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THE SOLUTION FOR A KIRCHHOFF-TYPE BEAM EQUATIONS WITH NONLINEAR BOUNDARY

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Abstract: In this paper we establish one kind of equations of nonlinear beam subjected to axial forces and transverse load. We also prove the existence and uniqueness of global solutions.

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1. Introduction

This problem is based on the equation

$$u_{tt} + u_{xxxx} - (\alpha + \beta \int_0^l |u_x(s,t)|^2 ds) u_{xx} = 0,$$

which was proposed by Woioowsky-Krieger, see [8], as a model for vibrating beams with hinged ends. One of the first mathematical analysis for the Kirchhoff-type beam equation

$$u_{tt} + u_{xxxx} - M(\int_0^l |u_x|^2 dx) u_{xx} = 0,$$

was done by Ball [1], which was later extended to an abstract setting by

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Medeiros [6]. Tucsnak [7] considered the beam equation

$$u_{tt} + \triangle^2 u - M(||\nabla u||_2^2)\triangle u = 0 \text{ in } \Omega \in \mathbf{R}^n$$

with clamped boundary. He obtained the exponential decay of the energy when a damping of the type $a(x)u_t$ is effective near the boundary. the Cauchy problem and initial-boundary value problem with linear boundary condition for the Kirchhoff-type beam equation were studied, using different methods, by the Biler [2], Brito [3], Kouemou Patcheu [4], etc. T.F. Ma [5] considered the Kirchhoff-type beam which is clamped at $x = 0$ and $x = L$ by a bearing with nonlinear response characterized by the function f, he proved the existence and decay rates. In this paper, we are concerned with the existence and uniqueness for the solution of the nonlinear beam equation with load

$$u_{tt} + u_{xxxx} - M(\int_0^l |u_x|^2 dx)u_{xx} = Q(t), \tag{1.1}$$

with nonlinear boundary conditions

$$u(0,t) = u_x(0,t) = u_{xx}(l,t) = 0, \tag{1.2}$$

$$u_{xxx}(l,t) - M(\int_0^l |u_x|^2 dx)u_x(l,t) = f(u(l,t)) + g(u_t(l,t)), \tag{1.3}$$

and initial conditions

$$u(0,t) = u^0(x), \quad u_t(0,t) = u^1(x), \tag{1.4}$$

where $u = u(x,t)$ is transverse deflection of nonlinear beam.

2. Assumptions and the Result

In this paper, we use only standard notations, sometimes, a function $u = u(x,t)$ will simply be denoted by $u(t)$ for the sake of simplicity. Our analysis is based on the Sobolev spaces

$$V = \{u \in H^2(0,l) | u(0) = u_x(0) = 0\},$$

equipped with the norm $\|w\|_V = \|w_{xx}\|_2$, and

$$W = \{u \in V \cap H^4(0,l) | u_{xx}(l) = 0\},$$

equipped with the norm $\|w\|_W = \|w_{xxx}\|_2 + \|w_{xxxx}\|_2$, where $||\cdot||_p$ denotes the L^p norms

We assume that $f, g : \mathbf{R} \to \mathbf{R}$ are continuously differentiable functions such that

$$f(s)s \geq 0 \ \text{ and } \ f(s)s - 2\hat{f}(s) \geq 0, \ \ \forall s \in \mathbf{R}, \tag{2.1}$$

where $\hat{f}(s) = \int_0^s f(z)dz$, and

$$g(0) = 0, \quad (g(r) - g(s))(r - s) \geq \rho|r - s|^2, \ \ \forall s \in \mathbf{R}, \tag{2.2}$$

for some $\rho > 0$.

Our result is the following theorem.

Theorem. *Let $M \in C^1([0, \infty))$ be a non-negative function, $Q \in L^2([0, \infty); L^2(0, l))$ and assume that* (2.1)*and* (2.2) *hold. Then for any $u^0, u^1 \in W$ satisfying the compatibility condition*

$$u^0_{xxx}(l) - M(||u^0_x||^2_2)u^0_x(l) = f(u^0(l)) + g(u^1(l)), \tag{2.3}$$

there exists unique function

$$u \in L^2(0, \infty; W) \cap C^0([0, \infty; V) \cap W^{2,\infty}(0, \infty; L^2(0, l)) \tag{2.4}$$

satisfying (1.1)–(1.4).

