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ON THE DISTRIBUTION OF GENERALIZED THRESHOLD ARCH STOCHASTIC PROCESSES

E. Gonçalves[1], N. Mendes-Lopes[2 §]
[1,2]Department of Mathematics
Faculty of Science and Technology
University of Coimbra
Ap. 3008, Coimbra, 3001-454, PORTUGAL
[1]e-mail: esmerald@mat.uc.pt
[2]e-mail: nazare@mat.uc.pt

Abstract: We study the marginal distribution function of a GTARCH process $\varepsilon = (\varepsilon_t, t \in Z)$ for which we obtain bounds based on the distribution function of the independent white noise, Z, associated to process ε. We point out that even if the marginal law of ε presents effectivelly different characteristics from the marginal law of Z it is, in some regions, strongly controlled by the law of the white noise associated. Those regions are evaluated for noise distributions particularly useful in applications, and with different properties, namely in what concerns the behaviour of the corresponding tails. The laws of finite dimension of the absolute value of the process ε are also evaluated in terms of the Z law; the bounds obtained for these laws are related to the run length of control charts.

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1. Introduction

Let $\varepsilon = (\varepsilon_t,\ t \in \mathbb{Z})$ be a real stochastic process. For any $t \in \mathbb{Z}$, we define $\varepsilon_t^+ = \varepsilon_t \mathbb{I}_{\{\varepsilon_t > 0\}}$, $\varepsilon_t^- = \varepsilon_t \mathbb{I}_{\{\varepsilon_t < 0\}}$ and $\underline{\varepsilon}_t$ the $\sigma-$ field generated by $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...$

The process $\varepsilon = (\varepsilon_t,\ t \in \mathbb{Z})$ follows a generalized threshold auto-regressive conditionally heteroscedastic model with orders p and q, GTARCH(p, q), if for

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§Correspondence author

every $t \in \mathbb{Z}$,

$$\begin{cases} \varepsilon_t = \sigma_t Z_t \,, \\ \sigma_t = \alpha_0 + \sum\limits_{i=1}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum\limits_{i=1}^{q} \beta_i \varepsilon_{t-i}^{-} + \sum\limits_{j=1}^{p} \gamma_j \sigma_{t-j} \,, \end{cases} \tag{1}$$

where $\alpha_0 > 0,\ \alpha_i \geq 0,\ \beta_i \geq 0\ (i = 1, ..., q)\,, \gamma_j \geq 0\ (j = 1, ..., p)$ and with $Z = (Z_t, t \in \mathbb{Z})$, called generating process, satisfying the following hypothesis:

Hypothesis 1. *The stochastic process $Z = (Z_t,\ t \in \mathbb{Z})$ is a sequence of independent and identically distributed real random variables with zero mean and unit variance; moreover, Z_t is independent of $\underline{\varepsilon}_{t-1}$, for each $t \in \mathbb{Z}$.*

This class of nonlinear stochastic models (Zakoian [5], Rabemananjara and Zakoian [4]) has, over the GARCH models, the advantage of taking into account the asymmetries, so often found in some real series; in fact, in this formulation the conditional standard deviation of the process at time t is a linear piecewise function of past observations depending, in a way not necessarily symmetrical, of its positive and negative values.

Several probabilistic results have been established for this class of models and, in particular, if $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ follows a GTARCH model with orders $p = q = 1$, namely with $\sigma_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^{+} - \beta_1 \varepsilon_{t-1}^{-} + \gamma_1 \sigma_{t-1}$, a necessary and sufficient condition of stationarity (strict and wide) of ε is (Gonçalves and Mendes-Lopes, [2]) $E\left[\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)^2\right] < 1$. Under this condition, the variance of ε, $V(\varepsilon_t) = \sigma_\varepsilon^2$, exists and we have

$$\sigma_\varepsilon^2 = \frac{\alpha_0^2 \left[1 + E\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)\right]}{\left[1 - E\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)\right]\left\{1 - E\left[\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)^2\right]\right\}}.$$

In the next section we study the marginal distribution function of the process $\varepsilon, F_{\varepsilon_t}, t \in \mathbb{Z}$. We obtain bounds for this function using the distribution function of the independent white noise, Z, associated to ε. As we will see, the lower and upper bounds obtained for the value of this function are valuable in subsets of $\mathbb{R}$ only dependent on the law of the generating noise and on the model coefficients. More precisely, we obtain subsets of these regions only dependent of Z distribution. In Section 3, lower and upper bounds for the joint probability $P(|\varepsilon_t| \leq x_t, t = 1, ..., n)$ are obtained. An application of this result, namely useful in statistical process control, is derived in the last paragraph.

2. Bounds for the Distribution Function of GTARCH Processes

2.1. Theoretical Results

The following result, with a relatively simple proof, gives an upper (resp., lower) bound of the process ε marginal distribution function restricted to $\mathbb{R}^+$ (resp., $\mathbb{R}^-$).

Theorem 1. *Let $\varepsilon = (\varepsilon_t,\ t \in \mathbb{Z})$ be the GTARCH(p,q) process defined in (1). Under hypothesis 1 we have, for every $t \in \mathbb{Z}$,*

$$F_{\varepsilon_t}(x) = E\left[F_{Z_t}\left(\frac{x}{\sigma_t}\right)\right].$$

In particular,

a) if $x > 0$: $F_{\varepsilon_t}(x) \leq F_{Z_t}\left(\frac{x}{\alpha_0}\right)$;

b) if $x < 0$: $F_{\varepsilon_t}(x) \geq F_{Z_t}\left(\frac{x}{\alpha_0}\right)$.

Proof. We have, for x arbitrarily fixed in $\mathbb{R}$,

$$F_{\varepsilon_t}(x) = P(\varepsilon_t \leq x) = E\left[P\left(\sigma_t Z_t \leq x | \underline{\varepsilon}_{t-1}\right)\right] = E\left[F_{Z_t}\left(\frac{x}{\sigma_t}\right)\right],$$

as Z_t is independent of $\underline{\varepsilon}_{t-1}$. As $\sigma_t \geq \alpha_0$, we deduce that

$$\forall x > 0,\ F_{\varepsilon_t}(x) = E\left[F_{Z_t}\left(\frac{x}{\sigma_t}\right)\right] \leq F_{Z_t}\left(\frac{x}{\alpha_0}\right)$$

and

$$\forall x < 0,\ F_{\varepsilon_t}(x) = E\left[F_{Z_t}\left(\frac{x}{\sigma_t}\right)\right] \geq F_{Z_t}\left(\frac{x}{\alpha_0}\right).$$ □

These inequalities are strict ones whenever the process is really conditionally heteroscedastic (i.e., σ_t is not reduced to the constant α_0) and the marginal distribution function of Z is strictly increasing. Moreover, we point out that in this theorem the stationarity of ε is not needed. Nevertheless, this condition is essential to obtain the announced bounds for the marginal distribution function of ε.

