} = \Sigma_k^{*(r)} := \mathrm{Var}\left(\boldsymbol{W}_{k1}^{*(r)}\right),$$

$$\boldsymbol{\mu}_k^{(r,a)} = E\boldsymbol{W}_{k1}^{(r,a)} = \Gamma(r+1) \begin{bmatrix} 1 + a(r+1) \\ r+1 \end{bmatrix} = \boldsymbol{\mu}_k^{*(r,a)} := E\boldsymbol{W}_{k1}^{*(r,a)},$$

$$\Sigma^{(r,a)} := \Sigma_k^{(r,a)} = \mathrm{Var}\left(\boldsymbol{W}_{k1}^{(r,a)}\right) := \begin{bmatrix} a^{(r,a)} & b^{(r,a)} \\ b^{(r,a)} & c^{(r)} \end{bmatrix}$$
$$= \Sigma_k^{*(r,a)} := \mathrm{Var}\left(\boldsymbol{W}_{k1}^{*(r,a)}\right),$$

$$\boldsymbol{\mu}_k^{(r,b)} = E\boldsymbol{W}_{k1}^{(r,b)} = \Gamma(r+1) \begin{bmatrix} 1 \\ b+r+1 \end{bmatrix} = \boldsymbol{\mu}_k^{*(r,b)} := E\boldsymbol{W}_{k1}^{*(r,b)}$$

$$\Sigma^{(r,b)} := \Sigma_k^{(r,b)} = \mathrm{Var}\left(\boldsymbol{W}_{k1}^{(r,b)}\right) := \begin{bmatrix} a^{(r)} & b^{(r,b)} \\ b^{(r,b)} & c^{(r,b)} \end{bmatrix}$$
$$= \Sigma_k^{*(r,b)} := \mathrm{Var}\left(\boldsymbol{W}_{k1}^{*(r,b)}\right).$$

Write

$$\Delta^{(r)} := \det\left(\Sigma^{(r)}\right), \qquad \Delta^{*(r)} := \det\left(\Sigma^{*(r)}\right)$$
$$\Delta^{(r,a)} := \det\left(\Sigma^{(r,a)}\right), \quad \Delta^{*(r,a)} := \det\left(\Sigma^{*(r,a)}\right)$$
$$\Delta^{(r,b)} := \det\left(\Sigma^{(r,b)}\right), \quad \Delta^{*(r,b)} := \det\left(\Sigma^{*(r,b)}\right).$$

We see that

$$\Delta^{(r)} = \Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)$$

and

$$\Delta^{(r,a)} = \Delta^{*(r,a)} = \Delta^{(r,b)} = \Delta^{*(r,b)} = \Delta^{(r)}.$$

The CLT says that

$$\begin{aligned}
\sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right) &\xrightarrow{D} \boldsymbol{W}^{(r,a)} \sim N\left(\mathbf{0}, \Sigma_k^{(r,a)}\right),\\
\sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right) &\xrightarrow{D} \boldsymbol{W}^{(r,b)} \sim N\left(\mathbf{0}, \Sigma_k^{(r,b)}\right),\\
\sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right) &\xrightarrow{D} \boldsymbol{W}^{*(r,a)} \sim N\left(\mathbf{0}, \Sigma_k^{(r,a)}\right),\\
\sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right) &\xrightarrow{D} \boldsymbol{W}^{*(r,b)} \sim N\left(\mathbf{0}, \Sigma_k^{(r,b)}\right),
\end{aligned}$$

where

$$\overline{\boldsymbol{W}}_{kN}^{(r,a)} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{W}_{kj}^{(r,a)}, \qquad \overline{\boldsymbol{W}}_{kN}^{(r,b)} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{W}_{kj}^{(r,b)},$$

$$\overline{\boldsymbol{W}^*}_{kN}^{(r,a)} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{W^*}_{kj}^{(r,a)}, \qquad \overline{\boldsymbol{W}^*}_{kN}^{(r,b)} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{W^*}_{kj}^{(r,b)}.$$

Whence

$$\begin{aligned}
T_{kN}^{(r,a)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right)'\left(\Sigma_k^{(r,a)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right) \xrightarrow{D} \chi^2(2),\\
T_{kN}^{(r,b)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right)'\left(\Sigma_k^{(r,b)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right) \xrightarrow{D} \chi^2(2),
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
T_{kN}^{*(r,a)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{*(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right)'\left(\Sigma_k^{(r,a)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,a)} - \boldsymbol{\mu}_k^{(r,a)}\right) \xrightarrow{D} \chi^2(2),\\
T_{kN}^{*(r,b)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{*(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right)'\left(\Sigma_k^{(r,b)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,b)} - \boldsymbol{\mu}_k^{(r,b)}\right) \xrightarrow{D} \chi^2(2),
\end{aligned} \tag{3.2}$$

with

$$\begin{aligned}
\left(\Sigma^{(r,a)}\right)^{-1} &= \frac{1}{\Delta^{(r)}}\begin{bmatrix} c^{(r)} & -b^{(r,a)} \\ -b^{(r,a)} & a^{(r,a)} \end{bmatrix},\\
\left(\Sigma^{(r,b)}\right)^{-1} &= \frac{1}{\Delta^{(r)}}\begin{bmatrix} c^{(r,b)} & -b^{(r,b)} \\ -b^{(r,b)} & a^{(r)} \end{bmatrix}.
\end{aligned}$$

The test-statistics $T_{kN}^{(r,a)}$ and $T_{kN}^{(r,b)}$ in (3.1) can be written in extended forms:

$$T_{kN}^{(r,a)} = \frac{N}{\Delta^{(r)}}\left\{c^{(r)}\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(akh_{r+1}(U_{kj}) + h_r(U_{kj})\right)\right.\right.$$

$$
- [1 + a(r+1)]\Gamma(r+1)\Bigg)^2 - 2b^{(r,a)}\Bigg(\frac{k^r}{N}\sum_{j=1}^{N}(akh_{r+1}(U_{kj}) + h_r(U_{kj}))
$$

$$
- [1 + a(r+1)]\Gamma(r+1)\Bigg)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}(U_{kj}) - \Gamma(r+2)\right)
$$

$$
+ a^{(r,a)}\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}(U_{kj}) - \Gamma(r+2)\right)^2\Bigg\}
$$

and

$$
T_{kN}^{(r,b)} = \frac{N}{\Delta^{(r)}}\Bigg\{c^{(r,b)}\left(\frac{k^r}{N}\sum_{j=1}^{N}h_r(U_{kj}) - \Gamma(r+1)\right)^2
$$

$$
- 2b^{(r,b)}\left(\frac{k^r}{N}\sum_{j=1}^{N}(kh_{r+1}(U_{kj}) + bh_r(U_{kj})) - (b+r+1)\Gamma(r+1)\right)
$$

$$
\times\left(\frac{k^r}{N}\sum_{j=1}^{N}h_r(U_{kj}) - \Gamma(r+1)\right)
$$

$$
+ a^{(r)}\left(\frac{k^r}{N}\sum_{j=1}^{N}(kh_{r+1}(U_{kj}) + bh_r(U_{kj})) - (b+r+1)\Gamma(r+1)\right)^2\Bigg\}.
$$

Similar formulae can be written for $T_{kN}^{*(r,a)}$ and $T_{kN}^{*(r,b)}$, i.e.

$$
T_{kN}^{*(r,a)} = \frac{N}{\Delta^{(r)}}\Bigg\{c^{(r)}\Bigg(\frac{k^r}{N}\sum_{j=1}^{N}\left(akh_{r+1}^*(V_{kj}) + h_r^*(V_{kj})\right)
$$

$$
- [1 + a(r+1)]\Gamma(r+1)\Bigg)^2 - 2b^{(r,a)}\Bigg(\frac{k^r}{N}\sum_{j=1}^{N}\left(akh_{r+1}^*(V_{kj}) + h_r^*(V_{kj})\right)
$$

$$
- [1 + a(r+1)]\Gamma(r+1)\Bigg)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}^*(V_{kj}) - \Gamma(r+2)\right)
$$

$$
+ a^{(r,a)}\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}^*(V_{kj}) - \Gamma(r+2)\right)^2\Bigg\},
$$

$$T_{kN}^{*(r,b)} = \frac{N}{\Delta^{(r)}}\left\{c^{(r,b)}\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1)\right)^2\right.$$

$$-2b^{(r,b)}\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(kh_{r+1}^*(V_{kj}) + bh_r^*(V_{kj})\right) - (b+r+1)\Gamma(r+1)\right)$$

$$\times\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1)\right)$$

$$\left.+\,a^{(r)}\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(kh_{r+1}^*(V_{kj}) + bh_r^*(V_{kj})\right) - (b+r+1)\Gamma(r+1)\right)^2\right\}.$$

Note that for any values of a and b the tests $T_{kN}^{(r,a)}$, $T_{kN}^{(r,b)}$, $T_{kN}^{*(r,a)}$, $T_{kN}^{*(r,b)}$ are equal to $T_{kN}^{(r)}$ and $T_{kN}^{*(r)}$, respectively, i.e.

$$T_{kN}^{(r)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$

$$\times\left[\left(c^{(r)} - u_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right)^2\right.$$

$$-2\left(b^{(r)} - t_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right)$$

$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right)$$

$$\left.+\left(a^{(r)} - s_k^{(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right)^2\right]$$

$$T_{kN}^{*(r)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$

$$\times\left[\left(c^{(r)} - u_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1)\right)^2\right.$$

$$
-2\left(b^{(r)}-t_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}h_r^*(V_{kj})-\Gamma(r+1)\right)
$$

$$
\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}^*(V_{kj})-\Gamma(r+2)\right)
$$

$$
+\left(a^{(r)}-s_k^{(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}h_{r+1}^*(V_{kj})-\Gamma(r+2)\right)^2\Bigg]
$$

Hence we consider the following partitions of these tests

$$
\begin{aligned}
T_{kN}^{(r)} &= T_{kN}^{(r,a)} = T_{kN;c_1}^{(r,a)} + T_{kN;c_2}^{(r,a)} = T_{kN;c_3}^{(r)} + T_{kN;c_4}^{(r)}\,,\\
T_{kN}^{(r)} &= T_{kN}^{(r,b)} = T_{kN;c_1}^{(r)} + T_{kN;c_2}^{(r)} = T_{kN;c_3}^{(r,b)} + T_{kN;c_4}^{(r,b)}\,,\\
T_{kN}^{*(r)} &= T_{kN}^{*(r,a)} = T_{kN;c_1}^{*(r,a)} + T_{kN;c_2}^{*(r,a)} = T_{kN;c_3}^{*(r)} + T_{kN;c_4}^{*(r)}\,,\\
T_{kN}^{*(r)} &= T_{kN}^{*(r,b)} = T_{kN;c_1}^{*(r)} + T_{kN;c_2}^{*(r)} = T_{kN;c_3}^{*(r,b)} + T_{kN;c_4}^{*(r,b)}\,,
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
T_{kN;c_1}^{(r,a)} &= \frac{N}{a^{(r)} + 2ab^{(r)} + a^2c^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right.\\
&\quad \left. + a\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right)\right]^2,\\
T_{kN;c_2}^{(r,a)} &= \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right)\left(a^{(r)} + 2ab^{(r)} + a^2c^{(r)}\right)}\\
&\quad \times \left[\left(b^{(r)} + ac^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right)\right.\\
&\quad \left. - \left(a^{(r)} + ab^{(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right)\right]^2\\
T_{kN;c_3}^{(r)} &= \frac{N}{c^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right]^2\\
T_{kN;c_4}^{(r)} &= \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right)c^{(r)}}\left[c^{(r)}\left(\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right)\right.\\
&\quad \left. - b^{(r)}\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right)\right]^2\\
T_{kN;c_1}^{(r)} &= \frac{N}{a^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right]^2,
\end{aligned}
$$

$$T_{kN;c_2}^{(r)} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) a^{(r)}} \left[b^{(r)} \left(\frac{k^r}{N} \sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. - a^{(r)} \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2) \right) \right]^2,$$

$$T_{kN;c_3}^{(r,b)} = \frac{N}{c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}} \left[b \left(\frac{k^r}{N} \sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. + \frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2) \right]^2,$$

$$T_{kN;c_4}^{(r,b)} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) \left(c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} + bb^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. - \left(b^{(r)} + ba^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2) \right) \right]^2,$$

and

$$T_{kN;c_1}^{*(r,a)} = \frac{N}{a^{(r)} + 2ab^{(r)} + a^2 c^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1) \right.$$

$$\left. + a \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}^*(V_{kj}) - \Gamma(r+2) \right) \right]^2,$$

$$T_{kN;c_2}^{*(r,a)} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) \left(a^{(r)} + 2ab^{(r)} + a^2 c^{(r)}\right)}$$

$$\times \left[\left(b^{(r)} + ac^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. - \left(a^{(r)} + ab^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}^*(V_{kj}) - \Gamma(r+2) \right) \right]^2,$$

$$T^{*(r)}_{kN;c_3} = \frac{N}{c^{(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} h^*_{r+1}(V_{kj}) - \Gamma(r+2) \right]^2,$$

$$T^{*(r)}_{kN;c_4} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) c^{(r)}} \Bigg[c^{(r)} \left(\frac{k^r}{N} \sum_{j=1}^{N} h^*_r(V_{kj}) - \Gamma(r+1) \right)$$
$$- b^{(r)} \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h^*_{r+1}(V_{kj}) - \Gamma(r+2) \right) \Bigg]^2,$$

$$T^{*(r)}_{kN;c_1} = \frac{N}{a^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} h^*_r(V_{kj}) - \Gamma(r+1) \right]^2,$$

$$T^{*(r)}_{kN;c_2} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) a^{(r)}} \Bigg[b^{(r)} \left(\frac{k^r}{N} \sum_{j=1}^{N} h^*_r(V_{kj}) - \Gamma(r+1) \right)$$
$$- a^{(r)} \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h^*_{r+1}(V_{kj}) - \Gamma(r+2) \right) \Bigg]^2,$$

$$T^{*(r,b)}_{kN;c_3} = \frac{N}{c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}} \Bigg[b \left(\frac{k^r}{N} \sum_{j=1}^{N} h^*_r(V_{kj}) - \Gamma(r+1) \right)$$
$$+ \frac{k^{r+1}}{N} \sum_{j=1}^{N} h^*_{r+1}(V_{kj}) - \Gamma(r+2) \Bigg]^2,$$

$$T^{*(r,b)}_{kN;c_4} = \frac{N}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right)\left(c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}\right)}$$
$$\times \Bigg[\left(c^{(r)} + bb^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} h^*_r(V_{kj}) - \Gamma(r+1) \right)$$
$$- \left(b^{(r)} + ba^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} h^*_{r+1}(V_{kj}) - \Gamma(r+2) \right) \Bigg]^2.$$

Note that

$$T^{(r)}_{kN;c_3} = T^{(r+1)}_{kN;c_1} \quad \text{and} \quad T^{*(r)}_{kN;c_3} = T^{*(r+1)}_{kN;c_1}.$$

When $k = 1$ we have

$$T_{1n}^{(r)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \left[\left(c^{(r)} - u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1)\right)^2\right.$$
$$- 2\left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1)\right),$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2)\right)$$
$$\left. + \left(a^{(r)} - s_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2)\right)^2\right],$$

its representations $T_{1n}^{(r,a)}$ and $T_{1n}^{(r,b)}$ for partitions are

$$T_{1n}^{(r,a)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \left[\left(c^{(r)} - u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_r(X_j)\left(ah(X_j)+1\right) + \frac{1}{n}\sum_{j=1}^{n} h_r(X_j)\right.\right.$$
$$\left. - \left[1 + a(r+1)\right]\Gamma(r+1)\right)^2$$
$$-2\left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_r(X_j)\left(ah(X_j)+1\right) + \frac{1}{n}\sum_{j=1}^{n} h_r(X_j)\right.$$
$$\left. - \left[1 + a(r+1)\right]\Gamma(r+1)\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2)\right)$$
$$\left. + \left(a^{(r)} - s_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2)\right)^2\right],$$
$$T_{1n}^{(r,b)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$

$$\times\left[\left(c^{(r)}-u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)\right)^2\right.$$
$$-2\left(b^{(r)}-t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)+\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)\left(h(X_j)+b\right)\right.$$
$$\left.-(b+r+1)\Gamma(r+1)\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)\right)$$
$$+\left(a^{(r)}-s_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)+\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)\left(h(X_j)+b\right)\right.$$
$$\left.\left.-(b+r+1)\Gamma(r+1)\right)^2\right]$$

and their components in the partitions (3.3) are given by

$$T_{1n;c_1}^{(r,a)}=\frac{n}{a^{(r)}+2ab^{(r)}+a^2c^{(r)}}$$
$$\times\left[\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)+a\left(\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)-\Gamma(r+2)\right)\right]^2,$$
$$T_{1n;c_2}^{(r,a)}=\frac{n}{\left(a^{(r)}c^{(r)}-\left(b^{(r)}\right)^2\right)\left(a^{(r)}+2ab^{(r)}+a^2c^{(r)}\right)}$$
$$\times\left[\left(b^{(r)}+ac^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)\right)\right.$$
$$\left.-\left(a^{(r)}+ab^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)-\Gamma(r+2)\right)\right]^2,$$
$$T_{1n;c_3}^{(r)}=\frac{n}{c^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)-\Gamma(r+2)\right]^2,,$$
$$T_{1n;c_4}^{(r)}=\frac{n}{\left(a^{(r)}c^{(r)}-\left(b^{(r)}\right)^2\right)c^{(r)}}\left[c^{(r)}\left(\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)\right)\right.$$

$$- b^{(r)} \left(\frac{1}{n} \sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2) \right) \Bigg]^2,$$

$$T_{1n;c_1}^{(r)} = \frac{n}{a^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1) \right]^2,$$

$$T_{1n;c_2}^{(r)} = \frac{n}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) a^{(r)}} \Bigg[b^{(r)} \left(\frac{1}{n} \sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1) \right)$$

$$- a^{(r)} \left(\frac{1}{n} \sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2) \right) \Bigg]^2,$$

$$T_{1n;c_3}^{(r,b)} = \frac{n}{c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}} \Bigg[b \left(\frac{1}{n} \sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1) \right)$$

$$+ \frac{1}{n} \sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2) \Bigg]^2,$$

$$T_{1n;c_4}^{(r,b)} = \frac{n}{\left(a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2\right) \left(c^{(r)} + 2bb^{(r)} + b^2 a^{(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} + bb^{(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} h_r(X_j) - \Gamma(r+1) \right)$$

$$- \left(b^{(r)} + ba^{(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} h_{r+1}(X_j) - \Gamma(r+2) \right) \Bigg]^2.$$

Similar formulae can be written for $T_{1n}^{*(r)}$, $T_{1n}^{*(r,a)}$, $T_{1n}^{*(r,b)}$, $T_{1n;c_1}^{*(r,a)}$, $T_{1n;c_2}^{*(r,a)}$, $T_{1n;c_3}^{*(r)}$, $T_{1n;c_4}^{*(r)}$, $T_{1n;c_1}^{*(r)}$, $T_{1n;c_2}^{*(r)}$, $T_{1n;c_3}^{*(r,b)}$, $T_{1n;c_4}^{*(r,b)}$.