3. Proof of Theorem (Existence and Uniqueness)

Step 1. Approximating solution. Let us solve the variational problem associated with (1.1)-(1.4), which is given by: find $u(t) \in W$ such that

$$\int_0^l u_{tt}(t)wdx + \int_0^l u_{xx}(t)w_{xx}dx + M(||u_x(t)||^2_2)\int_0^l u_x(t)w_xdx$$
$$+ f(u(l, t))w(l) + g(u_t(l, t))w(l) = \int_0^l Q(t)wdx, \tag{3.1}$$

for all $w \in V$. This is done with the Galerkin approximations. Let $\{w^j\}$ be a complete orthogonal system of W for which

$$\{u^0, u^1\} \in \text{Span}\,\{w^1, w^2\}.$$

For each $m \in \mathbf{N}$, let us put $W_m = \text{Span}\,\{w^1, w^2, \cdots, w^m\}$. We search for a function

$$u^m(t) = \sum_{j=1}^m k^j(t)w^j,$$

such that for any $w \in W_m$, it satisfies the approximate equation

$$\int_0^l u_{tt}^m(t) w dx + \int_0^l u_{xx}^m(t) w_{xx} dx + M(||u_x^m(t)||_2^2) \int_0^l u_x^m(t) w_x dx \\ + f(u^m(l,t))w(l) + g(u_t^m(l,t))w(l) = \int_0^l Q(t) w dx, \quad (3.2)$$

with the initial conditions

$$u^m(0) = u^0, \quad u_t^m(0) = u^1\,, \quad (3.3)$$

which are possible since $u^0, u^1 \in W_m$ for $m \geq 2$. We note that (3.2) and (3.3) are in fact an $m \times m$ system of ODEs in the variable t, which is known to have a local solution $u^m(t)$ in an interval $[0, t_m)$. After the estimate below the approximate solution $u^m(t)$ will be extended to the interval $[0, T]$, for any given $T > 0$.

Step 2. A priori estimates. We define the energy of the system (1.1)–(1.4) by

$$E(t) = \frac{1}{2}||u_t(t)||_2^2 + \frac{1}{2}||u_{xx}(t)||_2^2 + \frac{1}{2}\hat{M}(||u_x(t)||_2^2) + \hat{f}(u(l,t)),$$

where $\hat{M}(s) = \int_0^s M(z)dz$.

Estimate 1. *There exists $M_1 > 0$ such that*

$$||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \int_0^T g(u_t^m(l,t)) u_t^m(l,t) dt \leq M_1, \quad (3.4)$$

for all $t \in [0, T]$ and for all $m \in \mathbf{N}$.

Proof. By integration of (3.2) with $w = u_t^m(t)$ we see that

$$\frac{1}{2}\frac{d}{dt}\{||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \hat{M}(||u_x^m(t)||_2^2) + 2\hat{f}(u^m(l,t))\} \\ + g(u_t^m(l,t)) u_t^m(l,t) = \int_0^l Q(t) u_t^m(t) dx.$$

Then integrating from 0 to $t < t_m$, we get

$$||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \hat{M}(||u_x^m(t)||_2^2) + 2\hat{f}(u^m(l,t)) \\ + 2\int_0^t g(u_t^m(l,s)) u_t^m(l,s) ds = 2E(0) + 2\int_0^t (\int_0^l Q(s) u_t^m(s) dx) ds.$$

Taking $s = 0$ in (2.2), we obtain that

$$g(r)r \geq \rho r^2, \quad \forall s \in \mathbf{R}.$$

Indeed, it follows from $g(r)r \geq \rho r^2$ that

$$||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \hat{M}(||u_x^m(t)||_2^2) + 2\hat{f}(u^m(l,t)) \\ + 2\int_0^t \rho(u_t^m(l,s))^2 ds \leq 2E(0) + 2\int_0^t (\int_0^l Q(s)u_t^m(s)dx)ds.$$

Therefore

$$||u_t^m(t)||_2^2 - \int_0^t ||u_t^m(t)||_2^2 ds + ||u_{xx}^m(t)||_2^2 + \hat{M}(||u_x^m(t)||_2^2) + 2\hat{f}(u^m(l,t)) \\ + 2\int_0^t \rho(u_t^m(l,s))^2 ds \leq 2E(0) + \int_0^t ||Q(s)||_2^2 ds.$$

