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# CAUSAL (ANTICAUSAL) CONVOLUTION PRODUCTS OF THE DISTRIBUTIONAL FAMILIES RELATED TO THE ULTRAHYPERBOLIC KLEIN GORDON OPERATOR, ULTRAHYPERBOLIC OPERATOR, LAPLACIAN OPERATOR AND THE DIAMOND OPERATOR

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**Abstract:** Let $E_{\alpha,\beta}$ and $T_{\alpha,\beta}$ the convolution distributional functions families defined by $E_{\alpha,\beta} = G_\alpha * H_\beta$ and $T_{\alpha,\beta} = G_\alpha * R_\beta$, where $G_\alpha = G_\alpha(P \pm i0, m, n)$ is the causal (anticausal) distribution defined by (8)(cf. [6]) and $H_\beta = H_\beta(P \pm i0, n)$ is causal (anticausal) analogues of the elliptic kernel of M.Riesz (cf. [6]) defined by (19) and $R_\beta$ is the elliptic kernel of Marcel Riesz defined by (30). In this paper we give a sense to convolution product of $E_{\alpha,\beta} * E_{\alpha^{'},\beta^{'}}$ and $T_{\alpha,\beta} * T_{\alpha^{'},\beta^{'}}$ for all $\alpha, \beta, \alpha^{'}, \beta^{'}$ complex numbers such that $\beta, \beta^{'}$ and $\beta + \beta^{'} \neq n + 2r, r = 0, 1, \ldots$. As consequence of our formula we give a sense to convolution product of: $K^k\delta * K^{k^{'}}\delta, L^k\delta * L^{k^{'}}\delta, \triangle^k\delta * \triangle^{k^{'}}\delta$ and $\Diamond^k\delta * \Diamond^{k^{'}}\delta$, where $K^k$ is the $n$-dimensions ultrahyperbolic Klein Gordon operator iterated $k$-times, $L^k$ is the ultrahyperbolic operator iterated $k$-times, $\triangle^k$ is the Laplacian operator iterated $k$-times and $\Diamond^k$ is the Diamond operator iterated $k$-times.

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## 1. Introduction

Let $x = (x_1, \ldots x_n)$ be a point of the $n$-dimensional euclidean space $R^n$. Consider a quadratic formin $n$ variables defined by

$$P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 \ldots x_{p+q}^2 \,, \tag{1}$$

where $p$ is the number of positive term, $q$ is the number of negative term of $P(x)$ and $p + q = n$ dimensions of the space.

The hypersurface $P = P(x) = 0$ is a hypercone with singular point (the vertex) at the origin.

We call $\varphi(x)$ the $C^\infty$ functions with compact support defined from $R^n$ to $R$.

From [5], p. 253, the distribution $P_+^\lambda$ is defined by

$$\left\langle P_+^\lambda, \varphi \right\rangle = \int\limits_{P>0} (P(x))^\lambda \, \varphi(x) dx \,, \tag{2}$$

where $\lambda$ is a complex number and $dx = dx_1 \ldots dx$.

For $\mathrm{Re}\,(\lambda) \geq 0$, this integral converges and is analytic function of $\lambda$. Analytic continuation to $\mathrm{Re}(\lambda) < 0$, can be used to extend the definition of $\left(P_+^\lambda, \varphi\right)$.

From [5], p. 245, $\left(P_+^\lambda, \varphi\right)$ has two sets of singularities nameley

$$\lambda = -1, -2, \ldots \tag{3}$$

and

$$\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \cdots - \frac{n}{2} - k, \quad k = 0, 1, 2, \ldots \,. \tag{4}$$

On the other hand, the distributional $(P \pm i0)^\lambda$ is defined the following formula

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} (P \pm i\varepsilon \mid x \mid^2)^\lambda \,, \tag{5}$$

where $\varepsilon > 0, \lambda$ is a complex number and

$$|x|^2 = x_1^2 + \cdots + x_n^2. \tag{6}$$

The distribution (5) are analityc in $\lambda$ everywhere except at

$$\lambda = -\frac{n}{2} - k, \quad k = 0, 1, 2, \ldots \,, \tag{7}$$

where they have simples poles (see [5], p. 275).