3.2. Hypotheses Involving Unknown Parameters

With the usual notation and assumptions F now has the form $F(x;\boldsymbol{\lambda})$ and $\boldsymbol{\lambda}(p\times 1)$ are unknown identifiable parameters with true value $\boldsymbol{\lambda}_0$ in the parameter space Λ, the pdf is denoted by $f(x;\boldsymbol{\lambda})$. In this case we need the following

Theorem 2. *Let* $\widehat{\boldsymbol{T}}_n = \boldsymbol{T}_n(X_1,\ldots,X_n;\widehat{\boldsymbol{\lambda}}_n)$, *where* $\widehat{\boldsymbol{\lambda}}_n = \widehat{\boldsymbol{\lambda}}_n(X_1,\ldots,X_n)$ *is an estimator of a parameter* $\boldsymbol{\lambda}$, *and moreover let* $\boldsymbol{T}_n = \boldsymbol{T}_n(X_1,\ldots,X_n;\boldsymbol{\lambda})$

(here $\boldsymbol{T}_n$, $\boldsymbol{\lambda}$ and $\widehat{\boldsymbol{\lambda}}_n$ may be vectors). Suppose that:

(i) *For each $\boldsymbol{\lambda}$,*

$$\sqrt{n}\begin{bmatrix}\boldsymbol{T}_n \\ \widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}\end{bmatrix} \xrightarrow{D} \boldsymbol{T} \sim N(\mathbf{0}, \boldsymbol{V}),$$

where

$$\boldsymbol{V} = \begin{bmatrix}\boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22}\end{bmatrix}$$

and $\boldsymbol{V}_{22}$ is nonsingular.

(ii) *There is a matrix $\boldsymbol{B}$, possibly depending continuously on $\boldsymbol{\lambda}$ such that*

$$\sqrt{n}\widehat{\boldsymbol{T}}_n = \sqrt{n}\boldsymbol{T}_n + \boldsymbol{B}\sqrt{n}\left(\widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}\right) + o_p(1).$$

(iii) *$\widehat{\boldsymbol{\lambda}}_n$ is asymptotically efficient.*

Then

$$\sqrt{n}\widehat{\boldsymbol{T}}_n \xrightarrow{D} \boldsymbol{T}^0 \sim N(\mathbf{0}, \boldsymbol{V}_{11} - \boldsymbol{B}\boldsymbol{V}_{22}\boldsymbol{B}')$$

(cf. Pierce [16]).

Note that (ii) is satisfied when $\boldsymbol{T}_n$ is differentiable in $\boldsymbol{\lambda}$, and then

$$\boldsymbol{B} = \lim_{n\to\infty} E\left[\frac{\partial}{\partial\boldsymbol{\lambda}}\boldsymbol{T}_n\right].$$

Next from 3.1. (CLT preceding (3.1)), when $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0$,

$$\begin{aligned}\boldsymbol{D}_{kN}^{(r,a)}(\boldsymbol{\lambda}_0) &:= \sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)}(\boldsymbol{\lambda}_0) - \boldsymbol{\mu}_k^{(r,a)}\right) \xrightarrow{D} \boldsymbol{D}_k^{(r,a)} \sim N\left(\mathbf{0}, \Sigma_{k1}^{(r,a)}\right), \\ \boldsymbol{D}_{kN}^{(r,b)}(\boldsymbol{\lambda}_0) &:= \sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)}(\boldsymbol{\lambda}_0) - \boldsymbol{\mu}_k^{(r,b)}\right) \xrightarrow{D} \boldsymbol{D}_k^{(r,b)} \sim N\left(\mathbf{0}, \Sigma_{k1}^{(r,b)}\right).\end{aligned}$$

When $\boldsymbol{\lambda}$ is replaced by an estimator $\hat{\boldsymbol{\lambda}}_n$ (here we use MLE) for which

$$\sqrt{n}\left(\hat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}_0\right) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{A}(\boldsymbol{\lambda}_0)),$$

where in the regular case $\boldsymbol{A}(\boldsymbol{\lambda}) = \mathcal{I}^{-1}(\boldsymbol{\lambda})$ and $\mathcal{I}$ is the expected information matrix based on a single observation, and in other cases $\boldsymbol{A}$ is singular, the quoted theorem gives

$$\begin{aligned}\Sigma_{k1}^{(r,a)} &= \Sigma^{(r,a)} - \frac{1}{k}\boldsymbol{B}_k^{(r,a)}\boldsymbol{A}(\boldsymbol{\lambda}_0)\left(\boldsymbol{B}_k^{(r,a)}\right)' = \Sigma^{(r,a)} - \boldsymbol{K}_k^{(r,a)}, \\ \Sigma_{k1}^{(r,b)} &= \Sigma^{(r,b)} - \frac{1}{k}\boldsymbol{B}_k^{(r,b)}\boldsymbol{A}(\boldsymbol{\lambda}_0)\left(\boldsymbol{B}_k^{(r,b)}\right)' = \Sigma^{(r,b)} - \boldsymbol{K}_k^{(r,b)},\end{aligned} \tag{3.4}$$

with

$$\boldsymbol{B}_k^{(r,a)}(2\times p) = E\left[\left(\frac{\partial \overline{\boldsymbol{W}}_{kN}^{(r,a)}(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}}\right)_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_0}\right],$$
$$\boldsymbol{B}_k^{(r,b)}(2\times p) = E\left[\left(\frac{\partial \overline{\boldsymbol{W}}_{kN}^{(r,b)}(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}}\right)_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_0}\right],$$

and

$$\begin{aligned}\boldsymbol{K}_k^{(r,a)} &= \boldsymbol{B}_k^{(r,a)}(k\mathcal{I})^{-1}\left(\boldsymbol{B}_k^{(r,a)}\right)', \quad \boldsymbol{K}_k^{(r,b)} = \boldsymbol{B}_k^{(r,b)}(k\mathcal{I})^{-1}\left(\boldsymbol{B}_k^{(r,b)}\right)',\\ \boldsymbol{K}_k^{(r)} &= \boldsymbol{B}_k^{(r)}(k\mathcal{I})^{-1}\left(\boldsymbol{B}_k^{(r)}\right)'\end{aligned} \tag{3.5}$$

in the regular case.

Now

$$\begin{aligned}E&\left[\frac{\partial \overline{W}_{kN1}^{(r,a)}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] = E\left[\frac{\partial}{\partial \lambda_j}\left(k^r h_r(X_{1:k};\boldsymbol{\lambda}) + a k^{r+1} h_{r+1}(X_{1:k};\boldsymbol{\lambda})\right)\right]\\
&= r k^r E\left[h_{r-1}(X_{1:k};\boldsymbol{\lambda})\frac{1}{1-F(X_{1:k};\boldsymbol{\lambda})}\frac{\partial F(X_{1:k};\boldsymbol{\lambda})}{\partial \lambda_j}\right]\\
&+ (r+1)k^{r+1} a E\left[h_r(X_{1:k};\boldsymbol{\lambda})\frac{1}{1-F(X_{1:k};\boldsymbol{\lambda})}\frac{\partial F(X_{1:k};\boldsymbol{\lambda})}{\partial \lambda_j}\right]\\
&= r k^{r+1}\int h_{r-1}(x)(1-F(x;\boldsymbol{\lambda}))^{k-2}\frac{\partial F(x;\boldsymbol{\lambda})}{\partial \lambda_j} f(x;\boldsymbol{\lambda})dx\\
&+ a(r+1)k^{r+2}\int h_r(x)(1-F(x;\boldsymbol{\lambda}))^{k-2}\frac{\partial F(x;\boldsymbol{\lambda})}{\partial \lambda_j} f(x;\boldsymbol{\lambda})dx\\
&= r b_k^{(r)}(\lambda_j) + a(r+1) b_k^{(r+1)}(\lambda_j),\end{aligned}$$

since $X_{1:k}$ has pdf $k(1-F)^{k-1}f$, where we set

$$b_k^{(r)}(\lambda_j) = k^{r+1} E\left(1-F(X;\boldsymbol{\lambda})\right)^{k-2} h_{r-1}(X)\frac{\partial F(X;\boldsymbol{\lambda})}{\partial \lambda_j},$$
$$j = 1,\ldots,p.$$

Similarly,

$$E\left[\frac{\partial \overline{W}_{kN2}^{(r,a)}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] = (r+1) b_k^{(r+1)}(\lambda_j).$$

Thus

$$\boldsymbol{B}_k^{(r)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{(r)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{(r+1)}\right)' \end{bmatrix},$$

$$\boldsymbol{B}_k^{(r,a)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{(r)}\right)' + a(r+1)\left(\boldsymbol{b}_k^{(r+1)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{(r+1)}\right)' \end{bmatrix},$$

$$\boldsymbol{B}_k^{(r,b)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{(r)}\right)' \\ br\left(\boldsymbol{b}_k^{(r)}\right)' + (r+1)\left(\boldsymbol{b}_k^{(r+1)}\right)' \end{bmatrix}, \qquad \boldsymbol{b}_k^{(r)} := \boldsymbol{b}_k^{(r)}(\boldsymbol{\lambda}).$$

Then simple asymptotic tests of H_0 are obtained from

$$\begin{aligned} \hat{T}_{kN}^{(r,a)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,a)}\right)'\left(\Sigma_{k1}^{(r,a)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{(r,a)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,a)}\right) \\ &\to \chi^2(2)\,, \\ \hat{T}_{kN}^{(r,b)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,b)}\right)'\left(\Sigma_{k1}^{(r,b)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{(r,b)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,b)}\right) \\ &\to \chi^2(2) \end{aligned} \tag{3.6}$$

on H_0, where

$$\overline{\boldsymbol{W}}_{kN}^{(r,a)} = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} R_{kj}^{(r)}(\boldsymbol{\lambda}) + aR_{kj}^{(r+1)}(\boldsymbol{\lambda}) \\ R_{kj}^{(r+1)}(\boldsymbol{\lambda}) \end{bmatrix},$$

$$\overline{\boldsymbol{W}}_{kN}^{(r,b)} = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} R_{kj}^{(r)}(\boldsymbol{\lambda}) \\ bR_{kj}^{(r)}(\boldsymbol{\lambda}) + R_{kj}^{(r+1)}(\boldsymbol{\lambda}) \end{bmatrix},$$

$$R_{kj}^{(r)}(\boldsymbol{\lambda}) = k^r\left[-\log(1 - F(U_{kj};\boldsymbol{\lambda}))\right]^r.$$

The corresponding dual tests, obtained by replacing $1-F(x;\boldsymbol{\lambda})$ by $F(x;\boldsymbol{\lambda})$, and $1-F(U_{kj};\boldsymbol{\lambda})$ by $F(V_{kj};\boldsymbol{\lambda})$ use

$$\begin{aligned} \hat{T}_{kN}^{*(r,a)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{*(r,a)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,a)}\right)'\left(\Sigma_{k1}^{*(r,a)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,a)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,a)}\right) \\ &\xrightarrow{D} \chi^2(2)\,, \\ \hat{T}_{kN}^{*(r,b)} &= N\left(\overline{\boldsymbol{W}}_{kN}^{*(r,b)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,b)}\right)'\left(\Sigma_{k1}^{*(r,b)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{*(r,b)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r,b)}\right) \\ &\xrightarrow{D} \chi^2(2) \end{aligned} \tag{3.7}$$

on H_0, where

$$\overline{\boldsymbol{W}}_{kN}^{*(r,a)}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} R_{kj}^{*(r)}(\boldsymbol{\lambda}) + aR_{kj}^{*(r+1)}(\boldsymbol{\lambda}) \\ R_{kj}^{*(r+1)}(\boldsymbol{\lambda}) \end{bmatrix},$$

$$\overline{\boldsymbol{W}}_{kN}^{*(r,b)}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} R_{kj}^{*(r)}(\boldsymbol{\lambda}) \\ bR_{kj}^{*(r)}(\boldsymbol{\lambda}) + R_{kj}^{*(r+1)}(\boldsymbol{\lambda}) \end{bmatrix},$$

$$R_{kj}^{*(r)}(\boldsymbol{\lambda}) = k^r\left[-\log(F(V_{kj};\boldsymbol{\lambda}))\right]^r,$$

$$\Sigma_{k1}^{*(r,a)} = \Sigma^{(r,a)} - \boldsymbol{K}_k^{*(r,a)}, \qquad \Sigma_{k1}^{*(r,b)} = \Sigma^{*(r,b)} - \boldsymbol{K}_k^{*(r,b)}, \tag{3.8}$$

with

$$\begin{aligned} \boldsymbol{K}_k^{*(r,a)} &= \frac{1}{k}\boldsymbol{B}_k^{*(r,a)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{*(r,a)}\right)', \quad \boldsymbol{K}_k^{*(r,b)} = \frac{1}{k}\boldsymbol{B}_k^{*(r,b)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{*(r,b)}\right)', \\ \boldsymbol{K}_k^{*(r)} &= \frac{1}{k}\boldsymbol{B}_k^{*(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{*(r)}\right)' \end{aligned} \tag{3.9}$$

and

$$\boldsymbol{B}_k^{*(r)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{*(r)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{*(r+1)}\right)' \end{bmatrix},$$

$$\boldsymbol{B}_k^{*(r,a)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{*(r)}\right)' + a(r+1)\left(\boldsymbol{b}_k^{*(r+1)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{*(r+1)}\right)' \end{bmatrix},$$

$$\boldsymbol{B}_k^{*(r,b)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{*(r)}\right)' \\ br\left(\boldsymbol{b}_k^{*(r)}\right)' + (r+1)\left(\boldsymbol{b}_k^{*(r+1)}\right)' \end{bmatrix},$$

where

$$b_k^{*(r)}(\lambda_j) = -k^{r+1}E\left[F^{k-2}(X;\boldsymbol{\lambda})h_{r-1}^{*}(X;\boldsymbol{\lambda})\frac{\partial F(X;\boldsymbol{\lambda})}{\partial \lambda_j}\right], \quad j = 1,\ldots,p.$$

We write

$$\mathcal{I}^{-1} := \begin{bmatrix} i_{11} & i_{12} \\ i_{12} & i_{22} \end{bmatrix},$$

$$\boldsymbol{K}_k^{(r)} := \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)} \end{bmatrix}, \qquad \boldsymbol{K}_k^{*(r)} := \begin{bmatrix} s_k^{*(r)} & t_k^{*(r)} \\ t_k^{*(r)} & u_k^{*(r)} \end{bmatrix}$$

$$\boldsymbol{K}_k^{(r,a)} := \begin{bmatrix} s_k^{(r,a)} & t_k^{(r,a)} \\ t_k^{(r,a)} & u_k^{(r)} \end{bmatrix}, \quad \boldsymbol{K}_k^{*(r,a)} := \begin{bmatrix} s_k^{*(r,a)} & t_k^{*(r,a)} \\ t_k^{*(r,a)} & u_k^{*(r)} \end{bmatrix},$$

$$\boldsymbol{K}_k^{(r,b)} := \begin{bmatrix} s_k^{(r)} & t_k^{(r,b)} \\ t_k^{(r,b)} & u_k^{(r,b)} \end{bmatrix}, \qquad \boldsymbol{K}_k^{*(r,b)} := \begin{bmatrix} s_k^{*(r)} & t_k^{*(r,b)} \\ t_k^{*(r,b)} & u_k^{*(r,b)} \end{bmatrix},$$

$$\begin{aligned}
\Sigma_{k1}^{(r,a)} &= \Sigma^{(r,a)} - \boldsymbol{K}_k^{(r,a)} := \begin{bmatrix} a_{k1}^{(r,a)} & b_{k1}^{(r,a)} \\ b_{k1}^{(r,a)} & c_{k1}^{(r)} \end{bmatrix}, \\
&= \begin{bmatrix} a^{(r,a)} - s_k^{(r,a)} & b^{(r,a)} - t_k^{(r,a)} \\ b^{(r,a)} - t_k^{(r,a)} & c^{(r)} - u_k^{(r)} \end{bmatrix}, \\
\Sigma_{k1}^{(r,b)} &= \Sigma^{(r,b)} - \boldsymbol{K}_k^{(r,b)} := \begin{bmatrix} a_{k1}^{(r)} & b_{k1}^{(r,b)} \\ b_{k1}^{(r,b)} & c_{k1}^{(r,b)} \end{bmatrix}, \\
&= \begin{bmatrix} a^{(r)} - s_k^{(r)} & b^{(r,b)} - t_k^{(r,b)} \\ b^{(r,b)} - t_k^{(r,b)} & c^{(r,b)} - u_k^{(r,b)} \end{bmatrix} \\
\Sigma_{k1}^{*(r,a)} &= \Sigma^{(r,a)} - \boldsymbol{K}_k^{*(r,a)} = \begin{bmatrix} a^{(r,a)} - s_k^{*(r,a)} & b^{(r,a)} - t_k^{*(r,a)} \\ b^{(r,a)} - t_k^{*(r,a)} & c^{(r,a)} - u_k^{*(r)} \end{bmatrix}, \\
\Sigma_{k1}^{*(r,b)} &= \Sigma^{(r,b)} - \boldsymbol{K}_k^{*(r,b)} = \begin{bmatrix} a^{(r)} - s_k^{*(r)} & b^{(r,b)} - t_k^{*(r,b)} \\ b^{(r,b)} - t_k^{*(r,b)} & c^{(r,b)} - u_k^{*(r,b)} \end{bmatrix}.
\end{aligned}$$

