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ON SOME NONEXPANSIVE TYPE MULTI-VALUED MAPPINGS

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Abstract: In this paper a class of nonexpansive type multi-valued mappings from a metric space X into a class of nonempty closed and bounded subsets of X, which satisfy the condition (3) or (16) below are introduced and investigated. For such class of mappings coincidence and fixed point theorems are obtained. These results generalize and extend corresponding results for single-valued mappings of Bogin, Ćirić and Rhoades.

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1. Introduction

Let (X, d) be a metric space and let T be a selfmapping on X. If T is such that for all x, y, in X

$$d(Tx, Ty) \leq hd(x, y), \tag{1.1}$$

where $0 < h < 1$, then T is said to be a *contraction mappinng.* If T satisfies (1.1) with $h = 1$, then T is called a *nonexpansive mapping.* If T satisfies any conditions of type

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$$\begin{aligned} d(Tx,Ty) &\leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) \\ &+ a_4 d(x,Ty) + a_5 d(y,Tx), \end{aligned} \tag{1.2}$$

where a_i $(i = 1,2,3,4,5)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T is said to be a *contractive type mapping.* If T satisfies (1.2) with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, then T is said to be a *nonexpansive type mapping.* Similar terminology is used for multi-valued mappings.

Fixed point theorems for contractive, nonexpansive, contractive type and nonexpansive type mappings provide techniques for solving a variety of applied problems in mathematical and engineering sciences. It is one of the reason that many authors have studied various classes of contractive type or nonexpansive type mappings. In many cases domen and codomen of such mappings are metric spaces, not necessary with convex structure. For Banach spaces the famous is Greguš's Fixed Point Theorem (see [10]) for non expansive type single valued mappings, which satisfy (1.2) with $a_4 = a_5 = 0$, $a_1 < 1$. Many extensions and generalizations of that result have appeared (cf. [4-8]).

Extensions of the Banach contraction mapping principle to multi-valued mapping were initiated independently by Markin [13] and Nadler [14]. Further results on fixed points of contractive type multi-valued mappings were given by Ćirić [2], Dube and Singh [9], Kubiaczyk [11], Kubiak [12], Ray [15], Rhoades [17] and others. Results on fixed points of nonexpansive type multi-valued mappings are not numeorus.

Purpose of this paper is to consider a class of nonexpansive type multi-valued mappings which satisfy nonexpansive type conditions (2.1) or (2.14) below. Our coincidence theorem and a fixed point theorem generalizes and extends many fixed point theorems for contractive type multi-valued mappings (see [2], [9], [11-15]) and nonexpansive type single-valued mappings (see [1], [3], [16]) on metric spaces, not necessary with convex structure.

2. Main Results

Let (X,d) be a metric space. Throughout the paper let H denote the Hausdorff metric on $CB(X)$ induced by the metric d, where $CB(X)$ is the collection of all non-empty closed and bounded subset of X. For these definitions, one may refer [2], [11-14] and [17]. For any $x \in X$ and $A \subset X$ we write

$$D(x,A) = \inf\{d(x,a) : a \in A\}.$$

We shall recall certain definition from Ćirić [2] and Rhoades et al [17]. In all that follows, let T be a multi-valued mapping from a metric space X to the collection $CB(X)$ of non-empty subset of X and A a single-valued self-map on X.

Definition 1. (see [2]) An orbit of the multi-valued map T at a point x_0 in X is a sequence $\{x_n : x_n \in Tx_{n-1}\}$. A space X is T-orbitally complete if every Cauchy sequence of the form $\{x_{n(i)} : x_{n(i)} \in Tx_{n(i)-1}\}$ converges in X.

Definition 2. (see [17]) If for a point x_0 in X, there exists a sequence $\{x_n\} \subset X$ such that $Ax_{n+1} \in Tx_n$ $(n = 0, 1, 2, \cdots)$, then $O_A(x_0) = \{Ax_n : n = 1, 2, \cdots\}$ is an orbit for (T, A) at x_0. A space X is called (T,A)-orbitally complete if every Cauchy sequence of the form $\{Ax_{n(i)} : Ax_{n(i)} \in Tx_{n(i)-1}\}$ converges in X.

