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INTRODUCTION TO A PAIR OF MANIFOLDS CHARTED ON ONE MINKOWSKI DOMAIN

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Abstract: This paper introduces the idea of a "combined manifold", which essentially is the graph of a diffeomorphism from one manifold to another. Following a previously published work by the author, this present article addresses a combined space-time 4-manifold by introducing "combined particles" $\{p = (x, y)\}$, where $\{x\}$ and $\{y\}$ separately establish their own distinct 4-manifolds following General Relativity; this is made possible by assuming that $\{y\}$ only exert gravitational forces. We have proved that this combined 4-manifold is unique thanks to a distinct weighted average of the two metrics from a pair of sets of Einstein Field Equations, with the weights determined by two gravitational constants. As an illustration, we have decomposed the recognized gravitational constant G and the Earth mass, and as a motivation, we tender the particle-wave duality as a theoretical support of our hypothetical combined 4-manifold.

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1. Introduction

This paper introduces the idea of a "combined manifold" as a result of parametrizing two manifolds on the same domain; an elemental example is that of a diagonal map, $f(x) = (x, x)$ (cf. [2], p. 95). We exclusively focus on

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a combined 4-manifold sharing a common parameter domain that is the well-known Minkowski 4-space, i.e., a combined space-time manifold, for which we hypothesize a "combined particle" (see [3]) with mass summed by two parts – each part producing gravitational forces onto its counterparts of other masses. By Einstein's General Relativity (see [1]), either set of the two parts engenders its own space-time curvatures, resulting in a distinct manifold; thus, there are two sets of geodesics and gravitational motions, but clearly one particle cannot have two motions. Our goal is then to combine the two manifolds into one by determining the metric for the combined manifold (for a recent related work that decomposes 4-manifolds topologically, see [5]).

Section 2 below first formally defines combined manifolds and combined particles, proves that combined particles necessarily lead to a combined space-time 4-manifold, establishes propositions for the Newtonian gravitational motions of combined particles, presents a theorem that yields a unique metric for the combined 4-manifold, and remark on several experimental aspects of the model. Section 3 concludes with a summary.

2. A Combined Space-Time 4-Manifold

Definition 1. Let $n \in \mathbb{N}$, $\mathcal{M}^{[1]}$ and $\mathcal{M}^{[2]}$ be two differentiable n-manifolds. Assume that there exists a diffeomorphism $h : \mathcal{M}^{[1]} \longrightarrow \mathcal{M}^{[2]}$; then the product manifold

$$\mathcal{M}^{[3]} := \left\{ \left(p^{[1]}, p^{[2]}\right) \in \mathcal{M}^{[1]} \times \mathcal{M}^{[2]} \mid h\left(p^{[1]}\right) = p^{[2]} \right\}$$

is a combined n-manifold.

Remark 1. By definition, the graph of any diffeomorphism $h : \mathcal{M}^{[1]} \longrightarrow \mathcal{M}^{[2]}$ is a combined manifold; also, if two manifolds, $\mathcal{M}^{[1]}$ and $\mathcal{M}^{[2]}$, are parametrized by $f^{[1]}$ and $f^{[2]}$ on the same domain $U \subset \mathbb{R}^n$, then

$$\mathcal{M}^{[3]} := \left\{ p^{[3]} := \left(p^{[1]}, p^{[2]}\right) = \left(f^{[1]}(u), f^{[2]}(u)\right) \mid u \in U \right\} \tag{2.1}$$

is a combined n-manifold; however, we note that the tangent spaces are related by (cf. [2], p. 93)

$$T_{p^{[3]}}\mathcal{M}^{[3]} \subset T_{\left(p^{[1]}, p^{[2]}\right)}\left(\mathcal{M}^{[1]} \times \mathcal{M}^{[2]}\right) \simeq T_{p^{[1]}}\mathcal{M}^{[1]} \oplus T_{p^{[2]}}\mathcal{M}^{[2]}. \tag{2.2}$$

Definition 2. A combined particle $P_j, j \in \mathbb{N}$, is a point-like object (see [3]) that carries energy

$$E_j^{[3]} = E_j^{[1]} + E_j^{[2]}, \quad \text{with } E_j^{[1]}, E_j^{[2]} > 0; \tag{2.3}$$

$E_j^{[1]}$ exerts forces only onto $E_k^{[1]} > 0, k \in \mathbb{N} - \{j\}$, and $E_j^{[2]}$ only exerts gravitational forces onto $E_k^{[2]} > 0$, where $E_k^{[1]} + E_k^{[2]} = E_k^{[3]}$ is the energy carried by combined particle P_k.