In what follows we consider

$$c = (\alpha_1 + \beta_1)\gamma_1 + \sum_{i=1}^{q}\left(\alpha_i^2 + \beta_i^2\right) + \sum_{j=1}^{p}\gamma_j^2$$

and

$$k = \begin{cases} 1, & q = 1,\ p = 0\,, \\ p+q-1, & \text{other values of } p \text{ and } q. \end{cases}$$

Theorem 2. *Let $\varepsilon = (\varepsilon_t,\, t \in \mathbb{Z})$ be the* GTARCH*(p,q) process defined by (1) with Z_t verifying hypothesis 1. If ε is stationary and Z_t is absolutely continuous with a differentiable density of probability f_{Z_t}, we have for every $t \in \mathbb{Z}$*

a) $F_{\varepsilon_t}(x) \geq F_{Z_t}\left(\dfrac{x}{\alpha_0 + \sigma_\varepsilon \sqrt{ck}}\right)$, *if* $x > 0$ *and* $h(x) \geq 0$;

b) $F_{\varepsilon_t}(x) \leq F_{Z_t}\left(\dfrac{x}{\alpha_0 + \sigma_\varepsilon \sqrt{ck}}\right)$, *if* $x < 0$ *and* $h(x) \geq 0$, *where*

$$h(x) = 2 f_{Z_t}\left(\frac{x}{m}\right) + \frac{x}{m} f'_{Z_t}\left(\frac{x}{m}\right)\,,$$

with $m = \alpha_0 + y$, $y \geq 0$, and f'_{Z_t} the derivative of f_{Z_t}.

Proof. We have, for x arbitrarily fixed in $\mathbb{R}$,

$$F_{\varepsilon_t}(x) = E\left[F_{Z_t}\left(\frac{x}{\alpha_0 + \sum\limits_{i=1}^{q} \alpha_i \varepsilon^+_{t-i} - \sum\limits_{i=1}^{q} \beta_i \varepsilon^-_{t-i} + \sum\limits_{j=1}^{p} \gamma_j \sigma_{t-j}}\right)\right].$$

Let us consider the function $R : [0, +\infty[\to [0,1]$ defined by

$$R(y) = F_{Z_t}\left(\frac{x}{\alpha_0 + y}\right).$$

As

$$\frac{\partial}{\partial y} R(y) = f_{Z_t}\left(\frac{x}{m}\right)\left(-\frac{x}{m^2}\right),$$

with $m = \alpha_0 + y$, we obtain

$$\frac{\partial^2}{\partial y^2} R(y) = \frac{2x}{m^3} f_{Z_t}\left(\frac{x}{m}\right) + \frac{x^2}{m^4} f'_{Z_t}\left(\frac{x}{m}\right) = \frac{x}{m^3} h(x).$$

Let us suppose that $x > 0$. If $h(x) \geq 0$, R is convex. As ε is a second order process, we can apply Jensen inequality and we obtain

$$F_{\varepsilon_t}(x) \;=\; E\left[R\left(\sum_{i=1}^{q} \alpha_i \varepsilon^+_{t-i} - \sum_{i=1}^{q} \beta_i \varepsilon^-_{t-i} + \sum_{j=1}^{p} \gamma_j \sigma_{t-j}\right)\right]$$

$$\geq \quad R\left[E\left(\sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^{+}-\sum_{i=1}^{q}\beta_i\varepsilon_{t-i}^{-}+\sum_{j=1}^{p}\gamma_j\sigma_{t-j}\right)\right]$$

$$= \quad F_{Z_t}\left(\frac{x}{\alpha_0+E\left(\sum\limits_{i=1}^{q}\alpha_i\varepsilon_{t-i}^{+}-\sum\limits_{i=1}^{q}\beta_i\varepsilon_{t-i}^{-}+\sum\limits_{j=1}^{p}\gamma_j\sigma_{t-j}\right)}\right).$$

From Appendix 2 we have, in particular,

$$E\left(\sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^{+}-\sum_{i=1}^{q}\beta_i\varepsilon_{t-i}^{-}+\sum_{j=1}^{p}\gamma_j\sigma_{t-j}\right)\leq\sigma_\varepsilon\sqrt{ck}\,,$$

where

$$c=(\alpha_1+\beta_1)\,\gamma_1+\sum_{i=1}^{q}\left(\alpha_i^2+\beta_i^2\right)+\sum_{i=j}^{p}\gamma_j^2$$

and $k=\begin{cases}1, & q=1,\ p=0\,,\\ p+q-1, & \text{other values.}\end{cases}$

We conclude that if $x>0$ and $h(x)\geq 0$ then

$$F_{\varepsilon_t}(x)\geq F_{Z_t}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}\right).$$

When $x<0$ and $h(x)\geq 0$ we obtain, in an analogous way,

$$F_{\varepsilon_t}(x)\leq F_{Z_t}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}\right). \qquad \square$$

We note that in the cases $q=1$ and $p=0$ or $p=1$ we have $E\left(\alpha_1 Z_{t-1}^{+}-\beta_1 Z_{t-1}^{-}+\gamma_1\right)^2<1$ (taking $\gamma_1=0$ if $p=0$), by the necessary and sufficient condition of stationarity of ε (Gonçalves, Mendes-Lopes [2]). So, in these cases we can take simply $c=1$ in these expressions.

Let us also point out that upper bounds for $E\left(\sigma_t-\alpha_0\right)$ different from those now deduced can be used. For example, we also have, for $(q,p)\in\mathbb{N}\times\mathbb{N}_0$,

$$E\left(\sigma_t-\alpha_0\right)=E\left(\sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^{+}-\sum_{i=1}^{q}\beta_i\varepsilon_{t-i}^{-}+\sum_{j=1}^{p}\gamma_j\sigma_{t-j}\right)$$

$$\leq \sigma_\varepsilon \left[\sum_{i=1}^{q} (\alpha_i + \beta_i) + \sum_{j=1}^{p} \gamma_j \right].$$

In fact, as $E\left(\varepsilon_t^+\right) = E\left(-\varepsilon_t^-\right)$ and

$$E\left(\varepsilon_0^+\right) \leq \left[E\left(\left(\varepsilon_0^+\right)^2\right)\right]^{\frac{1}{2}} = \left[E\left(\left(\sigma_0 Z_0^+\right)^2\right)\right]^{\frac{1}{2}}$$
$$= \left[E\left[E\left((\sigma_0)^2 \left(Z_0^+\right)^2 |\underline{\varepsilon}_{-1}\right)\right]\right]^{\frac{1}{2}} = \left[E\left[(\sigma_0)^2 E\left(\left(Z_0^+\right)^2 |\underline{\varepsilon}_{-1}\right)\right]\right]^{\frac{1}{2}}$$
$$= \left[E\left[(\sigma_0)^2 E\left(\left(Z_0^+\right)^2\right)\right]\right]^{\frac{1}{2}} \leq \left[E\left[(\sigma_0)^2\right]\right]^{\frac{1}{2}},$$

because $E\left(\left(Z_0^+\right)^2\right) \leq 1$, we obtain

$$E\left(\varepsilon_0^+\right) \leq \left[E\left[(\sigma_0)^2\right]\right]^{\frac{1}{2}} = \left[V\left[\varepsilon_0\right]\right]^{\frac{1}{2}} = \sigma_\varepsilon.$$

Moreover, as $\sigma_\varepsilon^2 = E\left(\varepsilon_t^2\right) = E\left(\sigma_t^2 Z_t^2\right) = E\left(\sigma_t^2\right)$, we also deduce that $E\left(\sigma_t\right) \leq \sigma_\varepsilon$.

So, another choice for $\sqrt{ck}$ is $\sum_{i=1}^{q} (\alpha_i + \beta_i) + \sum_{j=1}^{p} \gamma_j$. Nevertheless, we note that, under the stationarity condition, the lower bound chosen in the proof is, in general, more accurate.

As an obvious consequence of the previous theorems, we can state the following corollary.

