International Journal of Pure and Applied Mathematics
Volume 35 No. 3 2007, 423-426

A NOTE ON A TAUBERIAN THEOREM FOR (A,i) LIMITABLE METHOD

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Abstract: In this note the Tauberian Theorem for Abel limitable method proved by Çanak and Totur [3] is generalized to (A,i) limitable method.

AMS Subject Classification: 40E05
Key Words: Tauberian Theorem, general control modulo, (A,i) limitability

1. Introduction

Let $u=(u_n)$ be a sequence of real numbers. Define

$$\sigma_n^{(i)}(u)=\begin{cases}\frac{1}{n+1}\sum_{k=0}^{n}\sigma_k^{(i-1)}(u)\,, & i\geq 1\;,\\ u_n\,, & i=0\,,\end{cases}$$

for each integer $i\geq 0$ and for all nonnegative integers n. A sequence (u_n) is said to be (H,i) limitable to s if $\lim_{n\to\infty}\sigma_n^{(i)}(u)$ exists and equals s. The $(H,1)$ limitable method is the same as the $(C,1)$ limitable method. A sequence (u_n) is said to be (A,i) limitable to s if $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\sigma_n^{(i)}(u)x^n$ exists and equals s (in short, we write $u_n\to s\ (A,i)$). If $i=0$, then (A,i) limitability reduces to Abel limitability. It is clear that $u_n\to s\ (A,0)$ implies $u_n\to s\ (A,i)$ for each integer $i\geq 1$. The converse is not necessarily true. For example , in the case that $i=1$, the sequence (u_n) which is the Taylor coefficients of the

Received: January 26, 2007

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function f defined by $f(x) = \sin((1-x)^{-1})$ on $0 < x < 1$ is not Abel limitable, but (u_n) is $(A, 1)$ limitable.

Denote $\mathcal{S}$ by the class of all slowly oscillating sequences in the sense of Stanojević [9] as follows. A sequence (u_n) belongs to the class $\mathcal{S}$, or briefly $u \in \mathcal{S}$, if

$$\lim_{\lambda \to 1^+} \overline{\lim_{n}} \max_{n+1 \le k \le [\lambda n]} |u_k - u_n| = 0.$$

If $(\sigma_n^{(1)}(u))$ is slowly oscillating, we say that (u_n) is $(C, 1)$ slowly oscillating. Stanojević's Definition of slow oscillation is more operational in proofs than those of Landau [7] and Schmidt [8]. An equivalent definition of slow oscillation of a sequence in S is given in terms of generator sequence $(V_n^{(0)}(\Delta u))$ of (u_n) by Dik [4]. A sequence (u_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is slowly oscillating and bounded.

It had been difficult to recover convergence of (u_n) out of its Abel limitability and Tauberian conditions weaker than those such as Hardy-Littlewood [5], Landau [7] and Schmidt [8]. In this connection, Stanojević [10] introduced the concept of the general control modulo of the oscillatory behavior of integer order $m \ge 1$ of a real sequence (u_n), defined by inductively, for all nonnegative integers n,

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u)),$$

where $\omega_n^{(0)}(u) = n\Delta u_n$. Çanak and Totur [2] have expressed this concept in a more explicit form in terms of a generator sequence. For a sequence (u_n) we define $(n\Delta)_m u_n = (n\Delta)_{m-1}((n\Delta)u_n) = n\Delta((n\Delta)_{m-1}u_n)$ for each integer $m \ge 1$ and each nonnegative integer n, where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$. It is proved in [2] that for each integer $m \ge 1$, $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$, where $V_n^{(m)}(\Delta u) = \sigma_n^{(1)}(V^{(m-1)}(\Delta u))$ and $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$.

Stanojević [9] has extended the class $\mathcal{S}$ and introduced the following definition. A sequence (u_n) belongs to the class $\mathcal{M}$, or briefly $u \in \mathcal{M}$, if for $\lambda > 1$

$$\overline{\lim_{n}} \max_{n+1 \le k \le [\lambda n]} |u_k - u_n| < \infty.$$

It is straightforward to show that $\mathcal{S} \subset \mathcal{M}$.

Using the concept of the generator sequence [4] and a corollary to Karamata's Main Theorem [6], Çanak [1] proved the theorem well known as the generalized Littlewood-Tauberian Theorem that states if (u_n) is Abel limitable to s and slowly oscillating, then (u_n) converges to s.

Çanak and Totur [3] obtained a sufficient condition in terms of the general control modulo for the Abel limitable sequence (u_n) to be convergent.

Theorem A. (see [3]) *Let (u_n) be Abel limitable to s. If $(w_n^{(m)}(u))$ is (C,1) slowly oscillating, then (u_n) converges to s.*

We note that the generalized Littlewood-Tauberian Theorem is a corollary to Theorem A.

In this note we prove that the Tauberian condition in Theorem A is also a Tauberian condition for (A, i) limitable method for each integer $i \geq 1$.

Theorem 1. *Let $u_n \to s$ (A, i) for any fixed integer $i \geq 1$. If $(w_n^{(m)}(u))$ is $(C, 1)$ slowly oscillating, then $u_n \to s$.*

Proof. Let $\sigma_n^{(1)}(w^{(m)}(u)) = a_n$ for some $a = (a_n) \in \mathcal{S}$. Noticing that

$$\sigma_n^{(1)}(w^{(m)}(u)) = w_n^{(m)}(\sigma^{(1)}(u)),$$

we have, for every positive integer n,

$$w_n^{(m)}(\sigma^{(1)}(u)) = a_n\,,$$

and then $w_n^{(m)}(\sigma^{(j)}(u)) = \sigma_n^{(j-1)}(a)$, for $j = 1, 2, ..., i+1$. From the equivalent definition of a sequence in $\mathcal{S}$, it follows that

$$(\sigma_n^{(j-1)}(a)) \in S$$

for $j = 1, 2, ..., i+1$. Since $u_n \to s$ (A, i), we have $u_n \to s$ (H, i) by Theorem A. By the fact that every sequence $(C, 1)$ limitable is Abel limitable, we have

$$u_n \to s\,(A, i-1). \tag{1}$$

Since $(w_n^{(m)}(\sigma^{(i)}(u)) = (\sigma_n^{(i-1)}(a)) \in \mathcal{S}$ and $u_n \to s$ $(A, i-1)$, we obtain that $u_n \to s$ $(H, i-1)$. By the same reasoning in obtaining (1), we have

$$u_n \to s\,(A, i-2).$$

Continuing in this way, we obtain that $u_n \to s\,(A, 0)$. Together with $(\sigma_n^{(1)}(w^{(m)}(u)) = (a_n) \in \mathcal{S}$, we have $u_n \to s$. This completes the proof. □

Corollary 2. *Let $u_n \to s$ (A, i). If $(w_n^{(m)}(u)) \in \mathcal{S}$ (or $\in \mathcal{M}$), then $u_n \to s$.*

Proof. It easily follows from the definition of the class $\mathcal{S}$ and $\mathcal{M}$ that $(w_n^{(m)}(u)) \in \mathcal{S}$ (or $\in \mathcal{M}$) implies that $(\sigma_n^{(1)}(w^{(m)}(u)) \in \mathcal{S}$. □

We remark that we can only observe boundedness of (u_n) instead of convergence of (u_n) if we replace "belonging to $\mathcal{S}$ in mean" by "belonging to $\mathcal{M}$ in mean" in Theorem 1.

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