Note that

$$\begin{aligned}
s_k^{(r,a)} &= s_k^{(r)} + 2at_k^{(r)} + a^2 u_k^{(r)}, \qquad t_k^{(r,a)} = t_k^{(r)} + au_k^{(r)}, \\
t_k^{(r,b)} &= t_k^{(r)} + bs_k^{(r)}, \qquad u_k^{(r,b)} = u_k^{(r)} + 2bt_k^{(r)} + b^2 s_k^{(r)}, \\
a_{k1}^{(r,a)} &= a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right), \\
b_{k1}^{(r,a)} &= b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right), \\
c_{k1}^{(r)} &= c^{(r)} - u_k^{(r)}, \qquad a_{k1}^{(r)} = a^{(r)} - s_k^{(r)}, \\
b_{k1}^{(r,b)} &= b^{(r)} - t_k^{(r)} + b\left(a^{(r)} - s_k^{(r)}\right), \\
c_{k1}^{(r,b)} &= c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right).
\end{aligned}$$

Moreover, we see that in the most frequently used cases with $\boldsymbol{\lambda} = (\alpha)$ and $\boldsymbol{\lambda} = (\alpha, \beta)$ we have respectively

$$s_k^{(r)} = \frac{r^2}{k}\left(b_k^{(r)}(\alpha)\right)^2 i_{11}, \qquad t_k^{(r)} = \frac{r(r+1)}{k} b_k^{(r)}(\alpha) b_k^{(r+1)}(\alpha) i_{11},$$

$$u_k^{(r)} = \frac{(r+1)^2}{k}\left(b_k^{(r+1)}(\alpha)\right)^2 i_{11}\,,$$

with

$$b_k^{(r)}(\alpha) = k^{r+1}E\left[(1-F(X;\alpha))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha)} \times \frac{\partial F(X;\alpha)}{\partial\alpha}\right]$$

and

$$\begin{aligned}
s_k^{(r)} &= \frac{r^2}{k}\left[\left(b_k^{(r)}(\alpha)\right)^2 i_{11} + 2b_k^{(r)}(\alpha)b_k^{(r)}(\beta)i_{12} + \left(b_k^{(r)}(\beta)\right)^2 i_{22}\right],\\
t_k^{(r)} &= \frac{r(r+1)}{k}\Big[b_k^{(r)}(\alpha)b_k^{(r+1)}(\alpha)i_{11}\\
&\quad + \left(b_k^{(r)}(\alpha)b_k^{(r+1)}(\beta) + b_k^{(r+1)}(\alpha)b_k^{(r)}(\beta)\right)i_{12} + b_k^{(r)}(\beta)b_k^{(r+1)}(\beta)i_{22}\Big],\\
u_k^{(r)} &= \frac{(r+1)^2}{k}\left[\left(b_k^{(r+1)}(\alpha)\right)^2 i_{11} + 2b_k^{(r+1)}(\alpha)b_k^{(r+1)}(\beta)i_{12}\right.\\
&\quad \left. + \left(b_k^{(r+1)}(\beta)\right)^2 i_{22}\right],
\end{aligned}$$

with

$$\begin{aligned}
&b_k^{(r)}(\alpha)\\
&= k^{r+1}E\left[(1-F(X;\alpha,\beta))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha,\beta)} \times \frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right],\\
&b_k^{(r)}(\beta)\\
&= k^{r+1}E\left[(1-F(X;\alpha,\beta))^{k-2}\log^{r-1}\frac{1}{1-F(X;\alpha,\beta)} \times \frac{\partial F(X;\alpha,\beta)}{\partial\beta}\right].
\end{aligned}$$

Similar formulae can be written for $s_k^{*(r)}$, $t_k^{*(r)}$ and $u_k^{*(r)}$, i.e.

$$\begin{aligned}
s_k^{*(r)} &= \frac{r^2}{k}\left(b_k^{*(r)}(\alpha)\right)^2 i_{11}, \qquad t_k^{*(r)} = \frac{r(r+1)}{k}b_k^{*(r)}(\alpha)b_k^{*(r+1)}(\alpha)i_{11},\\
u_k^{*(r)} &= \frac{(r+1)^2}{k}\left(b_k^{*(r+1)}(\alpha)\right)^2 i_{11}\,,
\end{aligned}$$

with

$$b_k^{*(r)}(\alpha) = -k^{r+1}E\left[(F(X;\alpha))^{k-2}\log^{r-1}\frac{1}{F(X;\alpha)} \times \frac{\partial F(X;\alpha)}{\partial\alpha}\right]$$

and

$$s_k^{*(r)} = \frac{r^2}{k}\left[\left(b_k^{*(r)}(\alpha)\right)^2 i_{11} + 2b_k^{*(r)}(\alpha)b_k^{*(r)}(\beta)i_{12} + \left(b_k^{*(r)}(\beta)\right)^2 i_{22}\right],$$
$$t_k^{*(r)} = \frac{r(r+1)}{k}\Big[b_k^{*(r)}(\alpha)b_k^{*(r+1)}(\alpha)i_{11} + \Big(b_k^{*(r)}(\alpha)b_k^{*(r+1)}(\beta) + b_k^{*(r+1)}(\alpha)b_k^{*(r)}(\beta)\Big) i_{12} + b_k^{*(r)}(\beta)b_k^{*(r+1)}(\beta)i_{22}\Big]$$
$$u_k^{*(r)} = \frac{(r+1)^2}{k}\left[\left(b_k^{*(r+1)}(\alpha)\right)^2 i_{11} + 2b_k^{*(r+1)}(\alpha)b_k^{*(r+1)}(\beta)i_{12} + \left(b_k^{*(r+1)}(\beta)\right)^2 i_{22}\right],$$

with

$$b_k^{*(r)}(\alpha) = -k^{r+1}E\left[F^{k-2}(X;\alpha,\beta)\log^{r-1}\frac{1}{F(X;\alpha,\beta)} \times \frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right],$$
$$b_k^{*(r)}(\beta) = -k^{r+1}E\left[F^{k-2}(X;\alpha,\beta)\log^{r-1}\frac{1}{F(X;\alpha,\beta)} \times \frac{\partial F(X;\alpha,\beta)}{\partial\beta}\right].$$

Hence

$$\nabla_k^{(r)} := \det\left(\boldsymbol{K}_k^{(r)}\right) = s_k^{(r)}u_k^{(r)} - \left(t_k^{(r)}\right)^2,$$
$$\nabla_k^{*(r)} := \det\left(\boldsymbol{K}_k^{*(r)}\right) = s_k^{*(r)}u_k^{*(r)} - \left(t_k^{*(r)}\right)^2.$$

Write

$$\nabla_k^{(r,a)} := \det\left(\boldsymbol{K}_k^{(r,a)}\right), \qquad \nabla_k^{*(r,a)} := \det\left(\boldsymbol{K}_k^{*(r,a)}\right),$$
$$\nabla_k^{(r,b)} := \det\left(\boldsymbol{K}_k^{(r,b)}\right), \qquad \nabla_k^{*(r,b)} := \det\left(\boldsymbol{K}_k^{*(r,b)}\right),$$
$$\Delta_{k1}^{(r,a)} := \det\left(\Sigma_{k1}^{(r,a)}\right), \qquad \Delta_{k1}^{*(r,a)} := \det\left(\Sigma_{k1}^{*(r,a)}\right),$$
$$\Delta_{k1}^{(r,b)} := \det\left(\Sigma_{k1}^{(r,b)}\right), \qquad \Delta_{k1}^{*(r,b)} := \det\left(\Sigma_{k1}^{*(r,b)}\right).$$

We have

$$\nabla_k^{(r,a)} = \nabla_k^{(r,b)} = \nabla_k^{(r)}, \qquad \nabla_k^{*(r,a)} = \nabla_k^{*(r,b)} = \nabla_k^{*(r)},$$
$$\Delta_{k1}^{(r,a)} = \Delta_{k1}^{(r,b)} = \left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2 = \Delta_{k1}^{(r)},$$
$$\Delta_{k1}^{*(r,a)} = \Delta_{k1}^{*(r,b)} = \left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2 = \Delta_{k1}^{*(r)}.$$

Since in many special cases $\boldsymbol{K}_k^{(r,a)}$, $\boldsymbol{K}_k^{(r,b)}$ and $\boldsymbol{K}_k^{*(r,a)}$, $\boldsymbol{K}_k^{*(r,b)}$ do not depend on a parameter $\boldsymbol{\lambda}_0$ and in a consequently $\Sigma_{k1}^{(r,a)}$, $\Sigma_{k1}^{(r,b)}$ and $\Sigma_{k1}^{*(r,a)}$, $\Sigma_{k1}^{*(r,)}$ do not depend on $\boldsymbol{\lambda}_0$ we write $a_{k1}^{(r,a)}$, $a_{k1}^{(r)}$, $b_{k1}^{(r,a)}$, $b_{k1}^{(r,b)}$, $c_{k1}^{(r)}$ and $c_{k1}^{(r,b)}$ instead of $a_{k1}^{(r,a)}(\boldsymbol{\lambda}_0)$, $a_{k1}^{(r)}(\boldsymbol{\lambda}_0)$, $b_{k1}^{(r,a)}(\boldsymbol{\lambda}_0)$, $b_{k1}^{(r,b)}(\boldsymbol{\lambda}_0)$, $c_{k1}^{(r)}(\boldsymbol{\lambda}_0)$, $c_{k1}^{(r,b)}(\boldsymbol{\lambda}_0)$ and so on.

Using the above notations we get the following tests, which correspond to (3.3).

$$
\begin{aligned}
\hat{T}_{kN}^{(r)} &= \hat{T}_{kN}^{(r,a)} = \hat{T}_{kN;c_1}^{(r,a)} + \hat{T}_{kN;c_2}^{(r,a)} = \hat{T}_{kN;c_3}^{(r)} + \hat{T}_{kN;c_4}^{(r)}\,, \\
\hat{T}_{kN}^{(r)} &= \hat{T}_{kN}^{(r,b)} = \hat{T}_{kN;c_1}^{(r)} + \hat{T}_{kN;c_2}^{(r)} = \hat{T}_{kN;c_3}^{(r,b)} + \hat{T}_{kN;c_4}^{(r,b)}\,, \\
\hat{T}_{kN}^{*(r)} &= \hat{T}_{kN}^{*(r,a)} = \hat{T}_{kN;c_1}^{*(r,a)} + \hat{T}_{kN;c_2}^{*(r,a)} = \hat{T}_{kN;c_3}^{*(r)} + \hat{T}_{kN;c_4}^{*(r)}\,, \\
\hat{T}_{kN}^{*(r)} &= \hat{T}_{kN}^{*(r,b)} = \hat{T}_{kN;c_1}^{*(r)} + \hat{T}_{kN;c_2}^{*(r)} = \hat{T}_{kN;c_3}^{*(r,b)} + \hat{T}_{kN;c_4}^{*(r,b)}\,,
\end{aligned}
\tag{3.10}
$$

with

$$
\begin{aligned}
\hat{T}_{kN}^{(r)} = {} & \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2} \\
& \times \Bigg[\left(c^{(r)} - u_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right)^2 \\
& -2\left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right) \\
& \times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \\
& + \left(a^{(r)} - s_k^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right)^2 \Bigg]
\end{aligned}
$$

its representations $\hat{T}_{kN}^{(r,a)}$ and $\hat{T}_{kN}^{(r,b)}$ for partitions are

$$
\begin{aligned}
\hat{T}_{kN}^{(r,a)} = {} & \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2} \\
& \times \Bigg[\left(c^{(r)} - u_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(ak\hat{h}_{r+1}(U_{kj}) + \hat{h}_r(U_{kj}) \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&- [1+a(r+1)]\Gamma(r+1)\Bigg)^2 \\
&- 2\left(b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(ak\hat{h}_{r+1}(U_{kj}) + \hat{h}_r(U_{kj})\right)\right. \\
&\left.- [1+a(r+1)]\Gamma(r+1)\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}_{r+1}(U_{kj}) - \Gamma(r+2)\right) \\
&+ \left(a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)\right) \\
&\times \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}_{r+1}(U_{kj}) - \Gamma(r+2)\right)^2\Bigg],
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\hat{T}_{kN}^{(r,b)} &= \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2} \\
&\times \Bigg[\left(c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)\right) \\
&\times \left(\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}_r(U_{kj}) - \Gamma(r+1)\right)^2 \\
&- 2\left(b^{(r)} - t_k^{(r)} + b\left(a^{(r)} - s_k^{(r)}\right)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(k\hat{h}_{r+1}(U_{kj}) + b\hat{h}_r(U_{kj})\right)\right. \\
&\left.- (b+r+1)\Gamma(r+1)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}_r(U_{kj}) - \Gamma(r+1)\right) \\
&+ \left(a^{(r)} - s_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(k\hat{h}_{r+1}(U_{kj}) + b\hat{h}_r(U_{kj})\right)\right. \\
&\left.- (b+r+1)\Gamma(r+1)\right)^2\Bigg],
\end{aligned} \tag{3.12}
$$

and their components in the partitions (3.10) are given by

$$
\hat{T}_{kN;c_1}^{(r,a)} = \frac{N}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}
$$

$$\times \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) + a \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{(r,a)}_{kN;c_2} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right)\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. - \left(a^{(r)} - s_k^{(r)} + a\left(b^{(r)} - t_k^{(r)}\right)\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \right]^2$$

$$\hat{T}^{(r)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right]^2,$$

$$\hat{T}^{(r)}_{kN;c_4} = \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right)\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$\left. - \left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \right]^2$$

$$\hat{T}^{(r)}_{kN;c_1} = \frac{N}{a^{(r)} - s_k^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right]^2$$

$$\hat{T}^{(r)}_{kN;c_2} = \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right)\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right) \right.$$

$$- \left(a^{(r)} - s_k^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \Bigg]^2$$

$$\hat{T}_{kN;c_3}^{(r,b)} = \frac{N}{c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[b \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right) + \frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right]^2$$

$$\hat{T}_{kN;c_4}^{(r,b)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$

$$\times \frac{1}{c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} - u_k^{(r)} + b\left(b^{(r)} - t_k^{(r)}\right)\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right)$$

$$- \left(b^{(r)} - t_k^{(r)} + b\left(a^{(r)} - s_k^{(r)}\right)\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right) \Bigg]^2.$$

Next

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$

$$\times \Bigg[\left(c^{(r)} - u_k^{*(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1) \right)^2$$

$$-2\left(b^{(r)} - t_k^{*(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1) \right)$$

$$\times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}^*(V_{kj}) - \Gamma(r+2) \right)$$

$$+ \left(a^{(r)} - s_k^{*(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}^*(V_{kj}) - \Gamma(r+2) \right)^2 \Bigg]$$

its representatives $\hat{T}_{kN}^{*(r,a)}$ and $\hat{T}_{kN}^{*(r,b)}$ are

$$\begin{aligned}
\hat{T}_{kN}^{*(r,a)} = & \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2} \\
& \times \Bigg[\left(c^{(r)} - u_k^{*(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(ak\hat{h}_{r+1}^{*}(V_{kj}) + \hat{h}_r^{*}(V_{kj})\right)\right. \\
& \left. - [1 + a(r+1)]\Gamma(r+1)\right)^2 \\
- & 2\left(b^{(r)} - t_k^{*(r)} + a\left(c^{(r)} - u_k^{*(r)}\right)\right)\left(\frac{k^r}{N} \sum_{j=1}^{N} \left(ak\hat{h}_{r+1}^{*}(V_{kj}) + \hat{h}_r^{*}(V_{kj})\right)\right. \\
- & \left.[1 + a(r+1)]\Gamma(r+1)\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}_{r+1}^{*}(V_{kj}) - \Gamma(r+2)\right) \\
+ & \left(a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)\right) \\
\times & \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}_{r+1}^{*}(V_{kj}) - \Gamma(r+2)\right)^2 \Bigg]
\end{aligned} \tag{3.13}$$

$$\hat{T}_{kN}^{*(r,b)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$
$$\times \Bigg[\left(c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_K^{*(r)}\right)\right)$$
$$\times \left(\frac{k^r}{N}\sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1)\right)^2$$
$$- 2\left(b^{(r)} - t_k^{*(r)} + b\left(a^{(r)} - s_k^{*(r)}\right)\right)$$
$$\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(k\hat{h}_{r+1}^*(V_{kj}) + b\hat{h}_r^*(V_{kj})\right)\right. \tag{3.14}$$
$$\left. - (b+r+1)\Gamma(r+1)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1)\right)$$
$$+ \left(a^{(r)} - s_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(k\hat{h}_{r+1}^*(V_{kj}) + b\hat{h}_r^*(V_{kj})\right)\right.$$
$$\left. - (b+r+1)\Gamma(r+1)\right)^2\Bigg]$$

and their components in the partitions (3.10) are given by

$$\hat{T}_{kN;c_1}^{*(r,a)} = \frac{N}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)}$$
$$\times \left[\frac{k^r}{N}\sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1) + a\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N} \hat{h}_{r+1}^*(V_{kj}) - \Gamma(r+2)\right)\right]^2,$$
$$\hat{T}_{kN;c_2}^{*(r,a)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$
$$\times \frac{1}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)}$$
$$\times \Bigg[\left(b^{(r)} - t_k^{*(r)} + a\left(c^{(r)} - u_k^{*(r)}\right)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1)\right)$$

$$- \left(a^{(r)} - s_k^{*(r)} + a\left(b^{(r)} - t_k^{*(r)}\right)\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}^*_{r+1}(V_{kj}) - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}^{*(r)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{*(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}^*_{r+1}(V_{kj}) - \Gamma(r+2)\right]^2,$$

$$\hat{T}^{*(r)}_{kN;c_4}$$

$$= \frac{N}{\left(\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2\right)\left(c^{(r)} - u_k^{*(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} - u_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}^*_r(V_{kj}) - \Gamma(r+1)\right)$$

$$- \left(b^{(r)} - t_k^{*(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}^*_{r+1}(V_{kj}) - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}^{*(r)}_{kN;c_1} = \frac{N}{a^{(r)} - s_k^{*(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}^*_r(V_{kj}) - \Gamma(r+1)\right]^2,$$

$$\hat{T}^{*(r)}_{kN;c_2} = \frac{N}{\left(\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2\right)}$$

$$\times \frac{1}{a^{(r)} - s_k^{*(r)}}\Bigg[\left(b^{(r)} - t_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}^*_r(V_{kj}) - \Gamma(r+1)\right)$$

$$- \left(a^{(r)} - s_k^{*(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}^*_{r+1}(V_{kj}) - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}^{*(r,b)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_k^{*(r)}\right)}$$

$$\times \left[b\left(\frac{k^r}{N}\sum_{j=1}^{N}\hat{h}^*_r(V_{kj}) - \Gamma(r+1)\right) + \frac{k^{r+1}}{N}\sum_{j=1}^{N}\hat{h}^*_{r+1}(V_{kj}) - \Gamma(r+2)\right]^2,$$

$$\hat{T}^{*(r,b)}_{kN;c_4} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$

$$
\times \frac{1}{c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_k^{*(r)}\right)}
$$

$$
\times \Bigg[\left(c^{(r)} - u_k^{*(r)} + b\left(b^{(r)} - t_k^{*(r)}\right)\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1) \right)
$$

$$
- \left(b^{(r)} - t_k^{*(r)} + b\left(a^{(r)} - s_k^{*(r)}\right)\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}^*(V_{kj}) - \Gamma(r+2) \right) \Bigg]^2 .
$$

Here $\hat{h}_r(\cdot) = h_r\left(\cdot, \hat{\boldsymbol{\lambda}}\right)$ and $\hat{h}_r^*(\cdot) = h_r^*\left(\cdot, \hat{\boldsymbol{\lambda}}\right)$. Moreover, we note that $\hat{T}_{kN;c_3}^{(r)} = \hat{T}_{kN;c_1}^{(r+1)}$ and $\hat{T}_{kN;c_3}^{*(r)} = \hat{T}_{kN;c_1}^{*(r+1)}$.

