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AN OPTIMAL MAINTENANCE POLICY WITH AGE-DEPENDENT MINIMAL REPAIR COST FOR A SYSTEM UNDER PERIODIC OVERHAUL

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Abstract: A maintenance policy for determining the optimal number (N^*-1) of overhauls and the optimal period (T^*) between overhauls which incorporate minimal repair, overhauls, replacement. The cost of minimal repair depends on age and the system is replaced at time N^*T^*. It is shown that there exists a unique optimal policy which minimizes the expected cost rate under certain conditions. Various cases are considered.

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1. Introduction

It is of great importance to avoid the failure of a system during actual operation when such an event is costly and/or dangerous. In such situations, one important area of interest in the reliability theory is the study of various maintenance policies in order to reduce the risk of a catastrophic breakdown and the operating cost.

A maintenance policy which includes minimal repairs and replacement has been first considered by Barlow and Hunter [1]. They considered two types of preventive maintenance policies, these two policies have been modified and generalized by Beichelt [2], Cl'eroux et al [10], Beichelt and Fisher [3], Nguyen

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and Murthy [14], Berg and Cl'eroux [5], Boland [8], Boland and Proschan [9], Block et al [6,7], Berg et al [4], Nakagawa [13], Sheu [17,18,19] and Sheu and Liou [20]. Various treatment methods and optimal policies on the imperfect maintenance are discussed and summarized by Pham and Wang [15].

Overhaul is performed to eliminate any probable failure of the system. However, an overhaul may affect only a limited number of components, it makes a system better than old. Seo and Bai [16] considered an optimal maintenance policy for a system under periodic overhaul, they proposed a model which not only decreases the hazard rate of a system at each overhaul time but also changes the hazard rate function after overhaul. But the cost structure of minimal repair does not depend on time.

In this paper, we consider a model in which the system undergoes minimal repair at failure with age-dependent cost $c(t)$, where $c(t)$ is a positive continuous non-decreasing function of t. As the system ages it becomes more expensive to perform minimal repair. The system is overhauled at time $iT(i = 1, 2, \cdots, N - 1)$ and it is replaced at time NT. After a replacement, the procedure is repeated.

The rest of this paper is organized as follows. In Section 2, a model describing the effect of overhaul is present. In Section 3, expected cost rate under our proposed model are obtained, and we have derived the optimal period and optimal number of overhauls. In Section 4 various special cases are detailed. In the last section a conclusion is given.

2. Our Model

Our model is constructed using the virtual age function and increasing the hazard rate function with the number of overhauls. Let t_i be the time of the ith overhaul, where $t_0 = 0$, that is $t_i = iT, i = 1, 2, \cdots, N$; but t_N be the time of replacement. An overhaul decreases the hazard rate but not to zero. The virtual age function $V(v, T)$ of the system is a function of two variables v and T, Kijima et al [11] and Kijima [12] measured the effect of the overhaul on the virtual age by a multiplier, where $0 \leq \theta \leq 1$. They used the virtual age function $V(v, X) = v + \theta X$. If the system has a hazard rate function $r_i(t)$ ($r_i(t)$ is increasing in i and t) in the i-th overhaul period and $v_i(T)$ is the virtual age of the system at i-th overhaul, then the hazard rate of the system is $r_i(V(v_{i-1}(T), T))$ right after t_i. So, we have $r_{i+1}(v_i(T)) = r_i(v_{i-1}(T) + \theta T)$. Our basic assumptions are as follows.

Assumption 1. $r_i(t)$ is strictly increasing in t, and $r_i(t) \to \infty$ as $t \to \infty$, for $i = 1, 2, \cdots, N$.

Assumption 2. $r_{i+1}(t) > r_i(t)$, for any $t > 0$ and $i = 1, 2, \cdots, N-1$.
Assumption 3. $r_{i+1}(v_i(T)) = r_i(v_{i-1}(T) + \theta T)$, for $i = 1, 2, \cdots, N-1$.