Let $G_\alpha(P \pm i0, m, n)$ be the causal (anticausal) distribution defined by

$$G_\alpha(P \pm i0, m, n) = a_\alpha(m,n)(P \pm i0)^{\frac{\alpha-n}{2}} K_{\frac{n-\alpha}{2}}(\sqrt[2]{m(P \pm i0)})\,, \tag{8}$$

where $m$ is a positive real number, $\alpha$ is a complex number, $K_\nu(z)$ designates the modified Bessel function of the third kind (see [13], p. 78, formulas (6) and (7)):

$$K_\nu(z) = \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{sen(\nu\pi)} \text{ if } \nu \neq \text{integer}, \tag{9}$$

$$I_\nu(z) = \sum_{k=o}^{\infty} \frac{(\frac{z}{2})^{2k+\nu}}{k!\Gamma(k+\nu+1)}\,. \tag{10}$$

$a_\alpha(m,n)$ is the constant defined by

$$\left(\Gamma(\frac{\alpha}{2})\right)^{-1} e^{\frac{\alpha\pi i}{2}} (2\pi)^{-\frac{n}{2}} 2^{1-\frac{\alpha}{2}} e^{\pm\frac{q\pi i}{2}} (m^2)^{\frac{n-\alpha}{4}} \tag{11}$$

and $(P \pm i0)^\lambda$ is defined by (5). The distribution $G_\alpha(P \pm i0, m, n)$ was introduce by S.E. Trione in [6].

**Lemma 1.** *The distributional family $G_\alpha(P \pm i0, m, n)$ has the following properties*

$$G_0(P \pm i0, m, n) = \delta(x), \tag{12}$$

$$G_\alpha(P \pm i0, m, n) * G_\beta(P \pm i0, m, n) = G_{\alpha+\beta}(P \pm i0, m, n), \tag{13}$$

$$G_\alpha(P \pm i0, m, n) * G_{-2k}(P \pm i0, m, n) = G_{\alpha-2k}(P \pm i0, m, n), \tag{14}$$

$$K^k\left\{G_\alpha(P \pm i0, m, n)\right\} = G_{\alpha-2k}(P \pm i0, m, n) \tag{15}$$

*and*

$$G_{-2k}(P \pm i0, m, n) = K^k\left\{\delta(x)\right\}\,, \tag{16}$$

*where*

$$K^k = \left\{\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} - m^2\right\}^k\,. \tag{17}$$

*is the $n$-dimension ultrahyperbolic Klein-Gordon operator iterated $k-$ times, $k \geq 1$ and $m > 0$.*

*The proof of Lemma 1 appears in [6].*

From (12) and (15) we have the following properties

$$K^k \{G_{2k}(P \pm i0, m, n)\} = \delta(x) \tag{18}$$

(cf. [6]) which we mean that $G_{2k}(P \pm i0, m, n)$ are elementary causal (anticausal) solutions of the ultrahyperbolic Klein Gordon operator iterated $k$-times.

On the other hand, let $H_\alpha = H_\alpha(P \pm i0, n)$ be the causal (anticausal) distribution defined by

$$H_\alpha = H_\alpha(P \pm i0, n) = \frac{e^{\mp \frac{\alpha\pi i}{2}} e^{\pm \frac{q\pi i}{2}} (P \pm i0)^{\frac{\alpha - n}{2}}}{D_n(\alpha)} \tag{19}$$

where

$$D_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \tag{20}$$

The distributional functions $H_\alpha$ are causal (anticausal) analogues of the elliptic kernel of M. Riesz (see [9], pp. 16-21) and was introduced by S.E. Trione in [6].