Proposition 1. *The set of all combined particles* $\mathcal{P} \equiv \left\{P_j \mid j = 1, 2, \cdots, n \in \mathbb{N}\right\}$ *establishes a combined space-time 4-manifold* $\mathcal{M}^{[3]}$.

Proof. By General Relativity (see [1]), $\forall i = 1, 2, \{E_j^{[i]} \mid j = 1, 2, \cdots, n \in \mathbb{N}\}$ produce an energy-momentum tensor $T_{\mu\nu}^{[i]}$, so that the Einstein Field Equations *(where the notations are as follows:* $\forall \{\mu, \nu\} \subset \{1, 2, 3, 4\}$, $R_{\mu\nu}^{[i]}$ *and* $R^{[i]}$ *are the Ricci curvature tensors of types* $(0, 2)$ *and* $(0, 0)$, *as dependent on the metric* $g^{[i]}$, *and* $G^{[i]} > 0$ *is a gravitational constant),*

$$R_{\mu\nu}^{[i]} - \frac{1}{2} R^{[i]} \cdot g_{\mu\nu}^{[i]} = \frac{-8\pi G^{[i]}}{c^2} T_{\mu\nu}^{[i]} \tag{2.4}$$

yield metric $g^{[i]}$ for space-time manifold $\mathcal{M}^{[i]}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$, i.e., $\mathbf{e}_i^T :=$ (Kronecker $\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4}$), and $c \equiv$ the speed of light in the vacuum; define $\mathbb{R}^{1+3} := (\mathbb{R}, \eta)$, where

$$\eta := \operatorname{diag}\left(1, -\frac{1}{c^2}, -\frac{1}{c^2}, -\frac{1}{c^2}\right)_{\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}}$$

is the Minkowski metric. Then relative to any arbitrary observer there is a coordinate system $\mathbb{R}^{1+3}$ for $\mathcal{P} \equiv \{P_j \mid j = 1, 2, \cdots, n \in \mathbb{N}\}$, and $\forall U \subset \mathbb{R}^{1+3}$ the observer can define diffeomorphisms $f^{[i]} : U \longrightarrow \mathcal{M}^{[i]}$; as such,

$$\mathcal{M}^{[3]} := \left\{p^{[3]} := \left(p^{[1]}, p^{[2]}\right) = \left(f^{[1]}(u), f^{[2]}(u)\right) \mid u \in U\right\}$$

is a combined space-time 4-manifold. ☐

Proposition 2. *Assume that the known* space-time *is a combined 4-manifold* $\mathcal{M}^{[3]}$ *and that the above parametrizations* $f^{[i]} \simeq I$ *(the identity map,* $i = 1, 2$*); then the gravitational motions in* space-time $\mathcal{M}^{[3]}$ *are given by*

$$\left\|m^{[3]}\right\| \cdot \mathbf{a}^{[3]} = -\frac{G^{[3]} M^{[3]} \bullet m^{[3]}}{r^2}, \tag{2.5}$$

where (notice the differences in the fonts of the letter "M" for different denotations)