3. Maintenance Policy

Suppose that the system undergoes minimal repair at failure, and failures over time interval $(v_{i-1}(T), v_{i-1}(T) + T]$ occur according to a non-homogeneous Poisson process $\{M_i(t); t \geq 0\}$ with failure rate $r_i(t)$ at time t (t hours from the last overhaul) in the i-th overhaul period and the cost of minimal repair is $c(t)$, where $c(t)$ is a positive continuous non-decreasing function of t. The system is overhauled at time iT $(i = 1, 2, \cdots, N-1)$, and it is replaced at time NT. The cost of overhaul is c_O, and the cost of replacement is c_R $(c_R \geq c_O)$. We assume overhaul and minimal repair times are negligible. Now, we want to derive the expected cost of minimal repairs incurred over $(v_{i-1}(T), v_{i-1}(T) + T]$. We also require the following lemma from Sheu [17].

Lemma 3.1. *Let $\{M(t); t \geq 0\}$ be a non-homogeneous Poisson process with intensity $\lambda(t), t \geq 0$ and $\Lambda(t) = E[M(t)] = \int_0^t \lambda(u)du$. Denote the successive arrival times by $S_1, S_2, \cdots$. Assume that at time S_i a cost of $g(C(S_i), c_i(S_i))$ is incurred. Suppose that $C(y)$ at age y is random variables with finite mean $E[C(y)]$ and g is a positive, non-decreasing and continuous function. If $A(t)$ is the total cost incurred over* [0, t], *then*

$$E[A(t)] = \int_0^t h(y)\lambda(y)dy, \tag{1}$$

where

$$h(y) = E_{M(y)}\left[E_{C(y)}\left[g(C(y), c_{M(Y)+1}(y))\right]\right]. \tag{2}$$

Theorem 3.1. *The expected cost of minimal repairs incurred over $(v_{i-1}(T), v_{i-1}(T) + T]$ is $\int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt$.*

Proof. In Lemma 3.1, if we put $g(C(y), c_{M(y)+1}(y)) = c(y), \lambda(y) = r_i(y)$, then the expected cost of minimal repairs incurred over $(v_{i-1}(T), v_{i-1}(T) + T]$ is

$$\begin{aligned} E[A(t)] &= \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} \sum_{k=1}^{\infty} c(y)P(M(y) = k-1)r_i(y)dy \\ &= \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)\sum_{k=1}^{\infty} P(M(t) = k-1)r_i(t)dt = \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt. \end{aligned}$$

Since the expected cost in a renewal cycle is $\sum_{i=1}^{N} \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt + (N-1)c_O + c_R$, and the expected length of a renewal cycle is NT. Thus, the expected cost rate over infinite time span is easily given by

$$C(N,T) = [\frac{\sum_{i=1}^{N} \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt + (N-1)c_O + c_R}{NT}]. \tag{3}$$

We want to find the optimal number N^* and the optimal period T^* which minimize $C(N,T)$ in (3). We see that the inequalities $C(N+1,T) \geq C(N,T)$ and $C(N,T) < C(N-1,T)$ if and only if

$$L(N,T) \geq c_R - c_O \quad \text{and} \quad L(N-1,T) < c_R - c_O\,, \tag{4}$$

where

$$L(N,T) = \begin{cases} N \displaystyle\int_{v_N(T)}^{v_N(T)+T} c(t)r_{N+1}(t)dt \\ - \displaystyle\sum_{i=1}^{N} \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt, & \text{for } N = 1,2,\cdots, \\ 0, \quad \text{for } N = 0. \end{cases}$$

Further,

$$L(N,T) - L(N-1,T) = N\left[\int_{v_N(T)}^{v_N(T)+T} c(t)r_{N+1}(t)dt - \int_{v_{N-1}(T)}^{v_{N-1}(T)+T} c(t)r_N(t)dt\right]. \tag{5}$$

We seek the optimal number N^* which minimizes $C(N,T)$ in (3) under certain conditions.