Corollary. *Under the hypotheses of Theorem 2, we have the following bounds for the marginal distribution function of the* ε *process*:

a) $F_{Z_t}\left(\dfrac{x}{\alpha_0 + \sigma_\varepsilon \sqrt{ck}}\right) \leq F_{\varepsilon_t}(x) \leq F_{Z_t}\left(\dfrac{x}{\alpha_0}\right)$, *if* $x > 0$ *and* $h(x) \geq 0$;

b) $F_{Z_t}\left(\dfrac{x}{\alpha_0}\right) \leq F_{\varepsilon_t}(x) \leq F_{Z_t}\left(\dfrac{x}{\alpha_0 + \sigma_\varepsilon \sqrt{ck}}\right)$, *if* $x < 0$ *and* $h(x) \geq 0$.

These results show that the law of ε is, in certain regions, strongly controlled by the law of the white noise associated. This fact is very relevant as we know that these laws have in general quite different characteristics (for example, the marginal law of ε is leptocurtic even if it does not happen with the Z marginal law).

2.2. Applications

The results presented in the previous section are valuable for a large class of probability laws of the process Z. Moreover, the upper and lower bounds obtained for the point value of the marginal distribution function are valuables in certain subsets of $\mathbb{R}$ only depending on the unrestricted parameter α_0 and on the Z_t law. So, we evaluate such sets considering several distributions particularly useful in the applications, and with different characteristics, namely in what concerns the behaviour of the corresponding tails.

Example 1. Let us consider Z_t distributed according to a standard Gaussian law. We have

$$h(x) = 2f_{Z_t}\left(\frac{x}{m}\right) + \frac{x}{m}f'_{Z_t}\left(\frac{x}{m}\right) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x}{m}\right)^2\right]\left[2-\left(\frac{x}{m}\right)^2\right].$$

Then $h(x) \geq 0$ if and only if $-m\sqrt{2} \leq x \leq m\sqrt{2}$. As $m \geq \alpha_0$ we obtain

$$x \in \left]-\alpha_0\sqrt{2}, \alpha_0\sqrt{2}\right] \Rightarrow h(x) \geq 0.$$

Example 2. If X is a real random variable following the student law with n degrees of freedom, we know that $E(X) = 0$, $n > 1$ and $V(X) = \frac{n}{n-2}$, $n > 2$. In order to have Z_t following a centered and unit variance law based on the student one, we have to make a normalization. So, we suppose that Z_t is absolutely continuous with density

$$f_{Z_t}(y) = \frac{1}{\sqrt{(n-2)\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{y^2}{n-2}\right)^{-\frac{n+1}{2}}, \quad y \in \mathbb{R},$$

where $n > 2$. We have

$$h(y) = 2f_{Z_t}\left(\frac{y}{m}\right) + \frac{y}{m}f'_{Z_t}\left(\frac{y}{m}\right) = f_{Z_t}\left(\frac{y}{m}\right)\left[\frac{-(n-1)y^2+2(n-2)m^2}{y^2+(n-2)m^2}\right].$$

Then

$$h(y) \geq 0 \Leftrightarrow -m\sqrt{\frac{2(n-2)}{n-1}} \leq y \leq m\sqrt{\frac{2(n-2)}{n-1}}.$$

As $m \geq \alpha_0$ we obtain the following region only dependent on α_0 and on the Z_t distribution:

$$y \in \left]-\alpha_0\sqrt{\frac{2(n-2)}{n-1}}, \alpha_0\sqrt{\frac{2(n-2)}{n-1}}\right] \Rightarrow h(y) \geq 0.$$

Example 3. We consider now that Z_t follows a bidirectional Pareto law, that is, Z_t is absolutely continuous with density

$$f_{Z_t}(y) = \frac{\alpha}{2}\left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha}\frac{1}{|y|^{\alpha+1}}\mathbb{I}_{\left(\left]-\infty,-\sqrt{\frac{\alpha-2}{\alpha}}\right[\cup\left]\sqrt{\frac{\alpha-2}{\alpha}},+\infty\right[\right)}(y),$$

where $y \in \mathbb{R}$, $\alpha > 2$.

For $\frac{y}{m}$ less than $-\sqrt{\frac{\alpha-2}{\alpha}}$ or greater than $\sqrt{\frac{\alpha-2}{\alpha}}$, we obtain

$$h(y) = 2f_{Z_t}\left(\frac{y}{m}\right) + \frac{y}{m}f'_{Z_t}\left(\frac{y}{m}\right) = f_{Z_t}\left(\frac{y}{m}\right)(1-\alpha) < 0.$$

We deduce that

$$y \in \left[-\alpha_0\sqrt{\frac{\alpha-2}{\alpha}}, \alpha_0\sqrt{\frac{\alpha-2}{\alpha}}\right] \Rightarrow h(y) \geq 0.$$

Example 4. We consider now a real random variable with a double exponential law, i.e., absolutely continuous with density

$$f_X(x) = \frac{\lambda}{2}\exp\left[-\lambda\left|x-\alpha\right|\right], \quad x \in \mathbb{R} \quad (\lambda > 0, \alpha \in \mathbb{R}).$$

As $E(X) = \alpha$ and $V(X) = \frac{2}{\lambda^2}$, we take Z_t following a double exponential law with parameters $\alpha = 0$ and $\lambda = \sqrt{2}$. Then

$$h(y) = 2f_{Z_t}\left(\frac{y}{m}\right) + \frac{y}{m}f'_{Z_t}\left(\frac{y}{m}\right) = f_{Z_t}\left(\frac{y}{m}\right)\left(2 - \frac{\sqrt{2}\left|y\right|}{m}\right).$$

We deduce that

$$y \in \left]-\alpha_0\sqrt{2}, \alpha_0\sqrt{2}\right] \Rightarrow h(y) \geq 0.$$

Once the regions where the function h is non negative are obtained, for each one of these particular laws, it is easy to deduce the corresponding bounds for the marginal distribution of ε. In particular, if Z is distributed according to a bidirectional Pareto law and ε follows a GTARCH(p,q) process we have

$$\forall x \in \left]-\alpha_0\sqrt{\frac{\alpha-2}{\alpha}}, 0\right[, \quad F_{\varepsilon_t}(x) \in \left[F_{Z_t}\left(\frac{x}{\alpha_0}\right), F_{Z_t}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}\right)\right],$$

$$\forall x \in \left]0, \alpha_0\sqrt{\frac{\alpha-2}{\alpha}}\right[, \quad F_{\varepsilon_t}(x) \in \left[F_{Z_t}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}\right), F_{Z_t}\left(\frac{x}{\alpha_0}\right)\right],$$

where the distribution function of Z_t is given by

$$F_{Z_t}(y) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha}(-y)^{-\alpha}, & y < -\sqrt{\frac{\alpha-2}{\alpha}}\,, \\ \frac{1}{2}, & y \in \left[-\sqrt{\frac{\alpha-2}{\alpha}}, \sqrt{\frac{\alpha-2}{\alpha}}\right[\,, \\ 1 - \frac{1}{2}\left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha} y^{-\alpha}, & y \geq \sqrt{\frac{\alpha-2}{\alpha}}\,. \end{cases}$$

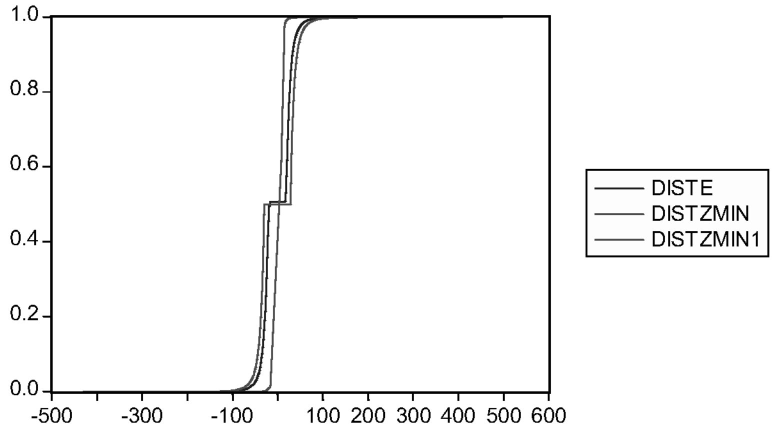

Figure 1: Plots of $F_{Z_t}\left(\frac{x}{\alpha_0}\right)$, the estimation of $F_{\varepsilon_t}(x)$ and $F_{Z_t}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}\right)$.