We observe that $H_\alpha$ has simple poles at $\alpha = n + 2l, l = 0, 1, 2, \ldots$ (which are due to $\Gamma$ which appear in the numerator). This is an essential difference between $H_\alpha$ and $G_\alpha$, which is an entire distribution.

**Lemma 2.** *Let $\alpha$ and $\beta$ be complex numbers such that $\alpha, \beta$, and $\alpha + \beta \neq n + 2r, r = 0, 1, \ldots$ and $n$-dimensions of the space, then the following formulae are valid*

$$H_0 = \delta(x), \tag{21}$$

$$H_\alpha * H_\beta = H_{\alpha+\beta}, \tag{22}$$

$$H_\alpha * H_{-2k} = H_{\alpha-2k}, \tag{23}$$

$$L^k \{H_\alpha\} = H_{\alpha-2k} \tag{24}$$

*and*

$$H_{-2k} = L^k \{\delta\} \ , \tag{25}$$

*where $L^s$ is the ultrahyperbolic operator iterated $s$ times, $s \geq 1$:*

$$L^s = \left\{ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^s . \tag{26}$$

*The proof of Lemma 2 appears in [6].*

From (21) and (24) we have the following properties

$$L^k \{H_{2k}\} = \delta \tag{27}$$

(cf. [6]) which we mean that $H_{2k} = H_{2k}(P \pm i0, n)$ are elementary causal (anticausal) solution of the ultrahyperbolic operator iterated $k$ times.

Now consider the convolution distributional function families defined by

$$E_{\alpha,\beta} = E_{\alpha,\beta}(P \pm i0, m, n) = G_\alpha * H_\beta \tag{28}$$

and

$$T_{\alpha,\beta} = T_{\alpha,\beta}(P \pm i0, m, n) = G_\alpha * R_\beta\,, \tag{29}$$

if $\beta \neq n + 2r, r = 0, 1, \dots$ and $n$-dimensions of the space. Here $G_\alpha = G_\alpha(P \pm i0, m, n)$ is defined by (8), $H_\alpha = H_\alpha(P \pm i0, n)$ by (19) and $R_\beta$ is the elliptic kernel of Marcel Riesz defined by

$$R_\alpha = R_\alpha(x) = \frac{|x|^{\alpha-n}}{D_n(\alpha)} \tag{30}$$

(see [9], p. 29), $D_n(\alpha)$ is defined by (21) and the symbol $*$ as is usual we mean convolution.

From (13), the convolution product of $G_\alpha * G_\beta$ exists for all $\alpha$ and $\beta$ complex numbers and from [12], we know that putting $m^2 = 0$ in (8) we have

$$G_\beta(P \pm i0, n) = H_\beta(P \pm i0, n)\,. \tag{31}$$

If $\beta \neq n + 2r, r = 0, 1, \dots$ (see [12], p. 49, formula (II, 7, 1)), therefore the distributional convolution families defined by (28) exists if $\beta \neq n + 2r, r = 0, 1, \dots$.

On the other hand, from [12], we know that putting $q = 0, p = n$ in the definition of $H_\beta(P \pm i0, n)$ the following formula is valid

$$H_\beta(P \pm i0, n)\,|_{q=0} = H_\beta(|x|^2, n) = R_\beta(x)\,. \tag{32}$$

If $\beta \neq n + 2r, r = 0, 1, \dots$ (see [12], p. 99, formula (V, 2, 1)), elliptic kernel of Marcel Riesz defined by (30), therefore the distribution convolution families defined by (29) exists if $\beta \neq n + 2r, r = 0, 1, \dots$.

## 2. The Distributional Convolution Product of $E_{\alpha,\beta} * E_{\alpha',\beta'}$

In this section we give a sense to distributional convolution product of $E_{\alpha,\beta} * E_{\alpha',\beta'}$ using the properties (13) and (22).