For $k = 1$ we have tests:

$$
\begin{aligned}
\hat{T}_{1n}^{(r)} &= \hat{T}_{1n}^{(r,a)} = \hat{T}_{1n;c_1}^{(r,a)} + \hat{T}_{1n;c_2}^{(r,a)} = \hat{T}_{1n;c_3}^{(r)} + \hat{T}_{1n;c_4}^{(r)} , \\
\hat{T}_{1n}^{(r)} &= \hat{T}_{1n}^{(r,b)} = \hat{T}_{1n;c_1}^{(r)} + \hat{T}_{1n;c_2}^{(r)} = \hat{T}_{1n;c_3}^{(r,b)} + \hat{T}_{1n;c_4}^{(r,b)} , \\
\hat{T}_{1n}^{*(r)} &= \hat{T}_{1n}^{*(r,a)} = \hat{T}_{1n;c_1}^{*(r,a)} + \hat{T}_{1n;c_2}^{*(r,a)} = \hat{T}_{1n;c_3}^{*(r)} + \hat{T}_{1n;c_4}^{*(r)} , \\
\hat{T}_{1n}^{*(r)} &= \hat{T}_{1n}^{*(r,b)} = \hat{T}_{1n;c_1}^{*(r)} + \hat{T}_{1n;c_2}^{*(r)} = \hat{T}_{1n;c_3}^{*(r,b)} + \hat{T}_{1n;c_4}^{*(r,b)} ,
\end{aligned} \tag{3.15}
$$

with

$$
\hat{T}_{1n}^{(r)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}
$$

$$
\begin{aligned}
&\times \Bigg[\left(c^{(r)} - u_1^{(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \hat{h}_r(X_j) - \Gamma(r+1) \right)^2 \\
&-2\left(b^{(r)} - t_1^{(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \hat{h}_r(X_j) - \Gamma(r+1) \right) \\
&\times \left(\frac{1}{n} \sum_{j=1}^{n} \hat{h}_{r+1}(X_j) - \Gamma(r+2) \right) \\
&+ \left(a^{(r)} - s_1^{(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \hat{h}_{r+1}(X_j) - \Gamma(r+2) \right)^2 \Bigg] ,
\end{aligned}
$$

its representations $\hat{T}_{1n}^{(r,a)}$ and $\hat{T}_{1n}^{(r,b)}$ for partitions are

$$\begin{aligned}
\hat{T}_{1n}^{(r,a)} &= \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2} \\
&\times \Bigg[\left(c^{(r)} - u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(a\hat{h}_{r+1}(X_j) + \hat{h}_r(X_j)\right)\right. \\
&\left. - [1 + a(r+1)]\Gamma(r+1)\right)^2 \\
&- 2\left(b^{(r)} - t_1^{(r)} + a\left(c^{(r)} - u_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(a\hat{h}_{r+1}(X_j) + \hat{h}_r(X_j)\right)\right. \\
&\left. - [1 + a(r+1)]\Gamma(r+1)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right) \\
&+ \left(a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)\right) \\
&\times \left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)^2 \Bigg],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\hat{T}_{1n}^{(r,b)} &= \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2} \\
&\times \Bigg[\left(c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)\right) \\
&\times \left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right)^2 \\
&- 2\left(b^{(r)} - t_1^{(r)} + b\left(a^{(r)} - s_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\hat{h}_{r+1}(X_j) + b\hat{h}_r(X_j)\right)\right. \\
&\left. - (b + r + 1)\Gamma(r+1)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right) \\
&+ \left(a^{(r)} - s_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\hat{h}_{r+1}(X_j) + b\hat{h}_r(X_j)\right)\right.
\end{aligned}$$

$$- (b+r+1)\Gamma(r+1)\Big)^2\Bigg], \tag{3.17}$$

and their components in the partitions (3.15) are given by

$$\hat{T}^{(r,a)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)}$$
$$\times \left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1) + a\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)\right]^2,$$
$$\hat{T}^{(r,a)}_{1n;c_2} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \frac{1}{a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)}$$
$$\times \left[\left(b^{(r)} - t_1^{(r)} + a\left(c^{(r)} - u_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right)\right.$$
$$\left. - \left(a^{(r)} - s_1^{(r)} + a\left(b^{(r)} - t_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)\right]^2,$$
$$\hat{T}^{(r)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right]^2,$$
$$\hat{T}^{(r)}_{1n;c_4} = \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2\right)\left(c^{(r)} - u_1^{(r)}\right)}$$
$$\times \left[\left(c^{(r)} - u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right)\right.$$
$$\left. - \left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)\right]^2,$$
$$\hat{T}^{(r)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right]^2,$$

$$\hat{T}^{(r)}_{1n;c_2} = \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2\right)\left(a^{(r)} - s_1^{(r)}\right)}$$
$$\times \left[\left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right)\right.$$
$$\left. - \left(a^{(r)} - s_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)\right]^2,$$
$$\hat{T}^{(r,b)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)}$$
$$\times \left[b\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right) + \frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right]^2,$$
$$\hat{T}^{(r,b)}_{1n;c_4} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \frac{1}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)}$$
$$\times \left[\left(c^{(r)} - u_1^{(r)} + b\left(b^{(r)} - t_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right)\right.$$
$$\left. - \left(b^{(r)} - t_1^{(r)} + b\left(a^{(r)} - s_1^{(r)}\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right)\right]^2.$$

Similar formulae can be written for $\hat{T}^{*(r)}_{1n}$, $\hat{T}^{*(r,a)}_{1n}$, $\hat{T}^{*(r,b)}_{1n}$, $\hat{T}^{*(r,a)}_{1n;c_1}$, $\hat{T}^{*(r,a)}_{1n;c_2}$, $\hat{T}^{*(r)}_{1n;c_3}$, $\hat{T}^{*(r)}_{1n;c_4}$, $\hat{T}^{*(r)}_{1n;c_1}$, $\hat{T}^{*(r)}_{1n;c_2}$, $\hat{T}^{*(r,b)}_{1n;c_3}$, $\hat{T}^{*(r,b)}_{1n;c_4}$. There are many options for a and b which can change powers of the above tests.

4. Special Cases

4.1. Exponential Distributions: $X \sim \mathrm{Exp}(\alpha)$

$$f(x) = \alpha e^{-\alpha x}, \quad x > 0; \quad \Lambda = \{\alpha : \alpha > 0\},$$

$$F(x;\alpha) = 1 - e^{-\alpha x}, \quad h(x) = \alpha x\,, \qquad \frac{\partial F}{\partial \alpha} = x e^{-\alpha x}, \qquad \mathcal{I}^{-1} = \alpha^2\,,$$

and

$$\hat{\alpha}_n = 1/\overline{X}_n.$$

Moreover,

$$\boldsymbol{b}_k^{(r)}(\alpha) = \frac{1}{\alpha}\Gamma(r+1)\,,$$
$$s_k^{(r)} = \frac{r^2\Gamma^2(r+1)}{k}, \quad t_k^{(r)} = \frac{r\Gamma^2(r+2)}{k}, \quad u_k^{(r)} = \frac{(r+1)^2\Gamma^2(r+2)}{k}.$$

Note that

$$\bigtriangledown_k^{(r,a)} = \bigtriangledown_k^{(r)} = \det\left(\boldsymbol{K}_k^{(r)}\right) = 0$$

and

$$\Delta_{k1}^{(1,a)} = \Delta_{k1}^{(1,b)} = \Delta_{k1}^{(1)} = 4\left(1 - \frac{1}{k}\right), \qquad \Delta_{11}^{(1,a)} = \Delta_{11}^{(1,b)} = 0.$$

Therefore, excluding the case $k = r = 1$ where $\Sigma_{k1}^{(1,a)}$ and $\Sigma_{k1}^{(1,b)}$ are singular we can use the following tests

$$\hat{T}_{kN}^{(r)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$
$$\times \left[\left(c^{(r)} - u_k^{(r)}\right)\left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^r \frac{1}{N}\sum_{j=1}^N U_{kj}^r - \Gamma(r+1)\right)^2\right.$$
$$-2\left(b^{(r)} - t_k^{(r)}\right)\left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^r \frac{1}{N}\sum_{j=1}^N U_{kj}^r - \Gamma(r+1)\right)$$
$$\times\left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^{r+1} \frac{1}{N}\sum_{j=1}^N U_{kj}^{r+1} - \Gamma(r+2)\right)$$
$$\left.+\left(a^{(r)} - s_k^{(r)}\right)\left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^{r+1} \frac{1}{N}\sum_{j=1}^N U_{kj}^{r+1} - \Gamma(r+2)\right)^2\right],$$

its representations $\hat{T}_{kN}^{(r,a)}$ and $\hat{T}_{kN}^{(r,b)}$ in (3.11) and (3.12), respectively, and their components in the partitions (3.10) which are given by

$$\hat{T}_{kN;c_1}^{(r,a)} = \frac{N}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} - \Gamma(r+1) \right. \right.$$

$$\left. \left. + a \left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right) \right]^{2},$$

$$\hat{T}_{kN;c_2}^{(r,a)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^{2}}$$

$$\times \frac{1}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^{2}\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right) \right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} - \Gamma(r+1) \right) \right.$$

$$- \left(a^{(r)} - s_k^{(r)} + a\left(b^{(r)} - t_k^{(r)}\right) \right)$$

$$\left. \times \left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right) \right]^{2},$$

$$\hat{T}_{kN;c_3}^{(r)} = \frac{N}{c^{(r)} - u_k^{(r)}} \left[\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right]^{2},$$

$$\hat{T}_{kN;c_4}^{(r)} = \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^{2} \right)\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_k^{(r)}\right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} - \Gamma(r+1) \right) \right.$$

$$\left. - \left(b^{(r)} - t_k^{(r)}\right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right) \right]^{2},$$

$$\hat{T}_{kN;c_1}^{(r)} = \frac{N}{a^{(r)} - s_k^{(r)}} \left[\left(\frac{k}{\overline{\overline{X}}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} - \Gamma(r+1) \right]^{2},$$

$$\hat{T}_{kN;c_2}^{(r)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^{2}}$$

$$\times \frac{1}{a^{(r)} - s_k^{(r)}} \left[\left(b^{(r)} - t_k^{(r)}\right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^r \frac{1}{N} \sum_{j=1}^{N} U_{kj}^r - \Gamma(r+1) \right) \right.$$

$$\left. - \left(a^{(r)} - s_k^{(r)}\right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}_{kN;c_3}^{(r,b)} = \frac{N}{c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[b \left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^r \frac{1}{N} \sum_{j=1}^{N} U_{kj}^r - \Gamma(r+1) \right) \right.$$

$$\left. + \left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right]^2,$$

$$\hat{T}_{kN;c_4}^{(r,b)} = \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right)}$$

$$\times \frac{1}{c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_k^{(r)} + b\left(b^{(r)} - t_k^{(r)}\right)\right) \left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^r \frac{1}{N} \sum_{j=1}^{N} U_{kj}^r - \Gamma(r+1) \right) \right.$$

$$- \left(b^{(r)} - t_k^{(r)} + b\left(a^{(r)} - s_k^{(r)}\right)\right)$$

$$\left. \times \left(\left(\frac{k}{\overline{\overline{X}}_{kN}}\right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right) \right]^2.$$

When $k = 1$ and $r \neq 1$ we have

$$\hat{T}_{1n}^{(r)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$

$$\times \left[\left(c^{(r)} - u_1^{(r)}\right) \left(\left(\frac{1}{\overline{\overline{X}}_n}\right)^r \frac{1}{n} \sum_{j=1}^{n} X_j^r - \Gamma(r+1) \right) \right.^2$$

$$-2\left(b^{(r)}-t_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^r\frac{1}{n}\sum_{j=1}^n X_j^r-\Gamma(r+1)\right)$$
$$\times\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^n X_j^{r+1}-\Gamma(r+2)\right)$$
$$\left.+\left(a^{(r)}-s_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^n X_j^{r+1}-\Gamma(r+2)\right)^2\right],$$

$\hat{T}_{1n}^{(r,a)}$ and $\hat{T}_{1n}^{(r,b)}$ as in (3.16) and (3.17), respectively, and their components in the partitions (3.15) are given by

$$\hat{T}_{1n;c_1}^{(r,a)}=\frac{n}{a^{(r)}-s_1^{(r)}+2a\left(b^{(r)}-t_1^{(r)}\right)+a^2\left(c^{(r)}-u_1^{(r)}\right)}$$
$$\times\left[\left(\frac{1}{\overline{X}_n}\right)^r\frac{1}{n}\sum_{j=1}^n X_j^r-\Gamma(r+1)\right.$$
$$\left.+a\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^n X_j^{r+1}-\Gamma(r+2)\right)\right]^2,$$
$$\hat{T}_{1n;c_2}^{(r,a)}=\frac{n}{\left(a^{(r)}-s_1^{(r)}\right)\left(c^{(r)}-u_1^{(r)}\right)-\left(b^{(r)}-t_1^{(r)}\right)^2}$$
$$\times\frac{1}{a^{(r)}-s_1^{(r)}+2a\left(b^{(r)}-t_1^{(r)}\right)+a^2\left(c^{(r)}-u_1^{(r)}\right)}$$
$$\times\left[\left(b^{(r)}-t_1^{(r)}+a\left(c^{(r)}-u_1^{(r)}\right)\right)\left(\left(\frac{1}{\overline{X}_n}\right)^r\frac{1}{n}\sum_{j=1}^n X_j^r-\Gamma(r+1)\right)\right.$$
$$-\left(a^{(r)}-s_1^{(r)}+a\left(b^{(r)}-t_1^{(r)}\right)\right)$$
$$\left.\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^n X_j^{r+1}-\Gamma(r+2)\right)\right]^2,$$
$$\hat{T}_{1n;c_3}^{(r)}=\frac{n}{c^{(r)}-u_1^{(r)}}\left[\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^n X_j^{r+1}-\Gamma(r+2)\right]^2,$$

$$\hat{T}^{(r)}_{1n;c_4} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$

$$\times \frac{1}{c^{(r)} - u_1^{(r)}} \left[\left(c^{(r)} - u_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n}\sum_{j=1}^n X_j^r - \Gamma(r+1)\right)\right.$$

$$\left. - \left(b^{(r)} - t_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1} \frac{1}{n}\sum_{j=1}^n X_j^{r+1} - \Gamma(r+2)\right)\right]^2,$$

$$\hat{T}^{(r)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{(r)}}\left[\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n}\sum_{j=1}^n X_j^r - \Gamma(r+1)\right]^2,$$

$$\hat{T}^{(r)}_{1n;c_2} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_1^{(r)}}\left[\left(b^{(r)} - t_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n}\sum_{j=1}^n X_j^r - \Gamma(r+1)\right)\right.$$

$$\left. - \left(a^{(r)} - s_1^{(r)}\right)\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1} \frac{1}{n}\sum_{j=1}^n X_j^{r+1} - \Gamma(r+2)\right)\right]^2,$$

$$\hat{T}^{(r,b)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)}$$

$$\times \left[b\left(\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n}\sum_{j=1}^n X_j^r - \Gamma(r+1)\right)\right.$$

$$\left. + \left(\frac{1}{\overline{X}_n}\right)^{r+1} \frac{1}{n}\sum_{j=1}^n X_j^{r+1} - \Gamma(r+2)\right]^2,$$

$$\hat{T}^{(r,b)}_{1n;c_4} = \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2\right)}$$

$$\times \frac{1}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_1^{(r)} + b\left(b^{(r)} - t_1^{(r)}\right)\right)\left(\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n}\sum_{j=1}^n X_j^r - \Gamma(r+1)\right)\right.$$

$$-\left(b^{(r)}-t_1^{(r)}+b\left(a^{(r)}-s_1^{(r)}\right)\right)\left(\left(\frac{1}{\overline{X}_n}\right)^{r+1}\frac{1}{n}\sum_{j=1}^{n}X_j^{r+1}-\Gamma(r+2)\right)\Bigg]^2,$$

where

$$\begin{aligned}
a^{(r)}-s_1^{(r)} &= \Gamma(2r+1)-(r^2+1)\Gamma^2(r+1)\,,\\
b^{(r)}-t_1^{(r)} &= \Gamma(2r+2)-(1+r(r+1))\Gamma(r+1)\Gamma(r+2)\,,\\
c^{(r)}-u_1^{(r)} &= \Gamma(2r+3)-(1+(r+1)^2)\Gamma^2(r+2).
\end{aligned}$$

Referring now to the dual tests, we see similarly that $h^*(x)=-\log(1-e^{-\alpha x})$,

$$\begin{aligned}
b_k^{*(r)}(\alpha) &= -k^{r+1}E\left[F^{k-2}(X;\alpha)\log^{r-1}\frac{1}{F(X;\alpha)}\times\frac{\partial F(X;\alpha)}{\partial\alpha}\right]\\
&= -\frac{1}{\alpha}k^{r+1}\int_0^1 y^{k-2}(1-y)\log\frac{1}{1-y}\log^{r-1}\frac{1}{y}dy\\
&= -\frac{1}{\alpha}k^{r+1}\left[\int_0^1\left(y^{k-1}-\sum_{n=1}^{\infty}\frac{y^{n+k-1}}{n(n+1)}\right)\log^{r-1}\frac{1}{y}dy\right]\\
&= -\frac{k^{r+1}}{\alpha}\Gamma(r)\left[\frac{1}{k^r}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+k)^r}\right] \text{ for } k>0,\ r>0.
\end{aligned}$$

For negative r we have

$$\begin{aligned}
b_k^{*(r)}(\alpha) &= -\frac{1}{\alpha}k^{r+1}\int_0^1 y^{k-2}(1-y)\log\frac{1}{1-y}\log^{r-1}\frac{1}{y}dy\\
&= -\frac{k^{r+1}}{\alpha}\frac{\Gamma(r+1)}{r}\left[\frac{1}{k^r}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+k)^r}\right]\\
&\text{for } k>0 \text{ and } 0>r>-1.
\end{aligned}$$

Therefore we have for $r>-1$

$$\begin{aligned}
s_k^{*(r)} &= k^{2r+1}\Gamma^2(r+1)\left(A_k^{(r)}\right)^2,\\
t_k^{*(r)} &= k^{2(r+1)}\Gamma(r+1)\Gamma(r+2)A_k^{(r)}A_k^{(r+1)},\\
u_k^{*(r)} &= k^{2r+3}\Gamma^2(r+2)\left(A_k^{(r+1)}\right)^2,
\end{aligned}$$

where

$$A_k^{(r)}=\frac{1}{k^r}-\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+k)^r}\,.$$

In calculations we can also use the above integral instead of the series.