The purpose of this paper is to investigated multi-valued mappings (not necessary upper or lower semicontinuous) which satisfy the following nonexpansive type condition:

$$\begin{aligned} H(Tx, Ty) \leq a \max\{&d(Ax, Ay), D(Ax, Tx), D(Ay, Ty), \\ &\frac{1}{2}[D(Ax, Ty) + D(Ay, Tx)]\} \\ &+ b \max\{D(Ax, Tx), D(Ay, Ty)\} + c[D(Ax, Ty) + D(Ay, Tx)] , \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$ are non-negative real functions from $X \times X$ into $[0, +\infty)$ satisfying

$$a(x, y) + b(x, y) + 2c(x, y) = 1. \tag{2.2}$$

Mappings which satisfy (2.1) with a, b, c satisfying (2.2) and $A = I$, the identity mapping, we call generalized nonexpansive type multi-valued mappings. It is clear that every generalized contractive multi-valued map (compare [2]) is also generalized nonexpansive, but converse is not true.

Now we shall prove the following main result.

Theorem 1. *Let (X, d) be a metric space, T a multi-valued mapping from X to $CB(X)$ and let there exist a selfmapping $A : X \to X$ such that $T(X) \subseteq A(X)$ and $A(X)$ is (T, A)-orbitally complete. If T and A satisfy the condition (2.1) with a, b, c satisfying (2.2) and if in addition*

$$\inf \Big\{ b(x, y) \cdot c(u, v) : x, y, u, v \in X \Big\} > 0, \tag{2.3}$$

then A and T have a coincidence, that is, there exists a point z in X such that $Az \in Tz$.

Proof. Set

$$s=\inf\{b(x,y)\cdot c(u,v): x,y,u,v\in X\}>0. \tag{2.4}$$

Let $x_0\in X$. We shall construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows. Put $y_0=Ax_0$. Since $T(X)\subseteq A(X)$, we can choose x_1 in X such that $Ax_1\in Tx_0$. Put $y_1=Ax_1$. We may assume that $y_1\notin Tx_1$, otherwise x_1 is a coincidence point for A and T. Choose now $y_2\in Tx_1$ such that

$$d(y_1,y_2)\le kD(y_1,Tx_1),$$

where

$$k=1+\frac{s}{2} \tag{2.5}$$

and s is defined by (2.4). Such choice is possible since $k>1$ by (2.3), (2.4) and (2.5). Since $y_2\in Tx_1$, there exists x_2 in X such that $y_2=Ax_2$. In general, if y_n is already chosen, choose y_{n+1} such that

$$y_{n+1}=Ax_{n+1}\in Tx_n$$

and

$$d(y_n,y_{n+1})\le kD(y_n,Tx_n), \tag{2.6}$$

where k is given by (2.5). From (2.1)

$$\begin{aligned}H(Tx_0,Tx_1)\le a\max\Big\{&d(Ax_0,Ax_1),D(Ax_0,Tx_0),D(Ax_1,Tx_1),\\&\frac{1}{2}\Big[D(Ax_0,Tx_1)+D(Ax_1,Tx_0)\Big]\Big\}\\+b\max\Big\{&D(Ax_0,Tx_0),D(Ax_1,Tx_1)\Big\}+c\Big[D(Ax_0,Tx_1)+D(Ax_1,Tx_0)\Big],\end{aligned}$$

where a, b, c are taken at (x_0, x_1). Since

$$D(Ax_0,Tx_0)\le d(y_0,y_1);\;\; D(Ax_1,Tx_1)\le H(Tx_0,Tx_1);\;\; D(Ax_1,Tx_0)=0\,,$$

and

$$D(Ax_0,Tx_1)\le d(y_0,y_1)+H(Tx_0,Tx_1),$$

we get

$$\begin{aligned}H(Tx_0,Tx_1)\le a\max\{&d(y_0,y_1),H(Tx_0,Tx_1),\\&\frac{1}{2}[d(y_0,y_1)+H(Tx_0,Tx_1)]\}\end{aligned}$$

$$+ b\max\{d(y_0,y_1), H(Tx_0,Tx_1)\} + c[d(y_0,y_1) + H(Tx_0,Tx_1)].$$

If we assume that $H(Tx_0,Tx_1) > d(y_0,y_1)$, then we have (as $c = c(x_0,x_1) > 0$),

$$\begin{aligned} H(Tx_0,Tx_1) &\le aH(Tx_0,Tx_1) + bH(Tx_0,Tx_1) \\ &+ c\Big[d(y_0,y_1) + H(Tx_0,Tx_1)\Big] < (a+b+2c)H(Tx_0,Tx_1) = H(Tx_0,Tx_1), \end{aligned}$$