We observe that putting $\alpha = -2k$ and $\beta = -2l, k, l = 0, 1, 2, \ldots$ in (28) we have

$$E_{-2k,-2l} = G_{-2k} * H_{-2l} = H_{-2l} * G_{-2k}. \tag{33}$$

Taking into account (16) and (25) $G_{-2k}$ and $H_{-2l}$ are a finite linear combination of $\delta$ and its derivatives, therefore $G_{-2k}$ and $H_{-2l}$ are distributions the class $O_c'$, where $O_c'$ is the space of rapidly decresing distributions. Therefore using the properties (16) and (25) $E_{-2k,-2l}$ is a distribution of the class $O_c'$ and have the following properties

$$K^k \left\{ L^l \delta \right\} = L^l \left\{ K^k \delta \right\} , \tag{34}$$

where $K^k$ is defined by (17) and $L^l$ by (26).

From (34) we have the following definition.

**Definition 3.** *Let $k, l$ be nonnegative integers and $O_l^k$ the operator defined by*

$$M_l^k = K^k \left\{ L^l \delta \right\} = L^l \left\{ K^k \delta \right\} . \tag{35}$$

**Lemma 4.** *The following properties are valid*

$$E_{0,0} = \delta, \tag{36}$$

$$M_l^k \left\{ E_{\alpha,\beta} \right\} = E_{\alpha-2k,\beta-2l}, \tag{37}$$

$$E_{-2k,-2l} = M_l^k \left\{ \delta \right\} \tag{38}$$

*and*

$$M_l^k \left\{ E_{2k,2l} \right\} = E_{0,0} = \delta\,. \tag{39}$$

*Proof.* From (28) and using (12) and (21) we obtain $E_{0,0} = G_0 * H_0 = \delta(x) * \delta(x) = \delta(x)$. From (28) and using (15), (24) and (31) we have

$$\begin{aligned} M_l^k \left\{ E_{\alpha,\beta} \right\} &= K^k \left\{ L^l \left\{ G_\alpha * H_\beta \right\} \right\} = L^l \left\{ K^k \left\{ G_\alpha * H_\beta \right\} \right\} \\ &= L^l \left\{ \left\{ K^k G_\alpha * H_\beta \right\} \right\} = L^l \left\{ \left\{ G_{\alpha-2k} * H_\beta \right\} \right\} = \left\{ \left\{ G_{\alpha-2k} * L^l H_\beta \right\} \right\} \\ &= \left\{ G_{\alpha-2k} * H_{\beta-2l} \right\} = E_{\alpha-2k,\beta-2l}. \end{aligned} \tag{40}$$

Now putting $\alpha = \beta = 0$ in(37) and using (36) we have $E_{-2k,-2l} = O_l^k \left\{ E_{0,0} \right\} = O_l^k \left\{ \delta \right\}$ and putting $\alpha = 2k, \beta = 2l$ in(37) and using (36) we have

$$M_l^k \left\{ E_{2k,2l} \right\} = E_{0,0} = \delta. \qquad \Box \tag{41}$$

**Lemma 5.** *Let $E_{\alpha,\beta}$ the distribution familie defined by (28) then the following formula is valid*

$$E_{\alpha,\beta} * E_{\alpha^{`},\beta^{`}} = E_{\alpha+\alpha^{`},\beta+\beta^{`}} \tag{42}$$

*for all $\alpha, \beta, \alpha^{`}$ and $\beta^{`}$ such that $\beta,^{`} \beta+\beta^{`} \neq n+2r, r = 0, 1, \ldots$ and $n$-dimensions of the space.*

*Proof.* Taking into account that $G_{-2k}(P \pm i0, m, n)$ and $H_{-2l}(P \pm i0, n)$ are in the class $O_c^{`}$ and considering that $G_\alpha \in S^{`}$ for all $\alpha \in C$, where $C$ is the set of complex numbers, $S^{`}$ is the dual of $S$ and $S$ is the Schwartz of functions (see [10], p. 233) and from(see [6]), $H_\beta \in S^{`}$ for all $\beta \in C$ such that $\beta \neq n + 2r, r = 0, 1, \ldots$ we conclude that

$$E_{-2k,\beta} \in O_c^{`} \tag{43}$$

if $\beta \neq n + 2r, r = 0, 1, \ldots$ and

$$E_{\alpha,-2l} \in O_c^{`} \tag{44}$$

for all $\alpha \in C$.