As $1-F_{Z_t}(y)=O\left(y^{-\alpha}\right), y \longrightarrow +\infty$, we deduce that the convergence rate for zero of $1-F_{\varepsilon_t}(y)$ is, at most, of order $y^{-\alpha}$.

In the following figure we illustrate the theoretical result established in Theorem 2 and its corollary using a simulation with a GTARCH(1,1) model with a noise Z_t following a bidirectional Pareto law, as in Example 3, with $\alpha = 2$. The parameters of the GTARCH model were chosen as $\alpha_0 = 10.0$, $\alpha_1 = \beta_1 = 0.5$ and $\gamma = 0.2$. The process ε is stationary with $\sigma_\varepsilon \simeq 31$. We estimated the distribution function of ε_t on x (represented by DISTE) using 10 000 realizations of the process. The distribution function of Z_t on $\frac{x}{\alpha_0}$ (DISTZMIN) and on $\frac{x}{\alpha_0+\sigma_\varepsilon\sqrt{ck}}$ (DISTZMIN1) is also plotted. In the interval $\left[-5\sqrt{2}, 5\sqrt{2}\right]$ we find the bounds for the distribution function of ε_t given in the results of the previous section. Similar conclusions were obtained with other simulated examples.

3. Bounds for $P\left(|\varepsilon_t| \leq x_t,\ t=1,...,n\right)$

In this section, we are interested in the evaluation of some joint probabilities for the process ε. In accordance to Pawlak and Schmid [3], in certain problems related to assessing the performance of control charts for GTARCH processes, it is important to evaluate the following probability

$$P\left(|\varepsilon_t| \leq x_t, t=1,...,n\right),$$

for some $n \geq 1$ and $x_t > 0,\ t = 1, ..., n$.

3.1. Theoretical Results

The following theorem gives an upper bound for $P\left(|\varepsilon_t| \leq x_t,\ t = 1, ..., n\right)$, for every $x_1, ..., x_n \in \left]0, +\infty\right[$.

Theorem 3. *Let $\varepsilon = (\varepsilon_t,\ t \in \mathbb{Z})$ be the* GTARCH*(p,q) process defined in (1). Then, for every $(x_1, ..., x_n) \in \left]0, +\infty\right[^n$,* we have

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}\left(x_1, ..., x_n\right) \leq \prod_{t=1}^{n}\left[F_{|Z_t|}\left(\frac{x_t}{\alpha_0}\right)\right].$$

Proof. As $\sigma_t \geq \alpha_0$, for every $t \in \mathbb{Z}$, we have the following relation

$$\left\{\sigma_1 |Z_1| \leq x_1, ..., \sigma_n |Z_n| \leq x_n\right\} \subseteq \left\{\alpha_0 |Z_1| \leq x_1, ..., \alpha_0 |Z_n| \leq x_n\right\}$$

from which we immediately deduce the result. □

In the following we deduce a lower bound for $P\left(|\varepsilon_t| \leq x_t,\ t = 1, ..., n\right)$, analogous to the bound established in Theorem 2, when ε is a GTARCH(p, q) process with $p = 0$ or $p = 1$ (higher values of p are treated in an analogous way, producing more complex expressions). In order to facilitate the notation, we denote the coefficient γ_1 by γ and consider $\alpha_i + \beta_i = s_i,\ i = 1, ..., q$.

Theorem 4. *Let $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ be the* GTARCH*(p,q) process defined in (1) with Z_t verifying hypothesis 1. We suppose ε stationary and Z_t absolutely continuous with a differentiable density of probability f_{Z_t}. Let us denote by $f^{'}_{|Z_t|}$ the derivative of $f_{|Z_t|}$. Let us suppose $q \leq n$. Then:*

a) if $p = 1$ and $q \geq 2$

$$\begin{aligned}
&F_{(|\varepsilon_1|,...,|\varepsilon_n|)}\left(x_1, ..., x_n\right) \\
&\geq F_{|Z_1|}\left(\frac{x_1}{\alpha_0 + \sigma_\varepsilon\left(\sum_{i=1}^{q} s_i + \gamma\right)}\right)\prod_{t=2}^{q} F_{|Z_t|}\left(\frac{x_t}{\alpha_0\sum_{j=0}^{t-1}\gamma^j + k_t + \sigma_\varepsilon\left[\sum_{j=0}^{t-2}\gamma^j\sum_{i=t-j}^{q} s_i + \gamma^{t-1}\left(\sum_{i=1}^{q} s_i + \gamma\right)\right]}\right) \\
&\prod_{t=q+1}^{n} F_{|Z_t|} \times \\
&\left(\frac{x_t}{\alpha_0\sum_{j=0}^{t-1}\gamma^j + \sum_{j=0}^{t-q-1}\gamma^j\sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q}k_q + \sigma_\varepsilon\left[\gamma^{t-q}\sum_{j=0}^{q-2}\gamma^j\sum_{i=q-j}^{q} s_i + \gamma^{t-1}\left(\sum_{i=1}^{q} s_i + \gamma\right)\right]}\right),
\end{aligned}$$

for every $(x_1, ..., x_n) \in \left]0, +\infty\right[^n$ such that

$$h_t\left(x_t\right) \geq 0,\ t = 1, ..., n,$$

where

$$
\begin{aligned}
k_t &= \sum_{j=0}^{t-2}\gamma^j \sum_{i=1}^{t-1-j} s_i x_{t-j-i}, \quad t\in\{2,...,q\}\ , \\
h_t(x) &= 2f_{|Z_t|}\left(\frac{x}{m_t}\right)+\frac{x}{m_t}f'_{|Z_t|}\left(\frac{x}{m_t}\right), \quad t\in\{1,...,n\}\ , \\
m_t &= \begin{cases} u, & t=1, \quad u\geq 0\,, \\ k_t+\sum\limits_{j=0}^{t-1}\gamma^j u_j, & t=2,...,q, \quad u_j\geq 0, \quad j=0,...,t-1\,, \\ c_t+\sum\limits_{j=0}^{q-1}\gamma^{t-q+j}v_j, & t=q+1,...,n, \quad v_j\geq 0, \quad j=0,...,q-1\,, \end{cases} \\
c_t &= \alpha_0\sum_{j=0}^{t-q-1}\gamma^j+\sum_{j=0}^{t-q-1}\gamma^j\sum_{i=1}^{q}s_i x_{t-j-i}+\gamma^{t-q}k_q, \quad t\in\{q+1,...,n\}\,.
\end{aligned}
$$

b) if $p=1$ *and* $q=1$*:*

$$
F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1,...,x_n)\geq F_{|Z_1|}\left(\frac{x_1}{\alpha_0+\sigma_\varepsilon(s_1+\gamma)}\right)
\times\prod_{t=2}^{n}F_{|Z_t|}\left(\frac{x_t}{\alpha_0\sum\limits_{j=0}^{t-1}\gamma^j+\sum\limits_{j=0}^{t-2}\gamma^j s_1 x_{t-j-1}+\sigma_\varepsilon\left[\gamma^{t-1}(s_1+\gamma)\right]}\right),
$$