Theorem 3.2. *Suppose that $r_i(t)$ is strictly increasing in i for any $t > 0$, and $c(t)$ is a positive continuous non-decreasing function of t. If $\lim_{N\to\infty} r_N(t) = \infty$, then for any $T > 0$,there exists a finite and unique N^* which satisfies*

$$L(N,T) \geq c_R - c_O \quad \text{and} \quad L(N-1,T) < c_R - c_O. \tag{6}$$

Proof. The inequalities $C(N+1,T) \geq C(N,T)$ and $C(N,T) < C(N-1,T)$ imply (6). Further, if the conditions of the theorem are satisfied, then from expression (5), it is easily seen that $L(N,T)$ is an increasing function of N. Evidently, from (5), we have

$$L(N,T) > \int_{v_N(T)}^{v_N(T)+T} c(t)r_{N+1}(t)dt - \int_0^T c(t)r_1(t)dt. \tag{7}$$

So, if $\lim_{N\to\infty} r_N(t) = \infty$, then $L(N,T)$ tends to ∞ as $N \to \infty$. Thus, a solution to (6) exists and, from the monotonicity of $L(N,T)$, it is unique. □

Now, we seek the optimal period T^* which minimizes $C(N,T)$ in (3) under certain conditions.

Theorem 3.3. *Suppose that $c(t)r_i(t)$ is a non-decreasing function of t and i, and $c''(t) \geq 0, r_i''(t) \geq 0, v_{i-1}''(T) \geq 0$, where $r_i''(t) = d^2r_i(t)/dt^2$, $v_{i-1}''(T) = d^2v_{i-1}(T)/dT^2$. If*

$$\lim_{T\to\infty} \sum_{i=1}^{N} \Big\{ \big[c(v_{i-1}(T)+T)r_i(v_{i-1}(T)+T)(v_{i-1}'(T)+1) - c(v_{i-1}(T))r_i(v_{i-1}(T))v_{i-1}'(T)\big]\, T - \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt \Big\} > (N-1)c_O + c_R, \tag{8}$$

then there exists a finite and unique T^* which minimizes $C(N,T)$ in (3) for any integer N.

Proof. We can differentiate $C(N,T)$ with respect to T, we see that $dC(N,T)/dT = 0$ if and only if

$$\sum_{i=1}^{N} \Big\{ \big[c(v_{i-1}(T)+T)r_i(v_{i-1}(T)+T)(v_{i-1}'(T)+1) - c(v_{i-1}(T))r_i(v_{i-1}(T))v_{i-1}'(T)\big]\, T - \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} c(t)r_i(t)dt \Big\} = (N-1)c_O + c_R. \tag{9}$$

Since the left-hand side of (9) is a strictly increasing function of T. So, if (8) holds, then there exists a finite and unique T^* which minimizes $C(N,T)$ in (3) for any integer N, and the resulting cost rate is

$$C(N,T^*) = \sum_{i=1}^{N} \big[c(v_{i-1}(T^*)+T^*)r_i(v_{i-1}(T^*)+T^*)(v_{i-1}'(T^*)+1) - c(v_{i-1}(T^*))r_i(v_{i-1}(T^*))v_{i-1}'(T^*)\big]/N.$$

Remark 3.1. If $r_i(t)$ is differentiable in t, and $r_i(t)$ is strictly increasing in t and i, $c(t)$ is a positive continuous non-decreasing function of t, (then $c(t)r_i(t)$ is a strictly increasing function of t and i), $c''(t) \geq 0, r_i''(t) \geq 0, v_{i-1}''(T) \geq 0$, and (8) holds, by Theorem 3.3, we know that there exists a finite and unique T^* which minimizes $C(N,T)$ in (3) for any integer N.

By using the following procedure, we can get the optimal number N^* and the optimal period T^*:

Step 1. Let $N_1 = 1$ and find $T = T_1$ satisfying (9), set $j = 2$.

Step 2. Let $T = T_{j-1}$ and find $N = N_j$ satisfying (4).

Step 3. Let $N = N_j$ and find $T = T_j$ satisfying (9).