In consequence from (43) and (44) we have

$$E_{-2k,\beta} * E_{\alpha,-2l} \in O_c^{`} \tag{45}$$

if $\beta \neq n + 2r, r = 0, 1, \ldots$

Now taking into account that $G_\alpha(P \pm i0, m, n)$ (see [6]) is an entire distributional functional of $\alpha$ and $H_\beta(P \pm i0, n)$ (see [6]) is a entire distributional function of $\beta$ such that $\beta \neq n + 2r, r = 0, 1, \ldots,$ then by appealing to the principle of analytical continuation we conclude that

$$E_{\alpha,\beta} * E_{\alpha^{`},\beta^{`}} \in O_c^{`} \tag{46}$$

for $\beta, \beta^{`}$ and $\beta + \beta^{`} \neq n + 2r, r = 0, 1, \ldots.$

From (28) and using the properties (13) and (22) we have

$$\begin{aligned} E_{\alpha,\beta} * E_{\alpha^{`},\beta^{`}} &= (G_\alpha * H_\beta) * \left(G_{\alpha^{`}} * H_{\beta^{`}}\right) = (G_\alpha * G_{\alpha^{`}}) * \left(H_\beta * H_{\beta^{`}}\right) \\ &= G_{\alpha+\alpha^{`}} * H_{\beta+\beta^{`}} = E_{\alpha+\alpha^{`},\beta+\beta^{`}}\,, \end{aligned} \tag{47}$$

for all $\alpha, \beta, \alpha^{`}$ and $\beta^{`}$ such that $\beta,^{`} \beta+\beta^{`} \neq n+2r, r = 0, 1, \ldots$ and $n$-dimensions of the space. ☐

In particular putting $\alpha = -2k, \beta = -2l, \alpha^{`} = -2k^{`}, \beta^{`} = -2l^{`}, k, l, k^{`}, l^{`}$ are nonnegative integers and using the properties (38) we have the following formula

$$M_l^k \{\delta\} * M_{l^{`}}^{k^{`}} \{\delta\} = M_{l+l^{`}}^{k+k^{`}} \{\delta\} . \tag{48}$$

Putting $l = l^{`} = 0$ in (48) and using (17), (26) and (35) we have

$$\begin{aligned}\left(L - m^2\right)^k \delta * \left(L - m^2\right)^{k^{`}} \delta = K^k \delta * K^{k^{`}} \delta = K^{k+k^{`}} \delta \\ = \left(L - m^2\right)^{k+k^{`}} \delta. \end{aligned} \tag{49}$$

Similarly putting $k = k^{`} = 0$ in (48) and using(35) we have

$$L^l \delta * L^{l^{`}} \delta = L^{l+l^{`}} \delta. \tag{50}$$

The formula (50) appear in [1], p. 347.

**Remark 6.** The properties (18) and (27) we mean that $G_{2k}\ (P \pm i0, m, n)$ are elmentary causal (anticausal) solution of the ultrahyperbolic Klein-Gordon operator iterated $k$-times and $H_{2k}(P \pm i0, n)$ are elemntary causal (anticausal) solution of the ultrahyperbolic operator iterated $k$ times. In the same form the properties (39) we mean that $E_{2k,2l} = E_{2k,2l}(P \pm i0, m, n) = (G_{2k}(P \pm i0, m, n) * H_{2k}(P \pm i0, n)$ are elemntary causal (anticausal) solution of the operator $M_l^k = K^k \left\{L^l \delta\right\} = L^l \left\{K^k \delta\right\}$.