The dual test is

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$
$$\times \Bigg[\left(c^{(r)} - u_k^{*(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^r - \Gamma(r+1) \right)^2$$
$$- 2\left(b^{(r)} - t_k^{*(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^r - \Gamma(r+1) \right)$$
$$\times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^{r+1} - \Gamma(r+2) \right)$$
$$+ \left(a^{(r)} - s_k^{*(r)}\right)$$
$$\times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^{r+1} - \Gamma(r+2) \right)^2 \Bigg],$$

$\hat{T}_{kN}^{*(r,a)}$ and $\hat{T}_{kN}^{*(r,b)}$ are as in (3.13) and (3.14), respectively, and their components in the partitions (3.10) are given by

$$\hat{T}_{kN;c_1}^{*(r,a)} = \frac{N}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)}$$
$$\times \Bigg[\left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^r - \Gamma(r+1) \right)$$
$$+ a \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2$$

$$\hat{T}_{kN;c_2}^{*(r,a)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$
$$\times \frac{1}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_k^{*(r)} + a \left(c^{(r)} - u_k^{*(r)} \right) \right) \right.$$

$$\times \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^r - \Gamma(r+1) \right)$$

$$- \left(a^{(r)} - s_k^{*(r)} + a \left(b^{(r)} - t_k^{*(r)} \right) \right)$$

$$\left. \times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}_{kN;c_3}^{*(r)}$$

$$= \frac{N}{c^{(r)} - u_k^{*(r)}} \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^{r+1} - \Gamma(r+2) \right)^2,$$

$$\hat{T}_{kN;c_4}^{*(r)}$$

$$= \frac{N}{\left(\left(a^{(r)} - s_k^{*(r)} \right) \left(c^{(r)} - u_k^{*(r)} \right) - \left(b^{(r)} - t_k^{*(r)} \right)^2 \right) \left(c^{(r)} - u_k^{*(r)} \right)}$$

$$\times \left[\left(c^{(r)} - u_k^{*(r)} \right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^r - \Gamma(r+1) \right) \right.$$

$$\left. - \left(b^{(r)} - t_k^{*(r)} \right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}_{kN;c_1}^{*(r)} = \frac{N}{a^{(r)} - s_k^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^r - \Gamma(r+1) \right]^2,$$

$$\hat{T}_{kN;c_2}^{*(r)} = \frac{N}{\left(\left(a^{(r)} - s_k^{*(r)} \right) \left(c^{(r)} - u_k^{*(r)} \right) - \left(b^{(r)} - t_k^{*(r)} \right)^2 \right) \left(a^{(r)} - s_k^{*(r)} \right)}$$

$$\times \left[\left(b^{(r)} - t_k^{*(r)} \right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^r - \Gamma(r+1) \right) \right.$$

$$\left. - \left(a^{(r)} - s_k^{*(r)} \right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-V_{kj}/\overline{X}_{kN}} \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{*(r,b)}_{kN;c_3} = \frac{N}{c^{(r)} - u^{*(r)}_k + 2b\left(b^{(r)} - t^{*(r)}_k\right) + b^2\left(a^{(r)} - s^{*(r)}_k\right)}$$
$$\times \left[b\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^r - \Gamma(r+1)\right)\right.$$
$$\left. + \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^{r+1} - \Gamma(r+2)\right)\right]^2,$$

$$\hat{T}^{*(r,b)}_{kN;c_4} = \frac{N}{\left(a^{(r)} - s^{*(r)}_k\right)\left(c^{(r)} - u^{*(r)}_k\right) - \left(b^{(r)} - t^{*(r)}_k\right)^2}$$
$$\times \frac{1}{c^{(r)} - u^{*(r)}_k + 2b\left(b^{(r)} - t^{*(r)}_k\right) + b^2\left(a^{(r)} - s^{*(r)}_k\right)}$$
$$\times \left[\left(c^{(r)} - u^{*(r)}_k + b\left(b^{(r)} - t^{*(r)}_k\right)\right)\right.$$
$$\times \left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^r - \Gamma(r+1)\right)$$
$$- \left(b^{(r)} - t^{*(r)}_k + b\left(a^{(r)} - s^{*(r)}_k\right)\right)$$
$$\left.\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-V_{kj}/\overline{X}_{kN}}\right)\right)^{r+1} - \Gamma(r+2)\right)\right]^2.$$

When $k = 1$ we have

$$\hat{T}^{*(r)}_{1n} = \frac{n}{\left(a^{(r)} - s^{*(r)}_1\right)\left(c^{(r)} - u^{*(r)}_1\right) - \left(b^{(r)} - t^{*(r)}_1\right)^2}$$
$$\times \left[\left(c^{(r)} - u^{*(r)}_1\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-X_j/\overline{X}_n}\right)\right)^r - \Gamma(r+1)\right)^2\right.$$
$$- 2\left(b^{(r)} - t^{*(r)}_1\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-X_j/\overline{X}_n}\right)\right)^r - \Gamma(r+1)\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-X_j/\overline{X}_n}\right)\right)^{r+1} - \Gamma(r+2)\right)$$

$$+\left(a^{(r)}-s_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)^2\Bigg],$$

$\hat{T}_{1n}^{*(r,a)}$ and $\hat{T}_{1n}^{*(r,b)}$ are as in (3.13) and (3.14), respectively, and their components in the partitions (3.15) are given by

$$\hat{T}_{1n;c_1}^{*(r,a)}=\frac{n}{a^{(r)}-s_1^{*(r)}+2a\left(b^{(r)}-t_1^{*(r)}\right)+a^2\left(c^{(r)}-u_1^{*(r)}\right)}$$
$$\times\Bigg[\Bigg(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r}-\Gamma(r+1)\Bigg)$$
$$+a\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)\Bigg]^2,$$
$$\hat{T}_{1n;c_2}^{*(r,a)}=\frac{n}{\left(a^{(r)}-s_1^{*(r)}\right)\left(c^{(r)}-u_1^{*(r)}\right)-\left(b^{(r)}-t_1^{*(r)}\right)^2}$$
$$\times\frac{1}{a^{(r)}-s_1^{*(r)}+2a\left(b^{(r)}-t_1^{*(r)}\right)+a^2\left(c^{(r)}-u_1^{*(r)}\right)}$$
$$\times\Bigg[\left(b^{(r)}-t_1^{*(r)}+a\left(c^{(r)}-u_1^{*(r)}\right)\right)$$
$$\times\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r}-\Gamma(r+1)\right)$$
$$-\left(a^{(r)}-s_1^{*(r)}+a\left(b^{(r)}-t_1^{*(r)}\right)\right)$$
$$\times\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)\Bigg]^2,$$
$$\hat{T}_{1n;c_3}^{*(r)}=\frac{n}{c^{(r)}-u_1^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right]^2$$
$$\hat{T}_{1n;c_4}^{*(r)}$$
$$=\frac{n}{\left(\left(a^{(r)}-s_1^{*(r)}\right)\left(c^{(r)}-u_1^{*(r)}\right)-\left(b^{(r)}-t_1^{*(r)}\right)^2\right)\left(c^{(r)}-u_1^{*(r)}\right)}$$

$$\times\left[\left(c^{(r)}-u_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^r-\Gamma(r+1)\right)\right.$$

$$\left.-\left(b^{(r)}-t_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)\right]^2,$$

$$\hat{T}_{1n;c_1}^{*(r)}=\frac{n}{a^{(r)}-s_1^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^r-\Gamma(r+1)\right]^2,$$

$$\hat{T}_{1n;c_2}^{*(r)}$$

$$=\frac{n}{\left(\left(a^{(r)}-s_1^{*(r)}\right)\left(c^{(r)}-u_1^{*(r)}\right)-\left(b^{(r)}-t_1^{*(r)}\right)^2\right)\left(a^{(r)}-s_1^{*(r)}\right)}$$

$$\times\left[\left(b^{(r)}-t_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^r-\Gamma(r+1)\right)\right.$$

$$\left.-\left(a^{(r)}-s_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)\right]^2,$$

$$\hat{T}_{1n;c_3}^{*(r,b)}=\frac{n}{c^{(r)}-u_1^{*(r)}+2b\left(b^{(r)}-t_1^{*(r)}\right)+b^2\left(a^{(r)}-s_1^{*(r)}\right)}$$

$$\times\left[b\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^r-\Gamma(r+1)\right)\right.$$

$$\left.+\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1-e^{-X_j/\overline{X}_n}\right)\right)^{r+1}-\Gamma(r+2)\right)\right]^2,$$

$$\hat{T}_{1n;c_4}^{*(r,b)}=\frac{n}{\left(a^{(r)}-s_1^{*(r)}\right)\left(c^{(r)}-u_1^{*(r)}\right)-\left(b^{(r)}-t_1^{*(r)}\right)^2}$$

$$\times\frac{1}{c^{(r)}-u_1^{*(r)}+2b\left(b^{(r)}-t_1^{*(r)}\right)+b^2\left(a^{(r)}-s_1^{*(r)}\right)}$$

$$\times\left[\left(c^{(r)}-u_1^{*(r)}+b\left(b^{(r)}-t_1^{*(r)}\right)\right)\right.$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-X_j/\overline{X}_n} \right) \right)^r - \Gamma(r+1) \right)$$
$$- \left(b^{(r)} - t_1^{*(r)} + b \left(a^{(r)} - s_1^{*(r)} \right) \right)$$
$$\left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-X_j/\overline{X}_n} \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2 ,$$

where

$$a^{(r)} - s_1^{*(r)} = \Gamma(2r+1) - \Gamma^2(r+1) \left(1 + \left(A_1^{(r)} \right)^2 \right) ,$$
$$b^{(r)} - t_1^{*(r)} = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \left(1 + A_1^{(r)} A_1^{(r+1)} \right) ,$$
$$c^{(r)} - u_1^{*(r)} = \Gamma(2r+3) - \Gamma^2(r+2) \left(1 + \left(A_1^{(r+1)} \right)^2 \right) \quad r > -\frac{1}{2} ,$$

and

$$a^{(1)} - s_1^{*(1)} = 1 - \left(\frac{\pi^2}{6} - 1 \right)^2 ,$$
$$b^{(1)} - t_1^{*(1)} = 2 \left(2 - \left(\frac{\pi^2}{6} - 1 \right) \left(\frac{\pi^2}{6} + \zeta(3) - 2 \right) \right) ,$$
$$c^{(1)} - u_1^{*(1)} = 4 \left(5 - \left(\frac{\pi^2}{6} + \zeta(3) - 2 \right)^2 \right) .$$

4.2. Inverse Exponential Distributions: $X \sim IExp(\theta)$

Here

$$f(x;\theta) = \frac{\theta e^{-\theta/x}}{x^2}, \quad x > 0, \quad \Lambda = \{\theta : \theta > 0\} ,$$
$$F(x;\theta) = e^{-\theta/x}, \qquad h(x) = -\log\left(1 - e^{-\theta/x}\right) ,$$
$$\frac{\partial F}{\partial \theta} = -f(x)\frac{x}{\theta}, \qquad \mathcal{I}^{-1} = \theta^2 ,$$

and the MLE $\hat{\theta}_n$ is obtained numerically.

Moreover, we have

$$b_k^{(r)}(\theta) = -k^{r+1} E \left[F^{k-2}(X;\theta) \log^{r-1} \frac{1}{F(X;\theta)} \times \frac{\partial F(X;\theta)}{\partial \theta} \right]$$

$$= -\frac{1}{\alpha}k^{r+1}\int_0^1 y^{k-2}(1-y)\log\frac{1}{1-y}\log^{r-1}\frac{1}{y}dy$$
$$= -\frac{k^{r+1}\Gamma(r)}{\theta}A_k^{(r)} \quad \text{for } r>0\,,$$

and

$$b_k^{(r)}(\theta) = -\frac{k^{r+1}}{\theta}\times\frac{\Gamma(r+1)}{r}A_k^{(r)} \quad \text{for } 0>r>-1\,,$$

where

$$A_k^{(r)} = \frac{1}{k^r} - \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+k)^r}.$$

Therefore we have for $r > -1$

$$s_k^{(r)} = k^{2r+1}\Gamma^2(r+1)\left(A_k^{(r)}\right)^2\,,$$
$$t_k^{(r)} = k^{2(r+1)}\Gamma(r+1)\Gamma(r+2)A_k^{(r)}A_k^{(r+1)}\,,$$
$$u_k^{(r)} = k^{2r+3}\Gamma^2(r+2)\left(A_k^{(r+1)}\right)^2.$$

The tests are as follows

$$\hat{T}_{kN}^{(r)} = \frac{N}{\left(a^{(r)}-s_k^{(r)}\right)\left(c^{(r)}-u_k^{(r)}\right)-\left(b^{(r)}-t_k^{(r)}\right)^2}$$
$$\times\Bigg[\left(c^{(r)}-u_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-e^{-\hat{\theta}_n/U_{kj}}\right)\right)^r-\Gamma(r+1)\right)^2$$
$$-2\left(b^{(r)}-t_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-e^{-\hat{\theta}_n/U_{kj}}\right)\right)^r-\Gamma(r+1)\right)$$
$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-e^{-\hat{\theta}_n/U_{kj}}\right)\right)^{r+1}-\Gamma(r+2)\right)$$
$$+\left(a^{(r)}-s_k^{(r)}\right)$$
$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-e^{-\hat{\theta}_n/U_{kj}}\right)\right)^{r+1}-\Gamma(r+2)\right)^2\Bigg],$$

its representations $\hat{T}_{kN}^{(r,a)}$ and $\hat{T}_{kN}^{(r,b)}$ are as in (3.11) and (3.12), respectively, and their components in the partitions (3.10) are given by

$$\hat{T}_{kN;c_1}^{(r,a)} = \frac{N}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$
$$\times \Bigg[\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^r - \Gamma(r+1)\right)$$
$$+a\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}_{kN;c_2}^{(r,a)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$
$$\times \frac{1}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$
$$\times \Bigg[\left(b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right)\right)$$
$$\times \left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^r - \Gamma(r+1)\right)$$
$$-\left(a^{(r)} - s_k^{(r)} + a\left(b^{(r)} - t_k^{(r)}\right)\right)$$
$$\times \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}_{kN;c_3}^{(r)}$$
$$= \frac{N}{c^{(r)} - u_k^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^{r+1} - \Gamma(r+2)\right]^2,$$

$$\hat{T}_{kN;c_4}^{(r)} = \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right)\left(c^{(r)} - u_k^{(r)}\right)}$$
$$\times \Bigg[\left(c^{(r)} - u_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1 - e^{-\hat{\theta}_n/U_{kj}}\right)\right)^r - \Gamma(r+1)\right)$$

$$- \left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2 ,$$

$$\hat{T}^{(r)}_{kN;c_1} = \frac{N}{a^{(r)} - s_k^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r} - \Gamma(r+1) \right]^2 ,$$

$$\hat{T}^{(r)}_{kN;c_2}$$

$$= \frac{N}{\left(\left(a^{(r)} - s_k^{(r)}\right) \left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2 \right) \left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \Bigg[\left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^{r}}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r} - \Gamma(r+1) \right)$$

$$- \left(a^{(r)} - s_k^{(r)}\right)$$

$$\times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2 ,$$

$$\hat{T}^{(r,b)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{(r)} + 2b \left(b^{(r)} - t_k^{(r)}\right) + b^2 \left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \Bigg[b \left(\frac{k^{r}}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r} - \Gamma(r+1) \right)$$

$$+ \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(- \log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2 ,$$

$$\hat{T}^{(r,b)}_{kN;c_4} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right) \left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$

$$\times \frac{1}{c^{(r)} - u_k^{(r)} + 2b \left(b^{(r)} - t_k^{(r)}\right) + b^2 \left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} - u_k^{(r)} + b \left(b^{(r)} - t_k^{(r)} \right) \right)$$

$$\times \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^r - \Gamma(r+1) \right)$$
$$- \left(b^{(r)} - t_k^{(r)} + b \left(a^{(r)} - s_k^{(r)} \right) \right)$$
$$\left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log \left(1 - e^{-\hat{\theta}_n / U_{kj}} \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2 .$$