$M^{[3]} := \left(M^{[1]}, M^{[2]}\right) := \left(E_M^1/c^2, E_M^2/c^2\right)$,
$m^{[3]} := \left(m^{[1]}, m^{[2]}\right) := \left(E_m^1/c^2, E_m^2/c^2\right)$,
$M^{[3]} \bullet m^{[3]} := M^{[1]}m^{[1]} + M^{[2]}m^{[2]}$,
$G^{[3]}$ is the recognized universal gravitational constant
$\approx 6.7 \times 10^{-11} \left(\frac{\mathrm{m}^3}{kg \cdot s^2}\right)$, m $\equiv$ *meter*, $kg \equiv kilogram$, $s \equiv$ *second*, as assessed by $\left(M^{[2]}/M^{[1]}\right), \left(m^{[2]}/m^{[1]}\right) \gtrapprox 0$,
$r > 0$ is the distance between $M^{[3]}$ and $m^{[3]}$ in $\mathbb{R}^3$,
$\mathbf{a}^{[3]} \equiv$ the acceleration of $m^{[3]}$ to $M^{[3]}$, and
$\left\|m^{[3]}\right\| := \left\|\left(m^{[1]}, m^{[2]}\right)\right\|_{l^1} := \left\|m^{[1]}\right\|_{l^1} + \left\|m^{[2]}\right\|_{l^1} := m^{[1]} + m^{[2]}$.

Proof. Since $f^{[i]} \simeq I$, we have the Newton's law of gravitation as an approximations. By the definition of combined particles, $M^{[1]}$ and $m^{[2]}$ do not interact, and likewise, $M^{[2]}$ and $m^{[1]}$; thus, we have the gravitational force

$$\mathbf{F}^{[3]} = -\frac{G^{[3]} M^{[3]} \bullet m^{[3]}}{r^2}, \tag{2.6}$$

where $G^{[3]}$ is the recognized universal gravitational constant as $\mathcal{M}^{[3]}$ represents the known *space-time*. For the left-hand-side of the equation, we have $\left\|m^{[3]}\right\|$ as the inertial mass of $m^{[3]}$ by invoking the Principle of Equivalence between an inertial mass of a particle and its passive gravitational mass $\left\|m^{[3]}\right\|$ on the force-side of the equation. □

Theorem 1. $G^{[3]} = \frac{G^{[1]}G^{[2]}}{G^{[1]}+G^{[2]}}$.

Proof. $\forall i = 1, 2, 3$, we have for the metric tensor $g^{[i]}$ of $\mathcal{M}^{[i]}$ (see [1], p. 817; [4], p. 288)

$$g_{11,r}^{[i]} \approx 1 - \frac{2G^{[i]}\left\|M^{[i]}\right\|}{rc^2}, \quad r > 0. \tag{2.7}$$

By considering the construction of a bilinear form for the direct sum of two vector spaces, we set

$$g^{[3]} = w^{[1]}g^{[1]} + w^{[2]}g^{[2]}, \tag{2.8}$$

for some $w^{[1]}, w^{[2]} \geq 0$; then

$$w^{[1]}\left(1 - \frac{2G^{[1]}\left\|M^{[1]}\right\|}{rc^2}\right) + w^{[2]}\left(1 - \frac{2G^{[2]}\left\|M^{[2]}\right\|}{rc^2}\right) = 1 - \frac{2G^{[3]}\left\|M^{[3]}\right\|}{rc^2}, \tag{2.9}$$

implying that

$$w^{[1]} + w^{[2]} = 1. \tag{2.10}$$

Accordingly,

$$\left(\frac{2}{rc^2}\right) \cdot \left(w^{[1]} G^{[1]} M^{[1]} + \left(1 - w^{[1]}\right) G^{[2]} M^{[2]}\right) = \left(\frac{2}{rc^2}\right) \cdot G^{[3]} \left(M^{[1]} + M^{[2]}\right); \tag{2.11}$$

since $M^{[1]}$ and $M^{[2]}$ are arbitrary, we have

$$G^{[3]} = w^{[1]} G^{[1]} = \left(1 - w^{[1]}\right) G^{[2]} = G^{[2]} - w^{[1]} G^{[2]}, \tag{2.12}$$

i.e.,

$$w^{[1]} = \frac{G^{[2]}}{G^{[1]} + G^{[2]}}, \tag{2.13}$$

$$w^{[2]} = \frac{G^{[1]}}{G^{[1]} + G^{[2]}}, \tag{2.14}$$

$$\text{and } G^{[3]} = \frac{G^{[1]} G^{[2]}}{G^{[1]} + G^{[2]}}. \tag{2.15}$$

□

Corollary 1. *The gravitational motions in $\mathcal{M}^{[3]}$ are given by the geodesics as measured by*

$$g^{[3]} := \frac{G^{[2]}}{G^{[1]} + G^{[2]}} \cdot g^{[1]} + \frac{G^{[1]}}{G^{[1]} + G^{[2]}} \cdot g^{[2]}. \tag{2.16}$$