On the other hand Putting $\alpha = 2, n = 4, q = 1$ in (8), the corresponding elementary solutions can be written

$$G_2(P + i0, m, 4) = -\frac{miK_1\left[\sqrt{m^2(P + i0)}\right]}{4\pi^2((P + i0))^{\frac{1}{2}}} \tag{51}$$

(see [6] and [12], p. 38) and

$$G_2(P + i0, m, 4) = \frac{miK_1\left[\sqrt{m^2(P + i0)}\right]}{4\pi^2((P + i0))^{\frac{1}{2}}} \tag{52}$$

(see [6] and [12], p. 38), where

$$K_l(z) = (-1)^{l+1} \log(\frac{z}{2}) I_l(z) + \frac{1}{2}\sum_{r=o}^{l-1} \frac{(-1)^r (l - r - 1)!}{r!} \left(\frac{z}{2}\right)^{l-2r}$$

$$+\frac{(-1)^l}{2}\sum_{r=o}^{\infty}\frac{\left(\frac{z}{2}\right)^{l-2r}}{(l+r)!}\left[\psi(l+r+1)+\psi(r+1)\right], \quad (53)$$

for $l = 1, 2, \ldots$ (see [2], p. 127, formula (109)), $I_l(z)$ is defined by (10) and $\psi(k)$ is given by

$$\psi(k) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}, \quad (54)$$

$k = 2, 3, ..; \gamma$ is Euler's constant. The formula(51) is a useful expression of the famous "magic function" or "causal propagator" of Feyman. S.E. Trione in [6]) and [12] say that it is the reason that decided to call "causal (anticausal)" the distribution $G_\alpha(P \pm i0, m, n)$ defined by (8). For the same reason we decided to call the "causal (anticausal) convolution products of the distributional families related to the ultrahyperbolic Klein Gordon operator, ultrahyperbolic operator, Laplacian operator and the Diamond operator".

## 3. The Distributional Convolution Product of $T_{\alpha,\beta} * T_{\alpha^{\text{‘}},\beta^{\text{‘}}}$

In this section we give a sense to distributional convolution product of $T_{\alpha,\beta} * T_{\alpha^{\text{‘}},\beta^{\text{‘}}}$ using the properties (13) and the following lemma.

**Lemma 7.** *Let $\alpha$ and $\beta$ be complex numbers such that $\alpha, \beta$, and $\alpha + \beta \neq n + 2r, r = 0, 1, \ldots$ and $n$-dimensions of the space and let $R_\alpha(x)$ be the elliptic kernel of Marcel Riesz (cf. [9]) defined by (30), then the following formulae are valid*

$$R_0(x) = \delta(x), \quad (55)$$

$$R_{-2k}(x) = (-1)^k \triangle^k \{\delta\}, \quad (56)$$

$$\triangle^k \{R_\alpha(x)\} = (-1)^k R_{\alpha-2k}(x) \quad (57)$$

*and*

$$R_\alpha(x) * R_\beta(x) = R_{\alpha+\beta}(x), \quad (58)$$

*where $\triangle^k$ is the Laplacian operator iterated $k$ times,*

$$\triangle^k = \left\{\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right\}^k. \quad (59)$$

The proof of the lemma is given by S.E. Trione in [11].

We observe that the properties (58) appear in [9] and [4] under conditions $\alpha > 0, \beta > 0$ and $\alpha + \beta < n$ and appear in [8], p. 5, under condition $\text{Re}(\alpha+\beta) < n$, $n$-dimensions of the space.

On the other hand we observe that putting $\alpha = -2k$ and $\beta = -2l, k, l = 0, 1, 2, \ldots$ in (29) we have

$$T_{-2k,-2l} = G_{-2k} * R_{-2l} = R_{-2l} * G_{-2k} \tag{60}$$

and using the properties (16) and (56) we have the following property

$$K^k \left\{ \triangle^l \, \delta \right\} = \triangle^l \left\{ K^k \delta \right\}. \tag{61}$$

From(61) we have the following definition.