for every $(x_1,...,x_n)\in\,]0,+\infty[^n$ *such that*

$$h_t(x_t)\geq 0,\ t=1,...,n,$$

where

$$h_t(x)=2f_{|Z_t|}\left(\frac{x}{m_t}\right)+\frac{x}{m_t}f'_{|Z_t|}\left(\frac{x}{m_t}\right), \quad t\in\{1,...,n\}\ ,$$

$$
\begin{aligned}
m_t &= \begin{cases} u, & t=1, \quad u\geq 0\,, \\ c_t+\gamma^{t-1}u, & t=2,...,n, \quad u\geq 0\,, \end{cases} \\
c_t &= \alpha_0\sum_{j=0}^{t-2}\gamma^j+\sum_{j=0}^{t-2}\gamma^j s_1 x_{t-j-1}, \quad t\in\{2,...,n\}\,.
\end{aligned}
$$

c) if $p=0$ *and* $q\geq 2$*:*

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1,...,x_n) \geq F_{|Z_1|}\left(\frac{x_1}{\alpha_0+\sigma_\varepsilon\sum\limits_{i=1}^{q}s_i}\right)$$

$$\times\prod_{t=2}^{q}F_{|Z_t|}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{t-1}s_ix_{t-i}+\sigma_\varepsilon\sum\limits_{i=t}^{q}s_i}\right)\prod_{t=q+1}^{n}F_{|Z_t|}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{q}s_ix_{t-i}}\right),$$

for every $(x_1,...,x_n)\in\,]0,+\infty[^n$ *such that*

$$h_t(x_t)\geq 0,\ \ t=1,...,q,$$

where

$$h_t(x) = 2f_{|Z_t|}\left(\frac{x}{m_t}\right)+\frac{x}{m_t}f'_{|Z_t|}\left(\frac{x}{m_t}\right),\quad t\in\{1,...,q\},$$

$$m_t = \begin{cases} u, & t=1,\quad u\geq 0, \\ \sum\limits_{i=1}^{t-1}s_ix_{t-i}+u, & t=2,...,q,\quad u\geq 0. \end{cases}$$

d) if $p=0$ *and* $q=1$*:*

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1,...,x_n)\geq F_{|Z_1|}\left(\frac{x_1}{\alpha_0+\sigma_\varepsilon s_1}\right)\prod_{t=2}^{n}F_{|Z_t|}\left(\frac{x_t}{\alpha_0+s_1x_{t-1}}\right)$$

for every $(x_1,...,x_n)\in\,]0,+\infty[^n$ *such that*

$$h_1(x_1)=2f_{|Z_1|}\left(\frac{x_1}{u}\right)+\frac{x_1}{u}f'_{|Z_1|}\left(\frac{x_1}{u}\right)\geq 0.$$

Proof. As the proof for the other cases is very similar, we prove the result in the case $p=1$ and $q\geq 2$.

From Appendix 1, we have the following upper bounds for σ_t, $t=1,...,n$, when $q\leq n$:

— for every $t\in\{2,...,q\}$

$$\sigma_t\leq k_t+\sum_{j=0}^{t-2}\gamma^j f_{t-j}\left(\underline{\varepsilon}_0\right)+\gamma^{t-1}\sigma_1,$$

with

$$k_t=\sum_{j=0}^{t-2}\gamma^j\sum_{i=1}^{t-1-j}s_ix_{t-j-i}, t\in\{2,...,q\},$$

$$f_t(\underline{\varepsilon}_0) = \begin{cases} \alpha_0 + \sum_{i=t}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=t}^{q} \beta_i \varepsilon_{t-i}^{-}, & t = 2, ..., q, \\ 0, & t \notin \{2, ..., q\}; \end{cases}$$

— for every $t \in \{q+1, ..., n\}$

$$\sigma_t \leq \alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1} \sigma_1.$$

We may now write, for every $(x_1, ..., x_n) \in]0, +\infty[^n$,

$$\begin{aligned} F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n) &= P(|\varepsilon_1| \leq x_1, ..., |\varepsilon_n| \leq x_n) \\ &= P\left(|Z_1| \leq \frac{x_1}{\sigma_1},\ |Z_t| \leq \frac{x_t}{\sigma_t},\ t = 2, ..., q,\ |Z_t| \leq \frac{x_t}{\sigma_t},\ t = q+1, ..., n\right) \\ &\geq P(|Z_1| \leq \frac{x_1}{\sigma_1}, |Z_t| \leq \frac{x_t}{k_t + \sum_{j=0}^{t-2} \gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}, \quad t = 2, ..., q, \end{aligned}$$

$$|Z_t| \leq \frac{x_t}{\alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}, \quad t = q+1, ..., n,$$

$$= E[P(|Z_1| \leq \frac{x_1}{\sigma_1}, \ |Z_t| \leq \frac{x_t}{k_t + \sum_{j=0}^{t-2} \gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}, \quad t = 2, ..., q,$$

$$|Z_t| \leq \frac{x_t}{\alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}, \quad t = q+1, ..., n | \underline{\varepsilon}_0],$$

according to the conditional expectation properties.

As Z_t is independent of $\underline{\varepsilon}_{t-1}$ and $\underline{\varepsilon}_0 \subseteq \underline{\varepsilon}_{t-1}$ $(t = 1, 2, ...)$ and as Z_1, Z_2, ..., Z_n are independent we have

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n)$$
$$\geq E[F_{|Z_1|}\left(\frac{x_1}{\sigma_1}\right) \prod_{t=2}^{q} F_{|Z_t|}\left(\frac{x_t}{k_t + \sum_{j=0}^{t-2}\gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}\right)$$
$$\prod_{t=q+1}^{n} F_{|Z_t|}\left(\frac{x_t}{\alpha_0 \sum_{j=0}^{t-q-1}\gamma^j + \sum_{j=0}^{t-q-1}\gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q}k_q + \gamma^{t-q}\sum_{j=0}^{q-2}\gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}\right)]$$
$$= E\left[F_{|Z_1|}\left(\frac{x_1}{\sigma_1}\right)\right] \prod_{t=2}^{q} E\left[F_{|Z_t|}\left(\frac{x_t}{k_t + \sum_{j=0}^{t-2}\gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}\right)\right] \prod_{t=q+1}^{n}$$
$$\times E\left[F_{|Z_t|}\left(\frac{x_t}{\alpha_0 \sum_{j=0}^{t-q-1}\gamma^j + \sum_{j=0}^{t-q-1}\gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q}k_q + \gamma^{t-q}\sum_{j=0}^{q-2}\gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1}\right)\right].$$

For future reference, we denote this lower bound of $F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n)$ by **lbound1**.

For t arbitrarily fixed in $\{2, ..., q\}$, let us introduce the function $R_t :]0, +\infty[^t \to [0, 1]$ defined by

$$R_t(u_0, ..., u_{t-1}) = F_{|Z_t|}\left(x_t \ / \ \left(c_t + \sum_{j=0}^{t-1} r_j u_j\right)\right)$$

where $c_t = k_t$ is determinist and positive and $r_j = \gamma^j, \quad j = 0, ..., t-1$.