Corollary 2. *If the parametrizations $f^{[i]} \simeq I, i = 1, 2,$ then the gravitational motions of $m^{[3]} = \left(m^{[1]}, m^{[2]}\right)$ is $\left(m^{[1]} + m^{[2]}\right) \mathbf{a}^{[3]}$*

$$= -\left(\frac{G^{[2]}}{G^{[1]} + G^{[2]}}\right) \cdot \left(\frac{G^{[1]} M^{[1]} m^{[1]}}{r^2}\right) - \left(\frac{G^{[1]}}{G^{[1]} + G^{[2]}}\right) \cdot \left(\frac{G^{[2]} M^{[2]} m^{[2]}}{r^2}\right), \tag{2.17}$$

or

$$\mathbf{a}^{[3]} = -\frac{G^{[3]} \left\| M^{[3]} \right\|}{r^2} \left(\frac{M^{[1]}}{\left\| M^{[3]} \right\|} \frac{m^{[1]}}{\left\| m^{[3]} \right\|} + \frac{M^{[2]}}{\left\| M^{[3]} \right\|} \frac{m^{[2]}}{\left\| m^{[3]} \right\|}\right). \tag{2.18}$$

Proof. Apply Theorem 1 to Proposition 2. □

Remark 2. Whereas $\forall i = 1, 2$ the Principle of Equivalence yields $\mathbf{a}^{[i]} = -\left(\frac{G^{[i]}M^{[i]}}{r^2}\right)$, independent of the passive gravitational mass $m^{[i]}$, in $\mathcal{M}^{[3]}$ the acceleration of $m^{[3]}$ depends however on the material compositions, as shown in $\left(\frac{M^{[1]}}{\|M^{[3]}\|}\right)$ and $\left(\frac{m^{[1]}}{\|m^{[3]}\|}\right)$.

Remark 3. By the definition of a combined particle, $E^{[2]}$ is experimentally undetectable except for its gravitational properties; as such, in the event of pair-annihilation by its anti-particle (see [3]), the observed $\left\|M^{[3]}\right\|$, which equals $\frac{E^{[3]}}{c^2} = \frac{E^{[1]}}{c^2} + \frac{E^{[2]}}{c^2}$, only reveals the observable $E^{[1]}$, less than $E^{[3]}$ as accounted for by $\left\|M^{[3]}\right\|$. Since experimental records did not report such discrepancy of $\left(E^{[1]} - E^{[3]}\right) < 0$, we conclude that $M^{[1]}c^2 = E^{[1]} \approx E^{[3]} = M^{[3]}c^2$, i.e.,

$$\kappa \equiv \left(\frac{M^{[1]}}{\left\|M^{[3]}\right\|}\right) \lessapprox 1. \tag{2.19}$$

Remark 4. $\left(\frac{M^{[1]}}{\|M^{[3]}\|}\right) \equiv \kappa$ can be experimentally determined by pairing $M^{[3]}$ with another $m^{[3]} = \left(m^{[1]}, m^{[2]}\right)$ that has the same $\kappa = \left(\frac{m^{[1]}}{\|m^{[3]}\|}\right)$ (by observing the same set of ratios among the contained neutrons, protons, and electrons) through solving the following quadratic equations

$$\mathbf{a}^{[3]} = -\frac{G^{[3]}\left\|M^{[3]}\right\|}{r^2}\left(\frac{M^{[1]}}{\left\|M^{[3]}\right\|}\frac{m^{[1]}}{\left\|m^{[3]}\right\|} + \frac{M^{[2]}}{\left\|M^{[3]}\right\|}\frac{m^{[2]}}{\left\|m^{[3]}\right\|}\right) \text{ (equation (2.18))}$$

$$= -\frac{G^{[3]}\left\|M^{[3]}\right\|}{r^2}\left(\kappa^2 + (1-\kappa)^2\right) \tag{2.20}$$

for $\kappa := \max\{\kappa_1, \kappa_2\}$ of the two roots of the equation (2.20), where the selection of the two roots is made by a consideration of $\kappa \approx 1$ (inequality (2.19)).