When $k = 1$ we have

$$\hat{T}_{1n}^{(r)} = \frac{n}{\left(a^{(r)} - s_1^{(r)} \right) \left(c^{(r)} - u_1^{(r)} \right) - \left(b^{(r)} - t_1^{(r)} \right)^2}$$
$$\times \left[\left(c^{(r)} - u_1^{(r)} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^r - \Gamma(r+1) \right)^2 \right.$$
$$- 2 \left(b^{(r)} - t_1^{(r)} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^r - \Gamma(r+1) \right)$$
$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^{r+1} - \Gamma(r+2) \right)$$
$$+ \left(a^{(r)} - s_1^{(r)} \right)$$
$$\times \left. \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^{r+1} - \Gamma(r+2) \right)^2 \right],$$

its representations $\hat{T}_{1n}^{(r,a)}$ and $\hat{T}_{1n}^{(r,b)}$ are as in (3.16) and (3.17) and their components in the partitions (3.15) are given by

$$\hat{T}_{1n;c_1}^{(r,a)} = \frac{n}{a^{(r)} - s_1^{(r)} + 2a \left(b^{(r)} - t_1^{(r)} \right) + a^2 \left(c^{(r)} - u_1^{(r)} \right)}$$
$$\times \left[\left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^r - \Gamma(r+1) \right) \right.$$
$$\left. + a \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(1 - e^{-\hat{\theta}_n / X_j} \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^2 ,$$

$$\hat{T}^{(r,a)}_{1n;c_2} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)}$$

$$\times \Bigg[\left(b^{(r)} - t_1^{(r)} + a\left(c^{(r)} - u_1^{(r)}\right)\right)$$

$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right)$$

$$- \left(a^{(r)} - s_1^{(r)} + a\left(b^{(r)} - t_1^{(r)}\right)\right)$$

$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right) \Bigg]^2,$$

$$\hat{T}^{(r)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{(r)}}$$

$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right)^2,$$

$$\hat{T}^{(r)}_{1n;c_4}$$

$$= \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2\right)\left(c^{(r)} - u_1^{(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} - u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right)$$

$$- \left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}^{(r)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right]^2,$$

$$\hat{T}^{(r)}_{1n;c_2}$$

$$
\begin{aligned}
&= \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2\right)\left(a^{(r)} - s_1^{(r)}\right)} \\
&\times \Bigg[\left(b^{(r)} - t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right) \\
&- \left(a^{(r)} - s_1^{(r)}\right) \\
&\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2, \\
\hat{T}_{1n;c_3}^{(r,b)} &= \frac{n}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)} \\
&\times \Bigg[b\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right) \\
&+ \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2, \\
\hat{T}_{1n;c_4}^{(r,b)} &= \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2} \\
&\times \frac{1}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)} \\
&\times \Bigg[\left(c^{(r)} - u_1^{(r)} + b\left(b^{(r)} - t_1^{(r)}\right)\right) \\
&\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^r - \Gamma(r+1)\right) \\
&- \left(b^{(r)} - t_1^{(r)} + b\left(a^{(r)} - s_1^{(r)}\right)\right) \\
&\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(1 - e^{-\hat{\theta}_n/X_j}\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2.
\end{aligned}
$$

Referring now to the dual tests, we see similarly that $h^*(x) = \theta/x$, and

$$
b_k^{*(r)}(\theta) := \frac{1}{\theta}\Gamma(r+1)\,,
$$

$$s_k^{*(r)} = \frac{r^2\Gamma^2(r+1)}{k}, \qquad t_k^{*(r)} = \frac{r\Gamma^2(r+2)}{k},$$
$$u_k^{*(r)} = \frac{(r+1)^2\Gamma^2(r+2)}{k}.$$

The case $k = r = 1$ has to be excluded, and we have

$$\begin{aligned}\hat{T}_{kN}^{*(r)} &= \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2} \\ &\times \left[\left(c^{(r)} - u_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r - \Gamma(r+1)\right)^2 - 2\left(b^{(r)} - t_k^{*(r)}\right)\right. \\ &\times \left(\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r - \Gamma(r+1)\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right) \\ &\left. + \left(a^{(r)} - s_k^{*(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)^2\right],\end{aligned}$$

its representations $\hat{T}_{kN}^{*(r,a)}$ and $\hat{T}_{kN}^{*(r,b)}$ are as in (3.13) and (3.14), respectively, and their components in the partitions (3.10) are given by

$$\begin{aligned}\hat{T}_{kN;c_1}^{*(r,a)} &= \frac{N}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)} \\ &\times \left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r - \Gamma(r+1)\right. \\ &\left. + a\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)\right]^2,\end{aligned}$$

$$\begin{aligned}\hat{T}_{kN;c_2}^{*(r,a)} &= \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2} \\ &\times \frac{1}{a^{(r)} - s_k^{*(r)} + 2a\left(b^{(r)} - t_k^{*(r)}\right) + a^2\left(c^{(r)} - u_k^{*(r)}\right)} \\ &\times \left[\left(b^{(r)} - t_k^{*(r)} + a\left(c^{(r)} - u_k^{*(r)}\right)\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r - \Gamma(r+1)\right)\right.\end{aligned}$$

$$- \left(a^{(r)} - s_k^{*(r)} + a\left(b^{(r)} - t_k^{*(r)}\right)\right)$$

$$\times \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}_{kN;c_3}^{*(r)} = \frac{N}{c^{(r)} - u_k^{*(r)}}\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)^2,$$

$$\hat{T}_{kN;c_4}^{*(r)} = \frac{N}{c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_k^{*(r)}\right)}$$

$$\times \frac{1}{c^{(r)} - u_k^{*(r)}}\Bigg[\left(c^{(r)} - u_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r} - \Gamma(r+1)\right)$$

$$- \left(b^{(r)} - t_k^{*(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}_{kN;c_1}^{*(r)} = \frac{N}{a^{(r)} - s_k^{*(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r} - \Gamma(r+1)\right]^2,$$

$$\hat{T}_{kN;c_2}^{*(r)} = \frac{N}{\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_k^{*(r)}}\Bigg[\left(b^{(r)} - t_k^{*(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r} - \Gamma(r+1)\right)$$

$$- \left(a^{(r)} - s_k^{*(r)}\right)\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}_{kN;c_3}^{*(r,b)} = \frac{N}{c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_k^{*(r)}\right)}$$

$$\times \Bigg[b\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r} - \Gamma(r+1)\right)$$

$$+ \frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\Bigg]^2,$$

$$\hat{T}^{*(r,b)}_{kN;c_4}$$
$$= \frac{N}{\left(\left(a^{(r)} - s_k^{*(r)}\right)\left(c^{(r)} - u_k^{*(r)}\right) - \left(b^{(r)} - t_k^{*(r)}\right)^2\right)}$$
$$\times \frac{1}{c^{(r)} - u_k^{*(r)} + 2b\left(b^{(r)} - t_k^{*(r)}\right) + b^2\left(a^{(r)} - s_k^{*(r)}\right)}$$
$$\times \Bigg[\left(c^{(r)} - u_k^{*(r)} + b\left(b^{(r)} - t_k^{*(r)}\right)\right)$$
$$\times \left(\frac{k^r}{N}\sum_{j=1}^{N}\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r - \Gamma(r+1)\right)$$
$$- \left(b^{(r)} - t_k^{*(r)} + b\left(a^{(r)} - s_k^{*(r)}\right)\right)$$
$$\times \left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2.$$

When $k = 1$ we have

$$\hat{T}^{*(r)}_{1n} = \frac{n}{\left(a^{(r)} - s_1^{*(r)}\right)\left(c^{(r)} - u_1^{*(r)}\right) - \left(b^{(r)} - t_1^{*(r)}\right)^2}$$
$$\times \Bigg[\left(c^{(r)} - u_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^r - \Gamma(r+1)\right)^2 - 2\left(b^{(r)} - t_1^{*(r)}\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^r - \Gamma(r+1)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^{r+1} - \Gamma(r+2)\right)$$
$$+ \left(a^{(r)} - s_1^{*(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^{r+1} - \Gamma(r+2)\right)^2\Bigg],$$

its representations $\hat{T}^{*(r,a)}_{1n}$ and $\hat{T}^{*(r,b)}_{1n}$ are as in (3.13) and (3.14) with $k = 1$, respectively, and their components in the partitions (3.15) are given by

$$\hat{T}^{*(r,a)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{*(r)} + 2a\left(b^{(r)} - t_1^{*(r)}\right) + a^2\left(c^{(r)} - u_1^{*(r)}\right)}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^r - \Gamma(r+1) + a \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{*(r,a)}_{1n;c_2} = \frac{n}{\left(a^{(r)} - s_1^{*(r)}\right)\left(c^{(r)} - u_1^{*(r)}\right) - \left(b^{(r)} - t_1^{*(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_1^{*(r)} + 2a\left(b^{(r)} - t_1^{*(r)}\right) + a^2\left(c^{(r)} - u_1^{*(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_1^{*(r)} + a\left(c^{(r)} - u_1^{*(r)}\right)\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^r - \Gamma(r+1) \right) \right.$$

$$\left. - \left(a^{(r)} - s_1^{*(r)} + a\left(b^{(r)} - t_1^{*(r)}\right)\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{*(r)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right]^2,$$

$$\hat{T}^{*(r)}_{1n;c_4} = \frac{n}{c^{(r)} - u_1^{*(r)} + 2b\left(b^{(r)} - t_1^{*(r)}\right) + b^2\left(a^{(r)} - s_1^{*(r)}\right)}$$

$$\times \frac{1}{c^{(r)} - u_1^{*(r)} + 2b\left(b^{(r)} - t_1^{*(r)}\right) + b^2\left(a^{(r)} - s_1^{*(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_1^{*(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^r - \Gamma(r+1) \right) \right.$$

$$\left. - \left(b^{(r)} - t_1^{*(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{*(r)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^r - \Gamma(r+1) \right]^2,$$

$$\hat{T}^{*(r)}_{1n;c_2} = \frac{n}{\left(a^{(r)} - s_1^{*(r)}\right)\left(c^{(r)} - u_1^{*(r)}\right) - \left(b^{(r)} - t_1^{*(r)}\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_1^{*(r)}} \times \left[\left(b^{(r)} - t_1^{*(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^r - \Gamma(r+1) \right) \right.$$

$$- \left(a^{(r)} - s_1^{*(r)}\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2,$$

$$\hat{T}^{*(r,b)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{*(r)} + 2b\left(b^{(r)} - t_1^{*(r)}\right) + b^2\left(a^{(r)} - s_1^{*(r)}\right)}$$

$$\times \left[b \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r} - \Gamma(r+1) \right) + \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right]^2,$$

$$\hat{T}^{*(r,b)}_{1n;c_4}$$

$$= \frac{n}{\left(\left(a^{(r)} - s_1^{*(r)}\right)\left(c^{(r)} - u_1^{*(r)}\right) - \left(b^{(r)} - t_1^{*(r)}\right)^2\right)\left(c^{(r)} - u_1^{*(r)}\right)}$$

$$\times \Bigg[\left(c^{(r)} - u_1^{*(r)} + b\left(b^{(r)} - t_1^{*(r)}\right)\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r} - \Gamma(r+1) \right)$$

$$- \left(b^{(r)} - t_1^{*(r)} + b\left(a^{(r)} - s_1^{*(r)}\right)\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2.$$

4.3. Normal Distributions: $X \sim N(\mu, \sigma^2)$

Here

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty;$$
$$\Lambda = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\, \sigma^2 > 0\},$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt,$$
$$\frac{\partial F(x;\mu,\sigma^2)}{\partial \mu} = -f(x), \qquad \frac{\partial F(x;\mu,\sigma^2)}{\partial \sigma^2} = -\frac{x-\mu}{2\sigma^2} f(x),$$

$$\mathcal{I}^{-1} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma^2 \end{bmatrix},$$

and the MLE are

$$\hat{\mu}_n = \overline{X}_n \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2.$$

Next we use

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

$$\begin{aligned} b_k^{(r)}(\mu) &= -k^{r+1} \frac{1}{\sigma} E \left[\phi(Z) \left(1 - \Phi(Z)\right)^{k-2} \log^{r-1} \frac{1}{1 - \Phi(Z)} \right], \\ &= -k^{r+1} \frac{1}{\sigma} E_1(k, r), \end{aligned}$$

where

$$E_1(k, r) = E \left[\phi(Z) \left(1 - \Phi(Z)\right)^{k-2} \log^{r-1} \frac{1}{1 - \Phi(Z)} \right], \quad Z \sim N(0, 1);$$

$$\begin{aligned} b_k^{(r)}(\sigma^2) &= -k^{r+1} \frac{1}{2\sigma^2} E \left[Z\phi(Z)(1 - \Phi(Z))^{k-2} \log^{r-1} \frac{1}{1 - \Phi(Z)} \right] \\ &= -k^{r+1} \frac{1}{2\sigma^2} E_2(k, r), \end{aligned}$$

where

$$E_2(k, r) = E \left[Z\phi(Z) \left(1 - \Phi(Z)\right)^{k-2} \log^{r-1} \frac{1}{1 - \Phi(Z)} \right], \quad Z \sim N(0, 1).$$

Also we have

$$\begin{aligned} s_k^{(r)} &= r^2 k^{2r+1} \left[E_1^2(k, r) + \frac{1}{2} E_2^2(k, r) \right], \\ t_k^{(r)} &= r(r+1) k^{2r+2} \left[E_1(k, r) E_1(k, r+1) + \frac{1}{2} E_2(k, r) E_2(k, r+1) \right], \\ u_k^{(r)} &= (r+1)^2 k^{2r+3} \left[E_1^2(k, r+1) + \frac{1}{2} E_2^2(k, r+1) \right]. \end{aligned}$$

Using the above quantities we get

$$\begin{aligned} \hat{T}_{kN}^{(r)} = &\frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2} \\ &\times \Bigg[\left(c^{(r)} - u_k^{(r)}\right) \Bigg(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F\left(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)\right)^r - \Gamma(r+1) \Bigg)^2 \end{aligned}$$

$$- 2\left(b^{(r)} - t_k^{(r)}\right)\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-F\left(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^r - \Gamma(r+1)\right)$$
$$\times\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-F\left(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r+1} - \Gamma(r+2)\right)$$
$$+\left(a^{(r)} - s_k^{(r)}\right)$$
$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-F\left(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r+1} - \Gamma(r+2)\right)^2\Bigg],$$

its representations $\hat{T}_{kN}^{(r,a)}$ and $\hat{T}_{kN}^{(r,b)}$ are as in (3.11) and (3.12), respectively, and their components in the partitions (3.10) are given by

$$\hat{T}_{kN;c_1}^{(r,a)} = \frac{N}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$
$$\times\Bigg[\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-F\left(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^r - \Gamma(r+1)$$
$$+ a\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-F\left(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$
$$\hat{T}_{kN;c_2}^{(r,a)} = \frac{N}{\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2}$$
$$\times\frac{1}{a^{(r)} - s_k^{(r)} + 2a\left(b^{(r)} - t_k^{(r)}\right) + a^2\left(c^{(r)} - u_k^{(r)}\right)}$$
$$\times\Bigg[\left(b^{(r)} - t_k^{(r)} + a\left(c^{(r)} - u_k^{(r)}\right)\right)$$
$$\times\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-F(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^r - \Gamma(r+1)\right)$$
$$-\left(a^{(r)} - s_k^{(r)} + a\left(b^{(r)} - t_k^{(r)}\right)\right)$$
$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-F(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2,$$

$$\hat{T}^{(r)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{(r)}}$$

$$\times \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2) \right]^2,$$

$$\hat{T}^{(r)}_{kN;c_4} = \frac{N}{\left[\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right]\left(c^{(r)} - u_k^{(r)}\right)}$$

$$\times \left[\left(c^{(r)} - u_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r} - \Gamma(r+1) \right) \right.$$

$$- \left(b^{(r)} - t_k^{(r)}\right)$$

$$\left. \times \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{(r)}_{kN;c_1} = \frac{N}{a^{(r)} - s_k^{(r)}} \left[\frac{1}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r} - \Gamma(r+1) \right]^2,$$

$$\hat{T}^{(r)}_{kN;c_2} = \frac{N}{\left[\left(a^{(r)} - s_k^{(r)}\right)\left(c^{(r)} - u_k^{(r)}\right) - \left(b^{(r)} - t_k^{(r)}\right)^2\right]\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[\left(b^{(r)} - t_k^{(r)}\right) \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F\left(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)\right)^{r} - \Gamma(r+1) \right) \right.$$

$$- \left(a^{(r)} - s_k^{(r)}\right) \left(\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F\left(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)\right)^{r+1} \right.$$

$$\left. \left. - \Gamma(r+2) \right) \right]^2,$$

$$\hat{T}^{(r,b)}_{kN;c_3} = \frac{N}{c^{(r)} - u_k^{(r)} + 2b\left(b^{(r)} - t_k^{(r)}\right) + b^2\left(a^{(r)} - s_k^{(r)}\right)}$$

$$\times \left[b \left(\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r} - \Gamma(r+1) \right) \right.$$

$$+\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-F(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r+1}-\Gamma(r+2)\Bigg]^2,$$

$$\hat{T}_{kN;c_4}^{(r,b)}=\frac{N}{\left(a^{(r)}-s_k^{(r)}\right)\left(c^{(r)}-u_k^{(r)}\right)-\left(b^{(r)}-t_k^{(r)}\right)^2}$$

$$\times\frac{1}{c^{(r)}-u_k^{(r)}+2b\left(b^{(r)}-t_k^{(r)}\right)+b^2\left(a^{(r)}-s_k^{(r)}\right)}$$

$$\times\Bigg[\left(c^{(r)}-u_k^{(r)}+b\left(b^{(r)}-t_k^{(r)}\right)\right)$$

$$\times\left(\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(1-F(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r}-\Gamma(r+1)\right)$$

$$-\left(b^{(r)}-t_k^{(r)}+b\left(a^{(r)}-s_k^{(r)}\right)\right)$$

$$\times\left(\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(1-F(U_{kj};\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r+1}-\Gamma(r+2)\right)\Bigg]^2.$$