As we have

$$\frac{\partial}{\partial u_i} R_t(u_0, ..., u_{t-1}) = f_{|Z_t|}\left(\frac{x_t}{m_t}\right)\left(-\frac{r_i x_t}{m_t^2}\right), i = 0, ..., t-1,$$

with $m_t = c_t + \sum_{j=0}^{t-1} r_j u_j$, we deduce that for $0 \leq i, j \leq t-1$

$$\begin{aligned} \frac{\partial^2}{\partial u_i \partial u_j} R_t(u_0, ..., u_{t-1}) &= \frac{r_i r_j x_t}{m_t^3}\left[2 f_{|Z_t|}\left(\frac{x_t}{m_t}\right) + \frac{x_t}{m_t} f'_{|Z_t|}\left(\frac{x_t}{m_t}\right)\right] \\ &= \frac{r_i r_j x_t}{m_t^3} h_t(x_t). \end{aligned}$$

So, if $h_t(x_t) \geq 0$, the R_t function is convex.

Proceeding in an analogous way for t arbitrarily fixed in $\{q+1,...,n\}$, we introduce the function $R_t :]0,+\infty[^q \to [0,1]$

$$R_t(v_0,...,v_{q-1}) = F_{|Z_t|}\left(x_t \Big/ \left(c_t + \sum_{j=0}^{q-1} w_j v_j\right)\right),$$

where c_t is determinist and positive and $w_j = \gamma^{t-q+j}, j = 0,...,q-1$.

As, for $0 \le i,j \le q-1$,

$$\begin{aligned}\frac{\partial^2}{\partial v_i \partial v_j} R_t(v_0,...,v_{q-1}) &= \frac{w_i w_j x_t}{m_t^3}\left[2 f_{|Z_t|}\left(\frac{x_t}{m_t}\right) + \frac{x_t}{m_t} f'_{|Z_t|}\left(\frac{x_t}{m_t}\right)\right] \\ &= \frac{w_i w_j x_t}{m_t^3} h_t(x_t)\end{aligned}$$

with $m_t = c_t + \sum_{j=0}^{q-1} w_j v_j$, we deduce that if $h_t(x_t) \ge 0$ the R_t function is convex.

Returning to the lower bound of $F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1,...,x_n)$ named **lbound1,** we point out that we have $n-1$ functions to which we can apply the Jensen inequality in the sets where the corresponding h_t are non negatives.

Moreover, if we consider the function $R_1 : \mathbb{R}^+ \longrightarrow [0,1]$ defined by $R_1(u) = F_{|Z_1|}\left(\frac{x_1}{u}\right)$, we prove that R_1 is convex if $h_1(x_1) = \left[2 f_{|Z_1|}\left(\frac{x_1}{u}\right) + \frac{x_1}{u} f'_{|Z_1|}\left(\frac{x_1}{u}\right)\right] \ge 0$. Under this condition, the Jensen inequality sets

$$E\left[F_{|Z_1|}\left(\frac{x_1}{\sigma_1}\right)\right] \ge F_{|Z_1|}\left(\frac{x_1}{E(\sigma_1)}\right).$$

So, we obtain

$$\begin{aligned}&F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1,...,x_n) \\ &\ge F_{|Z_1|}\left(\tfrac{x_1}{E(\sigma_1)}\right) \prod_{t=2}^{q} F_{|Z_t|}\left(\frac{x_t}{k_t + \sum_{j=0}^{t-2} \gamma^j E[f_{t-j}(\underline{\varepsilon}_0)] + \gamma^{t-1} E(\sigma_1)}\right) \prod_{t=q+1}^{n} \\ &\times F_{|Z_t|}\left(\frac{x_t}{\alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j E[f_{q-j}(\underline{\varepsilon}_0)] + \gamma^{t-1} E(\sigma_1)}\right).\end{aligned}$$

As $E\left(\varepsilon_t^+\right) = E\left(-\varepsilon_t^-\right) \le \sigma_\varepsilon$, we deduce that

$$E\left[f_{t-j}(\underline{\varepsilon}_0)\right] = E\left(\alpha_0 + \sum_{i=t-j}^{q} \alpha_i \varepsilon_{t-j-i}^+ - \sum_{i=t-j}^{q} \beta_i \varepsilon_{t-j-i}^-,\right) \le \alpha_0 + \sum_{i=t-j}^{q} s_i \sigma_\varepsilon.$$

Moreover, as referred after the proof of theorem 2, with $\gamma_1 = \gamma$ and $\gamma_j = 0$, $j = 2, ...,$ we may use the following upper bound for $E(\sigma_1)$

$$E(\sigma_1) \leq \alpha_0 + \sigma_\varepsilon \left(\sum_{i=1}^{q} s_i + \gamma \right) .$$

Finally, we have

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n) \geq F_{|Z_1|} \left(\frac{x_1}{\alpha_0 + \sigma_\varepsilon \left(\sum_{i=1}^{q} s_i + \gamma \right)} \right)$$

$$\prod_{t=2}^{q} F_{|Z_t|} \left(\frac{x_t}{k_t + \sum_{j=0}^{t-2} \gamma^j \left[\alpha_0 + \sum_{i=t-j}^{q} s_i \sigma_\varepsilon \right] + \gamma^{t-1} \left[\alpha_0 + \sigma_\varepsilon \left(\sum_{i=1}^{q} s_i + \gamma \right) \right]} \right) \prod_{t=q+1}^{n} F_{|Z_t|}$$

$$\times \left(\frac{x_t}{\alpha_0 \sum_{j=0}^{t-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} s_i x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j \left[\sum_{i=q-j}^{q} s_i \sigma_\varepsilon \right] + \gamma^{t-1} \sigma_\varepsilon \left(\sum_{i=1}^{q} s_i + \gamma \right)} \right)$$

for every $(x_1, ..., x_n) \in \left]0, +\infty\right[^n$ such that $h_t(x_t) \geq 0$, $t = 1, ..., n$. □

We point out that he analysis for the case $n < q$ follows the steps used in the situation $q \leq n$ and $t \in \{2, ..., q\}$. In fact,

— when $p = 1$ as, for every $t \in \{2, ..., n\}$,

$$\sigma_t \leq k_t + \sum_{j=0}^{t-2} \gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1} \sigma_1$$

with

$$k_t = \sum_{j=0}^{t-2} \gamma^j \sum_{i=1}^{t-1-j} s_i x_{t-j-i}, \quad t \in \{2, ..., n\}$$

and

$$f_t(\underline{\varepsilon}_0) = \begin{cases} \alpha_0 + \sum_{i=t}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=t}^{q} \beta_i \varepsilon_{t-i}^{-}, & t = 2, ..., n\,, \\ 0, & t \notin \{2, ..., n\}\,, \end{cases}$$

we obtain the following inequality

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n) \geq F_{|Z_1|} \left(\frac{x_1}{\alpha_0 + \sigma_\varepsilon \left(\sum_{i=1}^{q} s_i + \gamma \right)} \right)$$

$$\times \prod_{t=2}^{n} F_{|Z_t|} \left(\frac{x_t}{k_t + \sum\limits_{j=0}^{t-2} \gamma^j \left[\alpha_0 + \sum\limits_{i=t-j}^{q} s_i \sigma_\varepsilon \right] + \gamma^{t-1} \left[\alpha_0 + \sigma_\varepsilon \left(\sum\limits_{i=1}^{q} s_i + \gamma \right) \right]} \right);$$

— when $p = 0$, as $\sigma_t \leq \sum\limits_{i=1}^{t-1} s_i x_{t-i} + f_t(\underline{\varepsilon}_0)$ for every $t \in \{2, ..., n\}$, we deduce that

$$F_{(|\varepsilon_1|,...,|\varepsilon_n|)}(x_1, ..., x_n) \geq F_{|Z_1|} \left(\frac{x_1}{\alpha_0 + \sigma_\varepsilon \sum\limits_{i=1}^{q} s_i} \right) \times \prod_{t=2}^{n} F_{|Z_t|} \left(\frac{x_t}{\alpha_0 + \sum\limits_{i=1}^{t-1} s_i x_{t-i} + \sum\limits_{i=t}^{q} s_i \sigma_\varepsilon} \right).$$

These two inequalities are valuable for every $(x_1, ..., x_n) \in]0, +\infty[^n$ subject to conditions similar to those obtained in the previous theorem.