**Definition 8.** *Let $k, l$ be nonnegative integers and $N_l^k$ the operator defined by*

$$N_l^k = K^k \left\{ \triangle^l \, \delta \right\} = \triangle^l \left\{ K^k \delta \right\}, \tag{62}$$

*where $K^k$ is defined by (17) and $\triangle^l$ by (59).*

**Lemma 9.** *Let $T_{\alpha,\beta}$ be the distributional function families defined by (29) then the following properties are valid*

$$T_{0,0} = \delta(x), \tag{63}$$

$$N_l^k \{T_{\alpha,\beta}\} = T_{\alpha-2k,\beta-2l}, \tag{64}$$

$$T_{-2k,-2l} = N_l^k \{\delta(x)\} \tag{65}$$

*and*

$$N_l^k \{T_{2k,2l}\} = T_{0,0} = \delta(x). \tag{66}$$

*Proof.* From (29) and using (12) and (55) we obtain $T_{0,0} = G_0 * R_0 = \delta * \delta = \delta$. From (29) and using (15), (32) and (57) we have

$$\begin{aligned} N_l^k \{T_{\alpha,\beta}\} &= K^k \, \triangle^l \, \{G_\alpha * R_\beta\} = \triangle^l \, K^k \{G_\alpha * R_\beta\} \\ &= \triangle^l \left\{ K^k G_\alpha * R_\beta \right\} = \triangle^l \, \{G_{\alpha-2k} * R_\beta\} = \left\{ G_{\alpha-2k} * \triangle^l \, R_\beta \right\} \\ &= \{G_{\alpha-2k} * R_{\beta-2l}\} = \{T_{\alpha-2k,\beta-2l}\} = T_{\alpha-2k,\beta-2l}. \end{aligned} \tag{67}$$

Now putting $\alpha = \beta = 0$ in (64) and using (63) we obtain $T_{-2k,-2l} = N_l^k \{T_{0,0}\} = N_l^k \{\delta(x)\}$ and putting $\alpha = 2k, \beta = 2l$ in(64) and using(63) we obtain $N_l^k \{T_{2k,2l}\} = T_{0,0} = \delta(x)$. ☐

**Lemma 10.** *Let $T_{\alpha,\beta}$ be the distributional functions familie defined by (29) then the following formula is valid*

$$T_{\alpha,\beta} * T_{\alpha',\beta'} = T_{\alpha+\alpha',\beta+\beta'} \tag{68}$$

*for all $\alpha, \beta, \alpha', \beta'$ and $\beta, \beta'$ such that $\beta, \beta', \beta + \beta' \neq n + 2r, r = 0, 1, 2, \ldots$.*

*Proof.* Taking into account that $G_{-2k}(P \pm i0, m, n)$ and $R_{-2l}(x)$ are in the class $O_c^{'}$ and considering that $G_\alpha \in S^{'}$ for all $\alpha \in C$, and $R_\beta \in S^{'}$ for all $\beta \in C$ such that $\beta \neq n + 2r, r = 0, 1, \ldots$ we conclude that

$$T_{-2k,\beta} \in O_c^{'}, \tag{69}$$

if $\beta \neq n + 2r, r = 0, 1, \ldots$ and

$$T_{\alpha,-2l} \in O_c^{'} \tag{70}$$

for all $\alpha \in C$.

In consequence from (69) and (70) we have

$$T_{-2k,\beta} * T_{\alpha,-2l} \in O_c^{'} \tag{71}$$

if $\beta \neq n + 2r, r = 0, 1, \ldots$

Now taking into account that $G_\alpha(P \pm i0, m, n)$ (see [6]) is a entire distributional functional of $\alpha$ and $R_\beta(x)$ (see [11]) is an entire distributional function of $\beta$ such that $\beta \neq n + 2r, r = 0, 1, \ldots$, then by appealing to the principle of analytical continuation we conclude that

$$T_{\alpha,\beta} * T_{\alpha^{'},\beta^{'}} \in O_c^{'} \tag{72}$$

for $\beta, \beta^{'}$ and $\beta + \beta^{'} \neq n + 2r, r = 0, 1, \ldots$.