Step 4. If $N_{j-1} = N_j$, then $(N^*, T^*) = (N_j, T_j)$ and stop; otherwise, set $j = j+1$ and go to Step 2.

Now, we derive $C(N,T)$ in (3) for the Weibull hazard rate. Suppose the virtual age function satisfies Assumption 3, and the hazard rate in the i-th overhaul period is $r_i(t) = \alpha_i \beta t^{\beta-1}$ for $\beta > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_N$. Seo and Bai [16] have shown by mathematical induction method that

$$v_{i-1}(T) = \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\alpha_i}\right)^{1/(\beta-1)} \times \theta T. \tag{10}$$

Suppose $c(t) = a + bt, a > 0, b, t \geq 0$. From (3) and (10),

$$C(N,T) = [\frac{R(N,T) + (N-1)c_O + c_R}{NT}], \tag{11}$$

where

$$\begin{aligned} R(N,T) = aT^\beta \times \left\{ \sum_{i=1}^{N} \alpha_i \left[\left(\theta \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\alpha_i}\right)^{1/(\beta-1)} + 1 \right)^\beta \right.\right. \\ \left.\left. - \left(\theta \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\alpha_i}\right)^{1/(\beta-1)} \right)^\beta \right] \right\} \\ + [\frac{b\beta T^{\beta+1}}{\beta+1}] \times \left\{ \sum_{i=1}^{N} \alpha_i \left[\left(\theta \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\alpha_i}\right)^{1/(\beta-1)} + 1 \right)^{\beta+1} \right.\right. \\ \left.\left. - \left(\theta \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\alpha_i}\right)^{1/(\beta-1)} \right)^{\beta+1} \right] \right\}. \end{aligned}$$

4. Special Cases

Case 4.1. $c(t) = c_1$. This is the case considered by Seo and Bai [16]. If we put $c(t) = c_1$ in (3), then we get the following result (see [16]):

$$C(N,T) = [\frac{c_1 \sum_{i=1}^{N} \int_{v_{i-1}(T)}^{v_{i-1}(T)+T} r_i(t)dt + (N-1)c_O + c_R}{NT}].$$

Case 4.2. $v_{i-1}(T) = 0, i = 1, 2, \cdots, N$. (i.e $\theta = 0$) and $c(t) = c_M$. This is the case considered by Nakagawa [13]. If we put $v_{i-1}(T) = 0, i = 1, 2, \cdots, N$ and $c(t) = c_M$ in (3), then we get the following result (see [13]):

$$C(N,T) = [\frac{\sum_{i=1}^{N} c_M \int_0^T r_i(t)dt + (N-1)c_O + c_R}{NT}].$$

Case 4.3. $N = 1, r_i(t) = r(t)$. This is the case considered by Boland [8]. If we put $N = 1, r_i(t) = r(t)$ in (3), then we get the following result (see [8]):

$$C(1,T) = [\frac{\int_0^T c(t)r(t)dt + c_R}{T}].$$

Case 4.4. $N = 1, r_i(t) = r(t), c(t) = c_1$. This is the policy II considered by Barlow and Hunter [1]. The problem reduces to the classical periodic replacement problem with minimal repair at failure. If we put $N = 1, r_i(t) = r(t), c(t) = c_1$ in (3), then we get the following result (see [1]):

$$C(1,T) = [\frac{c_1 \int_0^T r(t)dt + c_R}{T}].$$

It can be seen that this model is an improvement on previously known policies.

5. Conclusion

In this article an optimal maintenance policy with age-dependent minimal repair cost for a system under periodic overhaul is proposed. Under such a policy an operating system is overhauled at time iT ($i = 1, 2, \cdots, N-1$), and the system is replaced at time NT. The system undergoes minimal repair at failure with age-dependent cost $c(t)$. The cost of overhaul is c_O, and the cost of replacement is c_R. We have proposed a unique optimal maintenance policy which minimizes the expected cost rate under certain conditions.

We can consider that the system undergoes minimal repair or overhaul at failure. The choice between these two possible actions is based on some random mechanism.

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