Referring now to the dual tests, we see similarly that

$$b_k^{*(r)}(\mu)=-k^{r+1}\frac{1}{\sigma}E\left[\phi(Z)\Phi^{k-2}(Z)\log^{r-1}\frac{1}{\Phi(Z)}\right]=-b_k^{(r)}(\mu)\,,$$

$$b_k^{*(r)}(\sigma^2)=-k^{r+1}\frac{1}{2\sigma^2}E\left[Z\phi(Z)\Phi^{k-2}(Z)\log^{r-1}\frac{1}{\Phi(Z)}\right]=-b_k^{(r)}\left(\sigma^2\right)\,,$$

and $\quad s_k^{*(r)}=s_k^{(r)},\qquad t_k^{*(r)}=t_k^{(r)},\qquad u_k^{*(r)}=u_k^{(r)}.$

We write these tests only for $k=1$.

$$\hat{T}_{1n}^{*(r)}=\frac{n}{\left(a^{(r)}-s_1^{(r)}\right)\left(c^{(r)}-u_1^{(r)}\right)-\left(b^{(r)}-t_1^{(r)}\right)^2}$$

$$\times\Bigg[\left(c^{(r)}-u_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r}-\Gamma(r+1)\right)^2$$

$$-2\left(b^{(r)}-t_1^{(r)}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r}-\Gamma(r+1)\right)$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}^2\right)\right)\right)^{r+1} - \Gamma(r+2) \right)$$
$$+ \left(a^{(r)} - s_1^{(r)}\right)$$
$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}^2\right)\right)\right)^{r+1} - \Gamma(r+2) \right)^2 \Bigg],$$

its representations $\hat{T}_{1n}^{*(r,a)}$ and $\hat{T}_{1n}^{*(r,b)}$ are as in (3.13) and (3.14) with $k = 1$, respectively, and their components in the partitions (3.15) are as follows

$$\hat{T}_{1n;c_1}^{*(r,a)} = \frac{n}{a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)}$$
$$\times \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r} - \Gamma(r+1)$$
$$+ a \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)\right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2,$$
$$\hat{T}_{1n;c_2}^{*(r,a)} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \frac{1}{a^{(r)} - s_1^{(r)} + 2a\left(b^{(r)} - t_1^{(r)}\right) + a^2\left(c^{(r)} - u_1^{(r)}\right)}$$
$$\times \Bigg[\left(b^{(r)} - t_1^{(r)} + a\left(c^{(r)} - u_1^{(r)}\right)\right)$$
$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r} - \Gamma(r+1) \right)$$
$$- \left(a^{(r)} - s_1^{(r)} + a\left(b^{(r)} - t_1^{(r)}\right)\right)$$
$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2,$$
$$\hat{T}_{1n;c_3}^{*(r)} = \frac{n}{c^{(r)} - u_1^{(r)}}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r+1} - \Gamma(r+2) \right]^2 ,$$

$$\hat{T}_{1n;c_4}^{*(r)} = \frac{n}{\left[\left(a^{(r)} - s_1^{(r)} \right) \left(c^{(r)} - u_1^{(r)} \right) - \left(b^{(r)} - t_1^{(r)} \right)^2 \right] \left(c^{(r)} - u_1^{(r)} \right)}$$

$$\times \left[\left(c^{(r)} - u_1^{(r)} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r} - \Gamma(r+1) \right) \right.$$

$$- \left(b^{(r)} - t_1^{(r)} \right)$$

$$\left. \times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^2 ,$$

$$\hat{T}_{1n;c_1}^{*(r)} = \frac{n}{a^{(r)} - s_1^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r} - \Gamma(r+1) \right]^2 ,$$

$$\hat{T}_{1n;c_2}^{*(r)} = \frac{n}{\left[\left(a^{(r)} - s_1^{(r)} \right) \left(c^{(r)} - u_1^{(r)} \right) - \left(b^{(r)} - t_1^{(r)} \right)^2 \right] \left(a^{(r)} - s_1^{(r)} \right)}$$

$$\times \left[\left(b^{(r)} - t_1^{(r)} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2 \right) \right) \right)^{r} - \Gamma(r+1) \right) \right.$$

$$- \left(a^{(r)} - s_1^{(r)} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2 \right) \right) \right)^{r+1} \right.$$

$$\left. \left. - \Gamma(r+2) \right) \right]^2 ,$$

$$\hat{T}_{1n;c_3}^{*(r,b)} = \frac{n}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)} \right) + b^2 \left(a^{(r)} - s_1^{(r)} \right)}$$

$$\times \left[b \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r} - \Gamma(r+1) \right) \right.$$

$$\left. + \frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) \right) \right)^{r+1} - \Gamma(r+2) \right]^2$$

$$\hat{T}^{*(r,b)}_{1n;c_4} = \frac{n}{\left(a^{(r)} - s_1^{(r)}\right)\left(c^{(r)} - u_1^{(r)}\right) - \left(b^{(r)} - t_1^{(r)}\right)^2}$$
$$\times \frac{1}{c^{(r)} - u_1^{(r)} + 2b\left(b^{(r)} - t_1^{(r)}\right) + b^2\left(a^{(r)} - s_1^{(r)}\right)}$$
$$\times \Bigg[\left(c^{(r)} - u_1^{(r)} + b\left(b^{(r)} - t_1^{(r)}\right)\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^r - \Gamma(r+1)\right)$$
$$- \left(b^{(r)} - t_1^{(r)} + b\left(a^{(r)} - s_1^{(r)}\right)\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n} \left(-\log\left(F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2)\right) \Bigg]^2$$

Note that in the above tests for normality we can use

$$F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) = \Phi\left(\frac{X_j - \hat{\mu}_n}{\hat{\sigma}_n}\right)$$

and

$$1 - F(X_j;\hat{\mu}_n,\hat{\sigma}_n^2) = \Phi\left(\frac{\hat{\mu}_n - X_j}{\hat{\sigma}_n}\right).$$

4.4. Inverse Gaussian Distributions: $X \sim IG(\mu, \lambda)$

Here

$$f(x) = \sqrt{\frac{\lambda}{2\pi}}\, x^{-3/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right),$$
$$x > 0;\ \Lambda = \{(\mu,\lambda) : \mu > 0, \lambda > 0\},$$
$$F(x) = \Phi\left[\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right] + e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{x}}\left(1 + \frac{x}{\mu}\right)\right]$$
$$= \Phi(R) + e^{2\lambda/\mu}\Phi(L),$$
$$1 - F(x) = \Phi(-R) - e^{2\lambda/\mu}\Phi(L),$$

with

$$R = -\sqrt{\frac{\lambda}{x}} + \frac{\sqrt{\lambda x}}{\mu}, \qquad L = -\sqrt{\frac{\lambda}{x}} - \frac{\sqrt{\lambda x}}{\mu},$$

$$
\begin{aligned}
\frac{\partial F(x;\mu,\lambda)}{\partial \mu} &= \frac{1}{\mu^2}\left\{-\sqrt{\lambda x}\phi(R) - 2\lambda e^{2\lambda/\mu}\Phi(L) + \sqrt{\lambda x}e^{2\lambda/\mu}\phi(L)\right\} \\
&= -\frac{2\lambda}{\mu^2}e^{2\lambda/\mu}\Phi(L)\,, \\
\frac{\partial F(x;\mu,\lambda)}{\partial \lambda} &= \frac{1}{\mu}\left\{\frac{x-\mu}{2\sqrt{\lambda x}}\phi(R) + 2e^{2\lambda/\mu}\Phi(L) - \frac{x+\mu}{2\sqrt{\lambda x}}e^{2\lambda/\mu}\phi(L)\right\} \\
&= -\frac{1}{\sqrt{\lambda x}}\phi(R) + \frac{2}{\mu}e^{2\lambda/\mu}\Phi(L), \\
\mathcal{I}(\mu,\lambda) &= \begin{bmatrix} (2\lambda^2)^{-1} & 0 \\ 0 & \lambda\mu^{-3} \end{bmatrix}, \qquad \mathcal{I}^{-1} = \begin{bmatrix} 2\lambda^2 & 0 \\ 0 & \lambda^{-1}\mu^3 \end{bmatrix},
\end{aligned}
$$

and the MLE estimators of μ and λ are

$$
\hat{\mu}_n = \overline{X}_n, \qquad \hat{\lambda}_n = \left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{X_j} - \frac{1}{\overline{X}_n}\right)\right)^{-1} \qquad \text{(cf. [7], [18])},
$$

respectively. Now

$$
\begin{aligned}
b_k^{(r)}(\mu) &= k^{r+1}E\left[(1-F(X;\mu,\lambda))^{k-2}\log^{r-1}\frac{1}{1-F(X;\mu,\lambda)}\right. \\
&\quad \left.\times\frac{\partial F(X;\mu,\lambda)}{\partial\mu}\right] \\
&= -k^{r+1}\frac{2\lambda}{\mu^2}e^{2\lambda/\mu}E\left[\left(\Phi\left(\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right)\right.\right. \\
&\quad \left.-e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)\right)^{k-2}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right) \\
&\quad \left.\times\log^{r-1}\frac{1}{\Phi\left(\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right) - e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)}\right] \\
&= -k^{r+1}\frac{2\lambda}{\mu^2}e^{2\lambda/\mu}G_1(k,r,\mu,\lambda),
\end{aligned}
$$

where

$$
G_1(k,r,\mu,\lambda) = E\left[\left(\Phi\left(\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right)\right.\right.
$$

$$-e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)\Bigg)^{k-2}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)$$

$$\times\log^{r-1}\frac{1}{\Phi\left(\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right)-e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)}\Bigg] .$$

Now

$$b_k^{(r)}(\lambda)=k^{r+1}E\Bigg[(1-F(X;\mu,\lambda))^{k-2}\log^{r-1}\frac{1}{1-F(X;\mu,\lambda)}$$

$$\times\frac{\partial F(X;\mu,\lambda)}{\partial\lambda}\Bigg]$$

$$=-k^{r+1}\frac{1}{\sqrt{\lambda}}E\left[\left(\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]\right.\right.$$

$$\left.-e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{X}\left(1+\frac{X}{\mu}\right)}\right]\right)^{k-2}\frac{1}{\sqrt{X}}\phi\left[\sqrt{\frac{\lambda}{X}}\left(\frac{X}{\mu}-1\right)\right]$$

$$\left.\times\log^{r-1}\frac{1}{\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]-e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)}\right]$$

$$+k^{r+1}\frac{2}{\mu}e^{2\lambda/\mu}E\left[\left(\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]\right.\right.$$

$$\left.-e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right]\right)^{k-2}\Phi\left[-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right]$$

$$\left.\times\log^{r-1}\frac{1}{\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]-e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right)}\right]$$

$$=-k^{r+1}\frac{1}{\sqrt{\lambda}}G_2(k,r,\mu,\lambda)+k^{r+1}\frac{2}{\mu}e^{2\lambda/\mu}G_1(k,r,\mu,\lambda)\,,$$

where

$$G_2(k,r,\mu,\lambda)=E\left[\left(\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]\right.\right.$$

$$-e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right]\Bigg)^{k-2}\frac{1}{\sqrt{X}}\phi\left[\sqrt{\frac{\lambda}{\mu}}\left(\frac{X}{\mu}-1\right)\right]$$

$$\times\log^{r-1}\frac{1}{\Phi\left[\sqrt{\frac{\lambda}{X}}\left(1-\frac{X}{\mu}\right)\right]-e^{2\lambda/\mu}\Phi\left[-\sqrt{\frac{\lambda}{X}}\left(1+\frac{X}{\mu}\right)\right]}\Bigg].$$

So we use

$$b_k^{(r)}(\mu)=-k^{r+1}\frac{2\lambda}{\mu^2}e^{2\lambda/\mu}G_1(k,r,\mu,\lambda)\,,$$
$$b_k^{(r)}(\lambda)=k^{r+1}\left(\frac{2}{\mu}e^{2\lambda/\mu}G_1(k,r,\mu,\lambda)-\frac{1}{\sqrt{\lambda}}G_2(k,r,\mu,\lambda)\right).$$

Hence

$$s_k^{(r)}(\mu,\lambda)=\frac{r^2}{k}\left[2\lambda^2\left(b_k^{(r)}(\mu)\right)^2+\frac{\mu^3}{\lambda}\left(b_k^{(r)}(\lambda)\right)^2\right],$$
$$t_k^{(r)}(\mu,\lambda)=\frac{r(r+1)}{k}\left[2\lambda^2 b_k^{(r)}(\mu)b_k^{(r+1)}(\mu)+\frac{\mu^3}{\lambda}b_k^{(r)}(\lambda)b_k^{(r+1)}(\lambda)\right],$$
$$u_k^{(r)}(\mu,\lambda)=\frac{(r+1)^2}{k}\left[2\lambda^2\left(b_k^{(r+1)}(\mu)\right)^2+\frac{\mu^3}{\lambda}\left(b_k^{(r+1)}(\lambda)\right)^2\right],$$

or

$$s_k^{(r)}(\mu,\lambda)=r^2k^{2r+1}\left[4e^{4\lambda/\mu}\left(2\frac{\lambda^4}{\mu^4}+\frac{\mu}{\lambda}\right)G_1^2(k,r,\mu,\lambda)\right.$$
$$\left.-\frac{4\mu^2}{\lambda\sqrt{\lambda}}G_1(k,r,\mu,\lambda)G_2(k,r,\mu,\lambda)+\frac{\mu^3}{\lambda^2}G_2^2(k,r,\mu,\lambda)\right],$$
$$t_k^{(r)}(\mu,\lambda)$$
$$=r(r+1)k^{2r+2}\left[4e^{4\lambda/\mu}\left(2\frac{\lambda^4}{\mu^4}+\frac{\mu}{\lambda}\right)G_1(k,r,\mu,\lambda)G_1(k,r+1,\mu,\lambda)\right.$$
$$-2e^{2\lambda/\mu}\frac{\mu^2}{\lambda\sqrt{\lambda}}\left(G_1(k,r,\mu,\lambda)G_2(k,r+1,\mu,\lambda)\right.$$
$$\left.+G_1(k,r+1,\mu,\lambda)G_2(k,r,\mu,\lambda)\right)$$
$$\left.+\frac{\mu^3}{\lambda^2}G_2(k,r,\mu,\lambda)G_2(k,r+1,\mu,\lambda)\right],$$
$$u_k^{(r)}(\mu,\lambda)=(r+1)^2k^{2r+3}\left[4e^{4\lambda/\mu}\left(2\frac{\lambda^4}{\mu^4}+\frac{\mu}{\lambda}\right)G_1^2(k,r+1,\mu,\lambda)\right.$$

$$-4\frac{\mu^2}{\lambda\sqrt{\lambda}}G_1(k,r+1,\mu,\lambda)G_2(k,r+1,\mu,\lambda)+\frac{\mu^3}{\lambda^2}G_2^2(k,r+1,\mu,\lambda)\Bigg].$$

Using the above quantities and letting $k=1$ we have the following goodness-of-fit tests for $H_0 : X \sim IG(\mu,\lambda)$.

$$\begin{aligned}
&\hat{T}_{1n}^{(r)}\\
&= \frac{n}{\left(a^{(r)}-s_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)\left(c^{(r)}-u_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)-\left(b^{(r)}-t_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)^2}\\
&\times\Bigg[\left(c^{(r)}-u_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)\Bigg(\frac{1}{n}\sum_{j=1}^{n}\Bigg(-\log\Bigg[\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(\frac{X_j}{\hat{\lambda}_n}-1\right)\right)\\
&-e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(1+\frac{X_j}{\hat{\mu}_n}\right)\right)\Bigg]\Bigg)^r-\Gamma(r+1)\Bigg)^2\\
&-2\left(b^{(r)}-t_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)\Bigg(\frac{1}{n}\sum_{j=1}^{n}\Bigg(-\log\Bigg[\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(\frac{X_j}{\hat{\lambda}_n}-1\right)\right)\\
&-e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(1+\frac{X_j}{\hat{\mu}_n}\right)\right)\Bigg]\Bigg)^r-\Gamma(r+1)\Bigg)\\
&\times\Bigg(\frac{1}{n}\sum_{j=1}^{n}\Bigg(-\log\Bigg[\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(\frac{X_j}{\hat{\lambda}_n}-1\right)\right)\\
&-e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(1+\frac{X_j}{\hat{\mu}_n}\right)\right)\Bigg]\Bigg)^{r+1}-\Gamma(r+2)\Bigg)\\
&+\left(a^{(r)}-s_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)\Bigg(\frac{1}{n}\sum_{j=1}^{n}\Bigg(-\log\Bigg[-\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(\frac{X_j}{\hat{\lambda}_n}-1\right)\right)\\
&-e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi\left(-\sqrt{\frac{\hat{\lambda}_n}{X_j}}\left(1+\frac{X_j}{\hat{\mu}_n}\right)\right)\Bigg]\Bigg)^{r+1}-\Gamma(r+2)\Bigg)^2\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\hat{T}_{1n;c_1}^{(r,a)}\\
&= \frac{n}{a^{(r)}-s_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)+2a\left(b^{(r)}-t_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)+a^2\left(c^{(r)}-u_1^{(r)}(\hat{\mu}_n,\hat{\lambda}_n)\right)}
\end{aligned}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r} - \Gamma(r+1) \right.$$

$$\left. +a \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^{2},$$

$$\hat{T}^{(r,a)}_{1n;c_2}$$

$$= \frac{n}{\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) - \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)^2}$$

$$\times \frac{1}{a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + 2a\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) + a^2\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)}$$

$$\times \left[\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + a\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) \right) \right.$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r} - \Gamma(r+1) \right)$$

$$- \left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + a\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) \right)$$

$$\left. \times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \right) \right]^{2},$$

$$\hat{T}^{(r)}_{1n;c_3} = \frac{n}{c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \right]^{2},$$

$$\hat{T}^{(r)}_{1n;c_4}$$

$$= \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) - \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)^2 \right)}$$

$$\times \frac{1}{\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)} \left[\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right) \right.$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^r) - \Gamma(r+1) \right)$$

$$- \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2,$$

$$\hat{T}^{(r)}_{1n;c_1} = \frac{n}{a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)}$$

$$\times \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^r - \Gamma(r+1) \right]^2,$$

$$\hat{T}^{(r)}_{1n;c_2}$$

$$= \frac{n}{\left(\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right) \left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right) - \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)^2 \right)}$$

$$\times \frac{1}{\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)} \Bigg[\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)$$

$$\times \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^r - \Gamma(r+1) \right)$$

$$- \left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)$$

$$\left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \right) \Bigg]^2,$$

$$\hat{T}^{(r,b)}_{1n;c_3}$$

$$= \frac{n}{c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + 2b \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right) + b^2 \left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) \right)}$$

$$\times \Bigg[b \left(\frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^r - \Gamma(r+1) \right)$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \left(-\log \left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n} \Phi(L) \right) \right)^{r+1} - \Gamma(r+2) \Bigg]^2,$$

$$\hat{T}_{1n;c_4}^{(r,b)}$$
$$= \frac{n}{\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) - \left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)^2}$$
$$\times \frac{1}{c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + 2b\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right) + b^2\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)}$$
$$\times \Bigg[\left(c^{(r)} - u_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + b\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi(L)\right)\right)^r - \Gamma(r+1)\right)$$
$$-\left(b^{(r)} - t_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n) + b\left(a^{(r)} - s_1^{(r)}(\hat{\mu}_n, \hat{\lambda}_n)\right)\right)$$
$$\times \left(\frac{1}{n}\sum_{j=1}^{n}\left(-\log\left(\Phi(-R) - e^{2\hat{\lambda}_n/\hat{\mu}_n}\Phi(L)\right)\right)^{r+1} - \Gamma(r+2)\right)\Bigg]^2.$$

Following the above consideration one can also get the duals tests.