From (29) and using the properties (13) and (58) we have

$$\begin{aligned} T_{\alpha,\beta} * T_{\alpha^{'},\beta^{'}} &= (G_\alpha * R_\beta) * \left(G_{\alpha^{'}} * R_{\beta^{'}}\right) = (G_\alpha * G_{\alpha^{'}}) * \left(R_\beta * R_{\beta^{'}}\right) \\ &= G_{\alpha+\alpha^{'}} * R_{\beta+\beta^{'}} = T_{\alpha+\alpha^{'},\beta+\beta^{'}}, \end{aligned} \tag{73}$$

if $\beta, \beta^{'}, \beta + \beta^{'} \neq n + 2r, r = 0, 1, 2, \ldots$ □

In particular, putting $\alpha = -2k, \beta = -2l, \alpha^{'} = -2k^{'}, \beta^{'} = -2l^{'}$ in (68), where $k, l, k^{'}, l^{'}$ are nonnegative integers and using the properties (65) and the definition (62), we have the following formula

$$K^k \left\{\triangle^l \delta\right\} * K^{k^{'}} \left\{\triangle^{l^{'}} \delta\right\} = K^{k+k^{'}} \left\{\triangle^{l+l^{'}} \delta\right\}. \tag{74}$$

Using (17) and (26) the formula (74) can be rewrite in the following form

$$\left(L - m^2\right)^k \left\{\triangle^l \delta\right\} * \left(L - m^2\right)^{k^{'}} \left\{\triangle^{l^{'}} \delta\right\} = \left(L - m^2\right)^{k+k^{'}} \left\{\triangle^{l+l^{'}} \delta\right\}. \tag{75}$$

Putting $k = k^{`} = 0$ in (74) we have

$$\triangle^{l}\{\delta\} * \triangle^{l^{`}}\{\delta\} = \triangle^{l+l^{`}}\{\delta\}\ , \tag{76}$$

where $\triangle^k$ is the Laplacian operator iterated $k$ times defined by (59). The formula (76) appear in [3], formula (26).

On the other hand, putting $m^2 = 0$ in (76) we have the following formula,

$$L^{k}\left\{\triangle^{l}\ \delta\right\} * L^{k^{`}}\left\{\triangle^{l^{`}}\ \delta\right\} = L^{k+k^{`}}\left\{\triangle^{l+l^{`}}\ \delta\right\}. \tag{77}$$

Putting $k = l$ and $k^{`} = l^{`}$ in (77) we have

$$L^{k}\left\{\triangle^{k}\ \delta\right\} * L^{k^{`}}\left\{\triangle^{k^{`}}\ \delta\right\} = L^{k+k^{`}}\left\{\triangle^{k+k^{`}}\ \delta\right\}. \tag{78}$$

Now taking into account the Diamond operator defined by

$$\Diamond^{k} = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\right)^2\right)^k = \Box^{k}\ \triangle^{k} = \triangle^{k}\ \Box^{k} \tag{79}$$

which appear in [7], where $p + q = n$ dimensions of the space, $\Box^k = L^k$ and $L^k$ is defined by (26), the formula (78) can be rewriten in the following form

$$\Diamond^{k}\{\delta\} * \Diamond^{k^{`}}\{\delta\} = \Diamond^{k+k^{`}}\{\delta\}\,. \tag{80}$$

Therefore our formulae (42) and (68) generalized convolution product connected with ultrahyperbolic Gordon Operator (see (49)), ultrahyperbolic operator (see (50)), Laplacian operator (see (76)) and Diamond operator (see (79)).

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