Finally, we note that if the law of Z_t is symmetrical around the origin, the functions h_t that appear in the last theorem may be expressed simply using the Z_t law. In fact, we have, with a convenient u_t,

$$h_t(x) = 2 \left[2 f_{Z_1} \left(\frac{x}{u_t} \right) + \frac{x}{u_t} f'_{Z_1} \left(\frac{x}{u_t} \right) \right].$$

3.2. Applications

Following Pawlak and Schmid [3], the bounds obtained in the last theorem may be related with the run length of control charts. Let us suppose, for example, that ε follows a TARCH(q) process (i.e., a GTARCH($1, q$) process with $\gamma_1 = 0$) and that $q \leq n$. Let us consider $x_1 = ... = x_n = x$. Then, for x under the convenient conditions stated in Theorem 3, we obtain

$$P \left(\max_{1 \leq t \leq n} |\varepsilon_t| \leq x \right) \geq F_{|Z_1|} \left(\frac{x}{\alpha_0 + \sigma_\varepsilon \sum\limits_{i=1}^{q} s_i} \right)$$

$$\times \prod_{t=2}^{q} F_{|Z_t|}\left(\frac{x}{\alpha_0+\sum_{i=1}^{t-1}s_i x+\sigma_\varepsilon\sum_{i=t}^{q}s_i}\right)\prod_{t=q+1}^{n}F_{|Z_t|}\left(\frac{x}{\alpha_0+\sum_{i=1}^{q}s_i x}\right)$$

$$= F_{|Z_1|}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sum_{i=1}^{q}s_i}\right)\prod_{t=2}^{q}F_{|Z_t|}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sum_{i=1}^{q}s_i+\sum_{i=1}^{t-1}s_i\left(x-\sigma_\varepsilon\right)}\right)$$

$$\prod_{t=q+1}^{n}F_{|Z_t|}\left(\frac{x}{\alpha_0+\sum_{i=1}^{q}s_i x}\right).$$

In particular, if $x<\sigma_\varepsilon$,

$$P\left(\max_{1\le t\le n}|\varepsilon_t|\le x\right)\ge\left[F_{|Z_1|}\left(\frac{x}{\alpha_0+\sigma_\varepsilon\sum_{i=1}^{q}s_i}\right)\right]^n$$

$$\ge\left[F_{|Z_1|}\left(\frac{x}{\sigma_\varepsilon+\sigma_\varepsilon\sum_{i=1}^{q}s_i}\right)\right]^n,$$

as $\alpha_0\le\sigma_\varepsilon$. Then, using also theorem 3, we have

$$\left[F_{|Z_1|}\left(\frac{x}{1+\sum_{i=1}^{q}s_i}\right)\right]^n\le P\left(\max_{1\le t\le n}\frac{|\varepsilon_t|}{\sigma_\varepsilon}\le x\right)\le\left[F_{|Z_1|}\left(\frac{\sigma_\varepsilon}{\alpha_0}x\right)\right]^n.$$

The last inequality may have an interesting interpretation in statistical quality control. In fact, if like in the ARCH process, $P\left(\max_{1\le t\le n}\frac{|\varepsilon_t|}{\sigma_\varepsilon}\le x\right)$ is the probability of no alarm until time n in the in-control state, $\left[F_{|Z_1|}\left(\frac{\sigma_\varepsilon}{\alpha_0}x\right)\right]^n$ is the analogical probability if the process becomes independent ($\alpha_i=\beta_i=0,\ i=1,...,q$). Thus, for a TARCH process the probability of no alarm is, for certain values of x, bounded by the probabilities indicated, related to the independent variables Z_t, $t\in\mathbb{Z}$.

References

[1] P.J. Brockwell, R.A. Davis, *Introduction to Time Series and Forecasting*, Second Edition, Springer Verlag (2002).

[2] E. Gonçalves, N. Mendes-Lopes, The generalized threshold ARCH model: Wide sense stationarity and asymptotic normality of the temporal aggregate, *Pub. Inst. Stat. Univ. Paris*, **XXXVIII**, No. 2 (1994), 19-35.

[3] M. Pawlak, W. Schmid, On the distributional properties of GARCH processes, *Journal of Time Series Analysis*, **22**, No. 3 (2001), 339-352.

[4] R. Rabemananjara, J.M. Zakoian, Threshold ARCH models and asymmetries in volatility, *Journal of Applied Econometrics*, **8** (1993), 31-49.

[5] J.M. Zakoian, *Modèles Autoregressifs à Seuil*, Thèse de Docteur de l'Université en Mathématiques Appliquées, Univ. Paris IX (1990).

Appendix 1: Bounds for σ_t

We begin by establishing some upper bounds for

$$\sigma_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^{-} + \gamma \sigma_{t-1},\ t = 1, ..., n.$$

We intend to separate, in σ_t, the values associated to positive times from those associated to non positive ones.

Let us analyse the case where $q \leq n$ with $q \geq 2$.

For every $t \in \{2, ..., q\}$ we may write

$$\sigma_t = \sum_{i=1}^{t-1} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=1}^{t-1} \beta_i \varepsilon_{t-i}^{-} + f_t\left(\underline{\varepsilon}_0\right) + \gamma \sigma_{t-1}$$

with

$$f_t\left(\underline{\varepsilon}_0\right) = \begin{cases} \alpha_0 + \sum\limits_{i=t}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum\limits_{i=t}^{q} \beta_i \varepsilon_{t-i}^{-}, & t = 2, ..., q\,, \\ 0\,, & t \notin \{2, ..., q\}\ . \end{cases}$$

Then

$$\sigma_t \leq \sum_{i=1}^{t-1} \left(\alpha_i + \beta_i\right) x_{t-i} + f_t\left(\underline{\varepsilon}_0\right) + \gamma \sigma_{t-1}$$

and, by recurrence, we obtain for every $t \in \{2, ..., q\}$

$$\sigma_t \leq \sum_{j=0}^{t-2} \gamma^j \sum_{i=1}^{t-1-j} (\alpha_i + \beta_i) x_{t-j-i} + \sum_{j=0}^{t-2} \gamma^j f_{t-j} (\underline{\varepsilon}_0) + \gamma^{t-1} \sigma_1.$$

In order to simplify some expressions, we make

$$k_t = \sum_{j=0}^{t-2} \gamma^j \sum_{i=1}^{t-1-j} (\alpha_i + \beta_i) x_{t-j-i}, \quad t \in \{2, ..., q\}.$$

In this way we have, for every $t \in \{2, ..., q\}$,

$$\sigma_t \leq k_t + \sum_{j=0}^{t-2} \gamma^j f_{t-j} (\underline{\varepsilon}_0) + \gamma^{t-1} \sigma_1.$$

Now, for $t \in \{q+1, ..., n\}$ we may write, with $A_t = \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^{-}$,

$$\sigma_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^{-} + \gamma \sigma_{t-1}$$
$$= \alpha_0 + A_t + \gamma (\alpha_0 + A_{t-1} + \gamma \sigma_{t-2}) = ...$$

Using this recurrence $t - q$ times in order to rewrite σ_t in terms of σ_q, we obtain

$$\sigma_t = \alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j A_{t-j} + \gamma^{t-q} \sigma_q.$$

As $A_t \leq \sum_{i=1}^{q} (\alpha_i + \beta_i) x_{t-i}$, we deduce that

$$\begin{aligned} \sigma_t &\leq \alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} (\alpha_i + \beta_i) x_{t-j-i} + \gamma^{t-q} \sigma_q \\ &\leq \alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} (\alpha_i + \beta_i) x_{t-j-i} \\ &\quad + \gamma^{t-q} \left[k_q + \sum_{j=0}^{q-2} \gamma^j f_{q-j} (\underline{\varepsilon}_0) + \gamma^{q-1} \sigma_1 \right] \end{aligned}$$

using the upper bound for σ_q.