Remark 5. Assume that the energies carried by light take the form of either $\left(E^{[1]}, 0\right)$ or $\left(0, E^{[2]}\right)$ (see [3]); then $\left(E^{[1]}, 0\right)$ is subject to the gravity of $G^{[1]}$ and $\left(0, E^{[2]}\right)$, that of $G^{[2]}$. As such, we define

$$G^{[3]} = G^{[1]} \text{ if } M^{[2]}m^{[2]} = 0, \tag{2.21}$$

$$\text{and } G^{[3]} = G^{[2]} \text{ if } M^{[1]}m^{[1]} = 0. \tag{2.22}$$

Since the deflection of light by gravity verified General Relativity as calculated via $G^{[3]}$, we conclude that $G^{[1]} \approx G^{[3]} = G^{[1]}\left(\frac{G^{[2]}}{G^{[1]}+G^{[2]}}\right)$ (from equation (2.15)), i.e.,

$$\left(\frac{G^{[2]}}{G^{[1]}+G^{[2]}}\right)=w^{[1]}\lessapprox 1 \text{ (from equation (2.13))}. \tag{2.23}$$

Example 1. Assume that $\kappa \equiv \left(\frac{M^{[1]}}{\left\|M^{[3]}\right\|}\right)=0.95$ for all combined particles and that $w^{[1]}=\frac{G^{[2]}}{G^{[1]}+G^{[2]}}=0.9$. We consider the gravity of Earth at its recognized radius $R\simeq 6.37\times 10^{6}$m. As is well-known,

$$\begin{aligned}
&\left|\mathbf{a}^{[3]}\right| \\
\approx\ & 9.8\left(\frac{\mathrm{m}}{s^2}\right) \\
\approx\ & \frac{G^{[3]}}{R^2}\times\left(5.98\times 10^{24}kg\right), \text{ where } 5.98\times 10^{24}kg \qquad (2.24)\\
& \text{is the deduced Earth mass by using } G^{[3]}, \\
\approx\ & \frac{G^{[3]}}{R^2}\times\left\{\left\|M^{[3]}\right\|\cdot\left[\kappa^2+(1-\kappa)^2\right]\right\} \quad \text{(by equation (2.20))} \\
=\ & \frac{G^{[3]}}{R^2}\times\left\{\left\|M^{[3]}\right\|\times 0.9050\right\}, \qquad (2.25)
\end{aligned}$$

implying that under the given assumptions the actual Earth mass

$$\begin{aligned}
\left\|M^{[3]}\right\| &= \frac{5.98\times 10^{24}kg}{0.9050}\approx 6.61\times 10^{24}kg, && (2.26)\\
\text{with } M^{[1]} &= 0.95\times\left\|M^{[3]}\right\|\approx 6.28\times 10^{24}kg, && (2.27)\\
\text{and } M^{[2]} &= 0.05\times\left\|M^{[3]}\right\|\approx 0.33\times 10^{24}kg. && (2.28)
\end{aligned}$$

Further,

$$\begin{aligned}
G^{[1]} &= \frac{G^{[3]}}{w^{[1]}}\approx\frac{6.7}{0.9}\times 10^{-11}\left(\frac{\mathrm{m}^3}{kg\cdot s^2}\right) && (2.29)\\
&\approx 7.4\times 10^{-11}\left(\frac{\mathrm{m}^3}{kg\cdot s^2}\right), \\
&\quad \text{as from equation (2.12)}, \\
\text{and } G^{[2]} &= \frac{G^{[3]}}{1-w^{[1]}}\approx 67\times 10^{-11}\left(\frac{\mathrm{m}^3}{kg\cdot s^2}\right). && (2.30)
\end{aligned}$$