which is a contradiction. Therefore, $H(Tx_0,Tx_1) \le d(y_0,y_1)$. This inequality implies that

$$H(Tx_{n-1},Tx_n) \le d(y_{n-1},y_n), \quad n = 1,2,\ldots. \tag{2.7}$$

By (2.6) and (2.7) we have

$$d(y_1,y_2) \le kD(y_1,Tx_1) \le kH(Tx_0,Tx_1) \le kd(y_0,y_1).$$

Similarly

$$d(y_n,y_{n+1}) \le kd(y_{n-1},y_n), \quad n = 1,2,\ldots. \tag{2.8}$$

From (2.8) and the triangle inequality we get

$$d(y_0,y_2) \le d(y_0,y_1) + d(y_1,y_2) \le (1+k)d(y_0,y_1).$$

From (2.1) again (and writing y_i instead of Ax_i) we have

$$\begin{aligned} H(Tx_0,Tx_2) \le a\max\Big\{&d(y_0,y_2), D(y_0,Tx_0), D(y_2,Tx_2), \\ &\frac{1}{2}\Big[D(y_0,Tx_2) + D(y_2,Tx_0)\Big]\Big\} \\ + b\max\Big\{D(y_0,Tx_0), D(y_2,Tx_2)\Big\} &+ c\Big[D(y_0,Tx_2) + D(y_2,Tx_0)\Big], \end{aligned} \tag{2.9}$$

where a, b, c are taken at (x_0,x_2). Using (2.6), (2.7) and (2.8) and the triangle inequality, we get

$$\begin{aligned} d(y_0,y_2) &\le d(y_0,y_1) + d(y_1,y_2) \le (1+k)d(y_0,y_1), \\ D(y_2,Tx_2) &\le H(Tx_1,Tx_2) \le d(y_1,y_2) \le kd(y_0,y_1), \\ D(y_0,Tx_2) &+ D(y_2,Tx_0) \le d(y_0,y_2) + D(y_2,Tx_2) + D(Tx_0,y_2) \\ &\le (1+k)d(y_0,y_1) + kd(y_0,y_1) + H(Tx_0.Tx_1) \\ &\le 2(1+k)d(y_0,y_1). \end{aligned}$$

Now from (2.9) we have

$$
\begin{aligned}
H(Tx_0,Tx_2) &\le a(1+k)d(y_0,y_1)+bkd(y_0,y_1)+2c(1+k)d(y_0,y_1) \\
&= \Big[(1+k)(a+b+2c)-b\Big]d(y_0,y_1)=(1+k-b)d(y_0,y_1). \quad (2.10)
\end{aligned}
$$

Therefore, using that $1<k$, we get

$$H(Tx_0,Tx_2)\le(2k-b)d(y_0,y_1) \tag{2.11}$$

where $b=b(x_0,x_2)$. Using (2.1) and (2.6) again we have

$$
\begin{aligned}
H(Tx_1,Tx_2) \quad \le \quad & a\max\Big\{d(y_1,y_2),D(y_1,Tx_1),D(y_2,Tx_2),\frac{1}{2}\Big[d(y_1,Tx_2)\Big]\Big\} \\
& +b\max\Big\{D(y_1,Tx_1),D(y_2,Tx_2)\Big\}+cD(y_1,Tx_2),
\end{aligned}
$$

where a, b, c are taken at (x_1,x_2). Since by (2.7), (2.10) and (2.11)

$$D(y_1,Tx_1)\le H(Tx_0,Tx_1)\le d(y_0,y_1),D(y_2,Tx_2)\le kd(y_0,y_1)$$

and

$$D(y_1,Tx_2)\le H(Tx_0,Tx_2)\le(2k-b)d(y_0,y_1)\,,$$

we have

$$
\begin{aligned}
H(Tx_1,Tx_2) \quad \le \quad & akd(y_0,y_1)+bkd(y_0,y_1)+c(2k-b)d(y_0,y_1) \\
= \quad & \Big[k(a+b+2c)-bc\Big]d(y_0,y_1)=(k-bc)d(y_0,y_1),
\end{aligned}
$$

where $bc=b(x_0,x_2)\cdot c(x_1,x_2)$. Thus, as by (2.4) $bc\ge s$, we have

$$H(Tx_1,Tx_2)\le(k-s)d(y_0,y_1). \tag{2.12}$$

From (2.6) and (2.12) we get

$$d(y_2,y_3)\le kD(y_2,Tx_2)\le kH(Tx_1,Tx_2)\le k(k-s)d(y_0,y_1).$$

Since by (2.5)