So, for every $t \in \{q+1, ..., n\}$

$$\sigma_t \leq \alpha_0 \sum_{j=0}^{t-q-1} \gamma^j + \sum_{j=0}^{t-q-1} \gamma^j \sum_{i=1}^{q} (\alpha_i + \beta_i)\, x_{t-j-i} + \gamma^{t-q} k_q + \gamma^{t-q} \sum_{j=0}^{q-2} \gamma^j f_{q-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1.$$

We note that if $q = 1$, the two inequalities obtained reduce to a simple one for every $t \in \{2, ..., n\}$, namely,

$$\sigma_t \leq \alpha_0 \sum_{j=0}^{t-2} \gamma^j + \sum_{j=0}^{t-2} \gamma^j (\alpha_1 + \beta_1)\, x_{t-j-1} + \gamma^{t-1}\sigma_1.$$

The analysis for the case $n < q$, $(n \geq 2)$, follows the steps used in the situation $t \in \{2, ..., q\}$; we obtain for every $t \in \{2, ..., n\}$

$$\sigma_t \leq k_t + \sum_{j=0}^{t-2} \gamma^j f_{t-j}(\underline{\varepsilon}_0) + \gamma^{t-1}\sigma_1$$

with

$$f_t(\underline{\varepsilon}_0) = \begin{cases} \alpha_0 + \sum_{i=t}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=t}^{q} \beta_i \varepsilon_{t-i}^{-}, & t = 2, ..., n\,, \\ 0\,, & t \notin \{2, ..., n\}\ . \end{cases}$$

Appendix 2: Bounds for $E(\sigma_t)$

Let us analyse

$$E(\sigma_t) = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^{-} + \sum_{j=1}^{p} \gamma_j \sigma_{t-j}.$$

We detail the analysis assuming that $q \geq 2$ and $p \geq 2$ but, as can be easily seen, the result is also valuable for the other possible values of q and p assuming as equal to zero the convenient coefficients. The TARCH(1) situation, *i.e.*, corresponding to $q = 1$ and $p = 0$, needs in this approach a litttle attention, as will be seen. We have

$$E(\sigma_t) - \alpha_0 = \sum_{i=1}^{q} \alpha_i E\left(\varepsilon_{t-i}^{+}\right) - \sum_{i=1}^{q} \beta_i E\left(\varepsilon_{t-i}^{-}\right) + \sum_{j=1}^{p} \gamma_j E\left(\sigma_{t-j}\right) \leq$$

$$\left[E\left(\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)\sigma_{t-1} + \sum_{i=2}^{q} \alpha_i \varepsilon_{t-i}^{+} - \sum_{i=2}^{q} \beta_i \varepsilon_{t-i}^{-} + \sum_{j=2}^{p} \gamma_j \sigma_{t-j}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[E\left(U^T V\right)^2\right]^{\frac{1}{2}},$$

where

$$\begin{aligned} U &= \left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1, \alpha_2, ..., \alpha_q, \beta_2, ..., \beta_q, \gamma_2, ..., \gamma_p\right)^T, \\ V &= \left(\sigma_{t-1}, \varepsilon_{t-2}^{+}, ..., \varepsilon_{t-q}^{+}, -\varepsilon_{t-2}^{-}, ..., -\varepsilon_{t-q}^{-}, \sigma_{t-2}, ..., \sigma_{t-p}\right)^T. \end{aligned}$$

But

$$E\left(U^T V\right)^2 = E\left(\left\|U^T V\right\|^2\right) \leq E\left(\|U\|^2 \|V\|^2\right) = E\left(\|U\|^2\right) E\left(\|V\|^2\right),$$

as Z_{t-1} is independent of $\underline{\varepsilon}_{t-2}$. Moreover,

$$\begin{aligned} E\left(\|U\|^2\right) & E\left(\|V\|^2\right) \\ &= E\left[\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)^2 + \sum_{i=2}^{q}\left(\alpha_i^2 + \beta_i^2\right) + \sum_{i=2}^{p} \gamma_i^2\right] \\ &\quad \times E\left[\sum_{i=2}^{q}\left[\left(\varepsilon_{t-i}^{+}\right)^2 + \left(\varepsilon_{t-i}^{-}\right)^2\right] + \sum_{i=1}^{p} \sigma_{t-i}^2\right] \\ &= \left[E\left(\alpha_1 Z_{t-1}^{+} - \beta_1 Z_{t-1}^{-} + \gamma_1\right)^2 + \sum_{i=2}^{q}\left(\alpha_i^2 + \beta_i^2\right) + \sum_{i=2}^{p} \gamma_i^2\right] \\ &\quad \times \left[\sum_{i=2}^{q} E\left(\varepsilon_{t-i}^2\right) + \sum_{i=1}^{p} E\left(\sigma_{t-i}^2\right)\right], \end{aligned}$$

as $\varepsilon_t = \varepsilon_t^{+} + \varepsilon_t^{-}$.

Moreover $E\left(\sigma_{t-i}^2\right) = E\left[E\left(\varepsilon_{t-i}^2 | \underline{\varepsilon}_{t-i-1}\right)\right] = E\left(\varepsilon_{t-i}^2\right) = \sigma_\varepsilon^2$.

So, under the stationarity of ε, we may write

$$E\left(\|U\|^2\right) E\left(\|V\|^2\right) = \begin{cases} c_1 \sigma_\varepsilon^2, & q = 1,\ p = 0, \\ c_1 (p + q - 1)\sigma_\varepsilon^2, & p \geq 1,\ q \geq 1 \text{ or } p = 0,\ q \geq 2, \end{cases}$$

where

$$c_1 = \alpha_1^2 E\left(Z_t^+\right)^2 + \beta_1^2 E\left(Z_t^-\right)^2 + \gamma_1^2 + 2\gamma_1\left[\alpha_1 E\left(Z_t^+\right) - \beta_1 E\left(Z_t^-\right)\right] + \sum_{i=2}^{q}\left(\alpha_i^2+\beta_i^2\right) + \sum_{i=2}^{p}\gamma_i^2 \le \left(\alpha_1+\beta_1\right)\gamma_1 + \sum_{i=1}^{q}\left(\alpha_i^2+\beta_i^2\right) + \sum_{i=1}^{p}\gamma_i^2 = c\,,$$

as $E\left(Z_t^+\right) = -E\left(Z_t^-\right) \le \frac{1}{2}$, $E\left(Z_t^+\right)^2 \le 1$ and $E\left(Z_t^-\right)^2 \le 1$.

Finally,

$$E\left(\sigma_t\right) \le \alpha_0 + \sigma_\varepsilon\sqrt{ck}$$

with

$$k = \begin{cases} 1, & q=1,\ p=0\,, \\ p+q-1, & p\ge 1,\ q\ge 1 \text{ or } p=0,\ q\ge 2. \end{cases}$$