5. Comparisions of Tests for Exponentiality with Other Tests

When $n = 20$ the test-statistics $\hat{T}_{kN}^{(r)}$ its representations $\hat{T}_{kN}^{(r,a)}$, $\hat{T}_{kN}^{(r,b)}$ and their components were investigated for $k = 1, 2, 4, 5$, and $r = -0.499, -0.495$,-0.45,-0.4,-0.2,-0.1, 0.1, 0.2, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 1.05, 1.1, 1.5, 1.7, 2.0, 2.5. We take in simulations

$$a = -\frac{2(b^{(r)} - t_1^{(r)})}{c^{(r)} - u_1^{(r)}}, \quad b = -\frac{2(b^{(r)} - t_1^{(r)})}{a^{(r)} - s_1^{(r)}}. \tag{5.1}$$

Powers of 5% tests based on 100000 simulations using empirical critical values with a and b in (5.1) are presented in table. We include here only sets of simulations where favorable omnibus tests are appearing. There are many tests which have powers greater than recommended tests given in the bottom of table. They are shown in boldface, while boldface with star denotes the maximum power. Among tests we omitted here there are ones having maximum power for particular alternatives. For instance, for Alt. $\mathrm{Ln}(\nu, 1)$ and $\mathrm{Ln}(\nu, 1.2)$ the test $\hat{T}_{n;c_1}^{(0.5,a)}$ has power 23 and 41, respectively. Moreover, we omitted simulations of powers for the tests with $k = 2, 4, 5$ as in general they are not good

	r	$W(\theta,\tau):\tau=$			$G(\lambda,\beta):\beta=$			$Ln(\nu,\sigma):\sigma=$			$Par(\theta,\alpha):\alpha=$			Av.
		2	1.5	0.5	2	1.5	0.5	0.775	1	1.2	1	2	4	
1	2	3			4			5			6			7
	-0.499	95	52	96	53	20	69	38	12	26	84	47	19	50.9
	-0.495	95	49	96	51	19	71	36	11	26	84	46	19	50.3
	-0.45	91	41	96	39	14	**75**	28	10	26	83	46	20	47.4
	-0.4	87	32	96	33	10	**75**	22	10	27	83	46	20	45.1
	-0.2	68	15	96	15	3	**76**	10	12	29	82	45	20	39.3
	-0.1	66	14	96	13	3	**77**	10	13	30	82	46	20	39.2
	0.1	73	18	**97***	19	5	**77**	17	17	33	83	**48**	**22**	42.4
	0.2	77	22	**97***	23	6	**77**	22	**18**	33	83	**48**	**22**	44.0
	0.4	83	29	96	30	9	**74**	31	**19**	**34**	84	**49**	**23**	46.8
	0.5	86	32	96	35	11	73	34	**20**	33	84	**49**	**23**	48.0
	0.6	87	34	96	37	12	71	36	**20**	33	84	**49**	**22**	48.4
$\hat{T}_{1n}^{(r)}$	0.7	89	37	96	39	13	70	37	**19**	33	84	**49**	**23**	49.1
	0.8	90	39	95	41	14	68	36	**19**	33	84	**49**	**23**	49.3
	0.9	91	40	95	42	14	66	36	**19**	33	84	**49**	**23**	49.3
	0.95	92	41	95	42	15	65	35	**19**	33	84	**49**	**23**	49.4
	1.05	92	42	94	43	15	64	35	**18**	**34**	84	**49**	**23**	49.4
	1.1	93	43	94	43	15	63	34	**18**	**34**	84	**49**	**23**	49.4
	1.5	93	44	92	42	15	57	29	17	**35**	**85***	**50**	**23**	48.5
	1.7	93	43	91	40	15	55	26	17	**36**	**85***	**50**	**23**	47.8
	2.0	92	40	89	36	13	51	22	17	**37**	84	**51**	**24**	46.3
	2.5	86	31	87	27	9	48	16	**19**	**39**	84	**52***	**25**	43.6
	-0.499	**96***	52	95	53	21	67	39	12	26	84	47	19	50.9
	-0.495	95	51	96	53	20	69	38	11	25	84	46	19	50.6
	-0.45	94	48	**97***	49	19	**76**	36	9	21	83	44	19	49.6
	-0.4	92	42	**97***	44	15	**78**	33	7	17	81	41	17	47.0
	-0.2	66	17	96	18	4	**79***	15	1	5	71	30	12	34.5
	-0.1	60	15	96	17	4	**79***	17	1	3	68	27	11	33.2
	0.1	71	23	95	29	9	**78**	35	5	3	62	22	10	36.8
	0.2	77	28	94	34	11	**77**	41	7	4	57	20	10	38.3
	0.4	85	37	90	43	15	72	**48**	10	7	47	20	10	40.3
	0.5	88	40	86	46	17	70	**48**	10	8	48	21	10	41.0
	0.6	91	43	84	49	18	68	**49***	11	9	53	23	10	42.3
$\hat{T}_{1n;c_2}^{(r,a)}$	0.7	92	46	82	51	19	65	**48**	11	10	55	24	10	42.8
	0.8	94	49	81	52	20	64	**48**	11	11	56	25	11	43.5
	0.9	94	50	79	54	21	62	**46**	10	12	58	26	11	43.6
	0.95	95	51	79	54	21	60	**44**	10	12	59	26	11	43.5
	1.05	95	52	78	54	21	58	**43**	10	13	60	27	11	43.5
	1.1	95	53	78	54	21	57	41	10	14	60	28	12	43.6
	1.5	**96***	53	77	51	21	52	34	9	18	65	32	13	43.4
	1.7	**96***	52	77	50	20	49	30	9	20	67	34	14	43.2
	2.0	95	49	78	45	18	47	26	10	24	70	37	15	42.8
	2.5	91	40	79	35	13	46	18	13	31	75	44	20	42.1

Table 1: Powers of 5% tests based on 100000 simulations using empirical critical values

candidates for omnibus tests for exponentiality. Nevertheless there are among them tests with maximum power for special alternatives. For instance, for Alt. Ln$(\nu, 0.775)$ the tests $\hat{T}_{5N;c_2}^{(0.8,a)}$, $\hat{T}_{5N;c_3}^{(0.8,b)}$ and $\hat{T}_{5N;c_4}^{(1.1)}$ have powers 57, 57 and 59

1	2	3			4			5			6			7
	-0.499	**96***	52	95	53	21	66	39	12	26	84	47	19	50.8
	-0.495	95	52	95	53	20	66	38	12	26	84	46	20	50.6
	-0.45	**96***	52	95	52	21	65	38	12	27	84	47	20	50.8
	-0.4	**96***	52	95	53	21	64	37	13	28	84	47	20	50.8
	-0.2	**96***	51	93	51	19	59	34	14	31	84	**49**	**21**	50.2
	-0.1	95	50	92	49	19	57	32	14	32	84	**49**	**22**	49.6
	0.1	94	47	91	45	17	54	28	16	**34**	84	**50**	**23**	49.6
	0.2	94	46	90	44	16	52	27	16	**35**	84	**50**	**23**	48.1
	0.4	93	43	88	39	14	49	24	17	**36**	84	**51**	**24**	46.8
	0.5	92	41	88	37	13	48	23	17	**37**	84	**52***	**24**	46.3
	0.6	91	38	87	35	12	47	21	**18**	**37**	84	**51**	**24**	45.4
$\hat{T}^{(r)}_{1n;c_3}$	0.7	90	37	86	33	12	46	20	**18**	**38**	84	**51**	**25**	45.0
	0.8	88	35	85	31	11	46	19	**18**	**38**	84	**51**	**25**	44.3
	0.9	87	32	85	28	9	45	17	**19**	**38**	84	**51**	**25**	43.3
	0.95	86	30	85	27	9	45	16	**19**	**38**	83	**51**	**25**	42.8
	1.05	83	27	84	24	8	44	16	**19**	**39**	83	**51**	**25**	41.9
	1.1	83	27	84	24	8	44	15	**19**	**39**	83	**51**	**26***	41.9
	1.5	68	15	82	13	4	41	10	**20**	**40**	83	**51**	**26***	37.8
	1.7	57	10	81	9	3	40	8	**20**	**40**	83	**51**	**26***	35.7
	2.0	24	2	79	2	1	39	5	**20**	**40**	82	**51**	**26***	30.9
	2.5	0	0	77	0	1	37	5	**21**	**40**	81	**50**	**26***	28.2
	-0.499	0	0	94	0	0	73	0	0	1	60	21	9	21.5
	-0.495	0	0	94	0	0	**74**	0	0	1	60	21	9	21.6
	-0.45	0	0	94	0	0	**75**	0	0	2	62	23	10	22.2
	-0.4	11	1	95	1	0	**76**	0	0	2	66	25	11	24.0
	-0.2	79	26	**97***	29	8	**79***	24	3	7	75	34	14	39.6
	-0.1	87	36	**97***	40	13	**79***	34	5	10	78	37	15	44.3
	0.1	93	47	**97***	51	19	**77**	41	8	16	81	42	17	49.1
	0.2	94	49	**97***	53	20	**75**	42	9	19	82	43	17	50.0
	0.4	95	51	96	53	21	69	40	11	24	83	45	19	50.6
	0.5	**96***	52	95	53	20	67	38	12	26	84	47	19	50.8
	0.6	**96***	52	95	52	20	64	37	12	28	84	47	20	50.6
$\hat{T}^{(r)}_{1n;c_1}$	0.7	**96***	51	94	52	20	61	35	13	29	84	**48**	**21**	50.3
	0.8	**96***	51	93	50	20	59	34	14	31	84	**48**	**21**	50.1
	0.9	95	50	92	49	19	57	32	14	32	84	**49**	**22**	49.6
	0.95	95	49	92	48	18	56	31	15	32	84	**49**	**22**	49.3
	1.05	95	48	91	46	18	54	30	16	**34**	84	**50**	**22**	49.0
	1.1	95	48	91	45	17	54	29	16	**34**	84	**50**	**23**	48.8
	1.5	92	40	88	37	13	48	23	**18**	**37**	84	**51**	**24**	46.3
	1.7	90	37	86	33	11	46	20	**18**	**38**	84	**51**	**25**	44.9
	2.0	85	30	84	25	8	44	16	**19**	**39**	83	**51**	**25**	42.4
	2.5	69	16	82	13	4	42	10	**20**	**40**	83	**51**	**26***	38.0

Table 1: Continuation

respectively. Also for Alt. $\Gamma(\lambda, 1.5)$ the tests $\hat{T}^{(1.7)}_{5N;c_4}$ and $\hat{T}^{(1.7)}_{5N;c_2}$ have powers 24 and 23, respectively.

The alternative distributions and recommended tests are taken from Ascher [2].

1	2	3			4			5			6			7
	-0.499	**96***	52	95	53	21	65	39	12	26	84	47	19	50.8
	-0.495	95	52	93	53	20	62	38	12	26	84	47	20	50.2
	-0.45	95	49	77	48	18	46	34	13	31	83	47	20	46.8
	-0.4	92	41	67	40	14	41	26	14	**34**	82	47	**21**	47.5
	-0.2	20	3	55	2	1	40	3	17	**37**	71	41	19	25.8
	-0.1	0	1	54	0	1	41	3	**18**	**37**	66	38	18	23.1
	0.1	0	0	52	0	1	43	6	**21**	**36**	61	35	18	22.8
	0.2	0	0	53	0	1	43	7	**22***	**34**	61	35	17	22.8
	0.4	0	0	55	0	1	44	10	**22***	32	60	34	17	22.9
	0.5	0	0	57	1	1	45	11	**22***	31	60	34	17	23.3
	0.6	0	0	59	1	1	46	11	**21**	29	60	33	17	23.2
$\hat{T}^{(r)}_{1n;c_2}$	0.7	3	0	60	1	2	47	12	**20**	28	61	34	17	23.8
	0.8	23	3	62	4	2	48	15	**19**	27	62	34	17	26.3
	0.9	45	8	63	10	3	49	20	**19**	26	63	34	17	29.8
	0.95	54	11	64	14	4	49	22	**18**	26	63	34	17	31.3
	1.05	69	18	66	21	6	49	26	**18**	25	64	34	17	34.4
	1.1	74	21	67	24	7	50	28	17	25	64	35	17	35.8
	1.5	90	39	70	39	13	50	31	14	24	67	36	17	40.8
	1.7	93	43	71	42	15	49	29	13	24	68	36	17	41.7
	2.0	94	46	73	42	16	47	26	12	24	69	37	17	41.9
	2.5	92	43	75	38	15	45	21	12	28	73	40	18	41.7
	-0.499	**96***	52	94	53	21	64	39	12	26	84	47	19	50.6
	-0.495	95	51	91	53	20	59	38	12	27	84	47	20	49.8
	-0.45	90	37	67	35	12	43	23	13	33	81	46	**21**	41.8
	-0.4	30	4	63	3	1	46	3	13	**34**	70	40	18	27.1
	-0.2	0	0	83	0	0	66	2	8	13	31	15	8	18.8
	-0.1	0	0	88	0	0	71	3	7	7	36	14	8	19.5
	0.1	51	13	94	17	5	**77**	27	7	4	52	18	9	31.2
	0.2	77	28	94	33	11	**77**	41	7	4	56	20	10	38.2
	0.4	90	43	95	49	18	**74**	**48**	8	6	63	25	11	44.2
	0.5	93	47	94	52	20	72	**47**	8	8	65	26	11	45.3
	0.6	94	50	94	54	21	69	**46**	8	10	67	29	11	46.1
$\hat{T}^{(r,b)}_{1n;c_3}$	0.7	95	52	94	**56***	**22***	68	**44**	8	12	67	31	13	46.8
	0.8	**96***	**54**	93	55	**22***	66	42	8	14	66	32	13	46.8
	0.9	**96***	**54**	93	**56***	**22***	64	41	8	16	65	35	14	47.0
	0.95	**96***	**55***	92	**56***	**22***	62	39	8	17	66	35	14	46.8
	1.05	**96***	**54**	92	54	**22***	60	37	9	20	67	37	15	46.9
	1.1	**96***	**54**	92	53	21	60	36	9	21	69	39	16	47.1
	1.5	95	50	90	47	18	54	29	13	31	77	47	**21**	47.7
	1.7	94	47	88	43	17	51	25	15	**34**	79	**48**	**22**	46.9
	2.0	91	40	87	35	13	47	20	17	**37**	80	**50**	**24**	45.1
	2.5	81	25	84	22	7	44	14	**19**	**39**	81	**51**	**26***	41.1
WE		94	48	78	43	17	35	25	17	33	79	44	20	44.4
G		95	51	91	48	19	55	31	12	30	84	47	20	48.6
$L(0.5)$		92	48	93	47	19	61	33	7	17	79	37	14	45.6
P		93	48	90	45	18	52	28	12	29	82	44	18	46.6
CO		**96***	53	96	**56***	**22***	73	42	11	22	82	44	18	51.3

Table 1: Continuation

Weibull. $X \sim W(\theta, \tau)$

$$f(x) = \frac{\tau}{\theta}\left(\frac{x}{\theta}\right)^{\tau-1} e^{-(x/\theta)^\tau}, \qquad \theta,\ \tau > 0\,.$$

Gamma. $X \sim G(\lambda, \beta)$

$$f(x) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x}, \qquad \lambda,\ \beta > 0\,.$$

Log-normal $X \sim \mathrm{Ln}(\nu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}\exp\left(-\frac{1}{2\sigma^2}(\log x - \nu)^2\right), \qquad \nu \in \mathbb{R},\ \sigma > 0\,.$$

Pareto. $X \sim$ Par (θ, α)

$$f(x) = \alpha\theta^\alpha \frac{1}{(x+\theta)^{\alpha+1}}, \qquad \theta,\ \alpha > 0.$$

Hahn and Spurrier's test.

$$WE = (n-1)S_n^2/n^2\overline{X}_n^2, \qquad \text{where } S_n^2 \text{ is the sample variance.}$$

Gini's statistic.

$$G = \sum_{i=1}^{n-1} iD_{i+1}/n(n-1)\overline{X}_n, \qquad D_{i+1} = (n-i)\left(X_{i+1:n} - X_{i:n}\right)\,.$$

Lorenz statistic. For given $p \in (0,1)$

$$L(p) = \sum_{i=1}^{[np]} X_{i:n}/n\overline{X}_n\,.$$

Pietra's test.

$$P = \sum_{i=1}^{n} |X_i - \overline{X}_n|/2n\overline{X}_n\,.$$

Cox and Oakes' test.

$$CO = n + \sum_{i=1}^{n}\log X_i - \left(\sum_{i=1}^{n} X_i \log X_i\right)/\overline{X}_n.$$

Our empirical study that compares the performances of the recommended tests (given in bottom of Table) for discussed alternatives shows that one can find in the presented class of tests at last one test with a power not less than the power of every recommended here test.

Our simplest omnibus test with Av. power 51 is

$$\hat{T}_{1n;c_1}^{(0.5)} = \frac{16n}{16-5\pi}\left[\frac{1}{n}\left(\frac{1}{\overline{X}_n}\right)^{1/2}\sum_{j=1}^{n} X_j^{1/2} - \frac{\sqrt{\pi}}{2}\right]^2 .$$

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