Thus, for an $m^{[3]}=(57kg,3kg)$ on the surface of Earth, we have

$$\left\|m^{[3]}\right\|\cdot\left|\mathbf{a}^{[3]}\right|$$

$$
\begin{aligned}
&\approx 60kg \times 9.8 \left(\frac{\mathrm{m}}{s^2}\right) \\
&\approx 0.9 \times \left(\frac{G^{[1]}M^{[1]}m^{[1]}}{R^2}\right) + 0.1 \times \left(\frac{G^{[2]}M^{[2]}m^{[2]}}{R^2}\right) && (2.31) \\
&\approx 0.9R^{-2}\left\{\left[7.4 \times 10^{-11}\left(\frac{\mathrm{m}^3}{kg \cdot s^2}\right)\right] \times \left[6.28 \times 10^{24}kg\right] \times 57kg\right\} \\
&+ 0.1R^{-2}\left\{\left[67 \times 10^{-11}\left(\frac{\mathrm{m}^3}{kg \cdot s^2}\right)\right] \times \left[0.33 \times 10^{24}kg\right] \times 3kg\right\}. && (2.32)
\end{aligned}
$$

Since the force of $1pound(lb) :\cong 0.4536kg \times 9.8\left(\frac{\mathrm{m}}{s^2}\right) \approx 4.448kg\left(\frac{\mathrm{m}}{s^2}\right)$, we have alternatively

$$
\begin{aligned}
& 60kg \times 9.8\left(\frac{\mathrm{m}}{s^2}\right) \\
\approx\ & 132.28lb \\
\approx\ & 0.9 \times 146.57lb + 0.1 \times 3.65lb, \text{ where} && (2.33) \\
& \text{(i) } 146.57lb \\
\approx\ & 57kg \times 11.44\left(\frac{\mathrm{m}}{s^2}\right) \\
\approx\ & 57kg \times R^{-2}\left[7.4 \times 10^{-11}\left(\frac{\mathrm{m}^3}{kg \cdot s^2}\right) \times 6.28 \times 10^{24}kg\right], && (2.34) \\
& \text{and (ii) } 3.65lb \\
\approx\ & 3kg \times 5.41\left(\frac{\mathrm{m}}{s^2}\right) \\
\approx\ & 3kg \times R^{-2}\left[67 \times 10^{-11}\left(\frac{\mathrm{m}^3}{kg \cdot s^2}\right) \times 0.33 \times 10^{24}kg\right]. && (2.35)
\end{aligned}
$$

Example 2. Consider a hypothetical situation where Earth were made up of "dark matter", of mass $\left\|\left(0, M^{[2]}\right)\right\| = M^{[2]} = 0.33 \times 10^{24}kg$, attracting the same $m^{[3]} = (57kg, 3kg)$; then we have

$$
60kg \cdot \left|\mathbf{a}^{[3]}\right| = \frac{G^{[2]}M^{[2]} \cdot 3kg}{R^2}, \tag{2.36}
$$

i.e.,

$$
\begin{aligned}
\left|\mathbf{a}^{[3]}\right| &= \frac{1}{20} \times R^{-2} \times \left(\frac{G^{[3]}}{0.1}\right) \times 0.33 \times 10^{24}kg && (2.37) \\
&\approx \frac{1}{2}\left[R^{-2}G^{[3]} \times \left(5.98 \times 10^{24}kg\right)\right] \times \frac{0.33 \times 10^{24}kg}{5.98 \times 10^{24}kg}
\end{aligned}
$$

$$\approx \frac{1}{2} \times 9.8 \left(\frac{\mathrm{m}}{s^2}\right) \times 0.055 \tag{2.38}$$

$$\approx 2.8\% \times 9.8 \left(\frac{\mathrm{m}}{s^2}\right)$$

$$\approx 0.27 \left(\frac{\mathrm{m}}{s^2}\right). \tag{2.39}$$

3. Summary

This paper decomposes the known *space-time* into two parts by breaking the mass carried by a particle into two components: the first component exerts forces only onto the first components of other masses, and the second component only exerts gravitational forces onto the second components of other masses. Consequently, the two sets of energies produce two space-time manifolds by Einstein's General Relativity. By taking a special weighted average of the two metrics as determined by two sets of Einstein Field Equations, we have unified the gravitational motions of the two space-time manifolds. We have also suggested ways to test our hypothesis by setting up experiments to pursue the ratios of the two gravitational constants as well as the distribution of a mass between its two components. Lastly, we regard the particle-wave duality as a particularly convincing theoretical support of our model.

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