$$k(k-s)=\left(1+\frac{s}{2}\right)\left(1+\frac{s}{2}-s\right)=\left(1+\frac{s}{2}\right)\left(1-\frac{s}{2}\right)=1-\frac{s^2}{4},$$

we have

$$d(y_2,y_3)\le\left(1-\frac{s^2}{4}\right)d(y_0,y_1).$$

Analogically

$$d(y_3, y_4) \le \left(1 - \frac{s^2}{4}\right) d(y_1, y_2).$$

If we continue the same procedure, we get

$$d(y_n, y_{n+1}) \le \left(1 - \frac{s^2}{4}\right)^{[\frac{n}{2}]} \max\Big\{d(y_0, y_1), d(y_1, y_2)\Big\}, \tag{2.13}$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Since $s > 0$, from (2.13) we have that $\{y_n\}$ is a Cauchy sequence in $A(X)$. Since $A(X)$ is (T, A)-orbitally complete, it has limit in $A(X)$, say p. Hence, there exists a point z in X such that $Az = p$.

Now by the triangle inequality and (2.1) we have

$$\begin{aligned} D(Az, Tz) &\le d(Az, Ax_{n+1}) + D(Ax_{n+1}, Tz) \le d(Az, Ax_{n+1}) + H(Tx_n, Tz) \\ &\le d(Az, Ax_{n+1}) + a \max\Big\{d(Ax_n, Az), D(Ax_n, Tx_n), D(Az, Tz), \\ &\qquad \frac{1}{2}\Big[D(Ax_n, Tz) + D(Az, Tx_n)\Big]\Big\} \\ &\quad + b \max\Big\{D(Ax_n, Tx_n), D(Az, Tz)\Big\} + c\Big[D(Ax_n, Tz) + D(Az, Tx_n)\Big] \\ &\le d(Az, y_{n+1}) + a \max\Big\{d(y_n, Az), d(y_n, y_{n+1}), D(Az, Tz), \\ &\qquad \frac{1}{2}\Big[D(y_n, Tz) + d(Az, y_{n+1})\Big]\Big\} \\ &\quad + b \max\Big\{d(y_n, y_{n+1}), D(Az, Tz),\Big\} + c\Big[D(y_n, Tz) + d(Az, y_{n+1})\Big], \end{aligned}$$

where a, b, c are taken at (x_n, z). If we suppose that $D(Az, Tz) > 0$, then we can choose n sufficience large such that we have

$$\begin{aligned} D(Az, Tz) &\le d(Az, y_{n+1}) + aD(Az, Tz) + bD(Az, Tz) \\ &\quad + c[D(Az, Tz) + d(Az, y_n) + d(Az, y_{n+1})] \\ &\quad = d(p, y_{n+1}) + (1 - c)(Az, Tz) + c[d(p, y_n) + d(p, y_{n+1})]. \end{aligned}$$

Taking the $\limsup$ when n tends to infinity we have

$$D(Az, Tz) \le D(Az, Tz) - \limsup_{n\to\infty} c(x_n, z) D(Az, Tz) < D(Az, Tz),$$

which is a contradiction. Therefore $Az \in Tz$, which completes the proof of the theorem. □

For $A = I$, the identify mapping, we obtain the following fixed point theorem.

Theorem 2. *Let (X, d) be a metric space and T a multi-valued map from X to $CB(X)$. If X is T-orbitally complete and for all $x, y \in X$*

$$\begin{aligned} H(Tx, Ty) \leq a \max \Big\{ d(x, y), D(x, Tx), D(y, Ty), \\ \frac{1}{2}\Big[D(x, Ty) + D(y, Tx)\Big]\Big\} \\ + b \max \Big\{D(x, Tx), D(y, Ty)\Big\} + c\Big[D(x, Ty) + D(y, Tx)\Big], \end{aligned} \tag{2.14}$$

where a, b, c are non negative real functions from $X \times X$ into R satisfying (2.2) and (2.3), then T has a fixed point.

Remark. Theorem 2 generalizes corresponding fixed point theorems for contractive type multi-valued mappings of Ćirić [2], Dube and Singh [9], Kubiaczyk [11], Kubiak [12], Ray [15] and several other authors. Also Theorem 2 generalizes and extends corresponding fixed point theorems of Bogin [1], Ćirić [3], and Rhoades [16] for nonexpansive type single-valued mappings.

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