Set $\int_0^\infty ||Q(s)||_2^2 ds = \|Q\|^2_{c^2([0,+\infty);L^2(0,l))} = c < +\infty$. We infer from (2.2) $\hat{f}(s) \geq 0$, using $\hat{f}(s) \geq 0$ and $M > 0$, it follows that $\hat{M}(\|u_x^m(t)\|_2^2) \geq 0$. Therefore, we deduce that

$$||u_t^m(t)||_2^2 - \int_0^t ||u_t^m(s)||_2^2 ds \leq 2E(0) + c.$$

Set $Y(t) = \|u_t^m(t)\|_2^2$, we have $Y(t) - \int_0^t Y(s)ds \leq 2E(0) + c$, then

$$Y(t) \leq (2E(0) + c)(1 + \exp(t) \leq (2E(0) + c)(1 + \exp(T)),$$

hence $\|u_t^m(t)\|_2^2 \leq (2E(0) + c)(1 + \exp(T))$. There exists $M_1 > 0$ such that

$$||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \int_0^T g(u_t^m(l,t))u_t^m(l,t)dt \leq M_1,$$

for all $t \in [0, T]$ and for all $m \in \mathbf{N}$. □

Estimate 2. *There exists $M_2 > 0$ such that*

$$||u_{tt}^m(0)||_2 \leq M_2, \quad \forall m \in \mathbf{N}. \tag{3.5}$$

Proof. Integrating by parts (3.2), with $w = u_{tt}^m(0)$ and $t = 0$, we get

$$||u_{tt}^m(0)||_2^2 + \int_0^l u_{xx}^0 u_{xxtt}^m(0)dx + M(||u_x^0||_2^2) + \int_0^l u_x^0 u_{xtt}^m(0)dx$$

$$+ (f(u^0(l) + g(u^1(l))u_{tt}^m(l,0) = \int_0^l Q(0)u_{tt}^m(0)dx.$$

Since

$$\begin{aligned}\int_0^l u_{xx}^0 u_{xxtt}^m(0)dx + M(||u_x^0||_2^2) + \int_0^l u_x^0 u_{xtt}^m(0)dx \\ = -(u_{xxx}^0 - M(||u_x^0||_2^2)u_x^0(l))u_{tt}^m(l,0) \\ + \int_0^l u_{xxxx}^0 u_{tt}^m(0)dx - M(||u_x^0||_2^2)\int_0^l u_{xx}^0 u_{tt}^m(0)dx,\end{aligned}$$

we infor from the compatibility (2.3)

$$\begin{aligned}||u_{tt}^m(0)||_2^2 = -\int_0^l u_{xxxx}^0 u_{tt}^m(0)dx \\ + M(||u_x^0||_2^2)\int_0^l u_{xx}^0 u_{tt}^m(0)dx + \int_0^l Q(0)u_{tt}^m(0)dx \\ \le (\|u_{xxxx}^0\|_2 + M(\|u_x^0\|_2^2)\|u_{xx}^0\|_2 + \|Q(0)\|_2)\|u_{tt}^m(0)\|_2,\end{aligned}$$

and there exists $M_2 > 0$ such that

$$||u_{tt}^m(0)|| \le M_2, \quad \forall m \in \mathbf{N}. \qquad \square$$

Estimate 3. *There exists $c > 0$ such that*

$$\begin{aligned}&||u_t^m(t)||_2, \;\; ||u_{xx}^m(t)||_2, \;\; ||u_{txx}^m(t)||_2, \;\; ||u_{tt}^m(t)||_2 \le c, \\ &\forall m \in \mathbf{N}, \;\; t \in [0,T],\end{aligned} \tag{3.6}$$

for all $t \in [0,T]$ and for all $m \in \mathbf{N}$.

Proof. Let us fix $t, \xi > 0$ such that $\xi < T - t$. Taking the difference of (3.2) $t = t + \xi$ and $t = t$, and replacing w by $u_t^m(t+\xi) - u_t^m(t)$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{||u_t^m(t+\xi) - u_t^m(t)||_2^2 + ||u_{xx}^m(t+\xi) - u_{xx}^m t(t)||_2^2\} + (g(u_t^m(l,t+\xi)) \\ - g(u_t^m(l,t)))(u_t^m(l,t+\xi) - u_t^m(l,t)) + (f(u^m(l,t+\xi)) - f(u^m(l,t)))(u_t^m(l,t+\xi) \\ - u_t^m(l,t)) + I_1 = \int_0^l (Q(t+\xi) - Q(t))(u_t^m(t+\xi) - u_t^m(t))dx \\ \le c||u_t^m(t+\xi) - u_t^m(t)||_2,\end{aligned} \tag{3.7}$$

where

$$||Q(t+\xi)-Q(t)||_2 \le c.$$

Set

$$I_1=\int_0^l (M(||u_x^m(t+\xi)||_2^2)u_x^m(t+\xi)-M(||u_x^m(t)||_2^2)u_x^m(t))(u_{xt}^m(t+\xi) \\ - u_{xt}^m(t))dx.$$

Let us estimate $|I_1|$. Since $u(0)=u_x(0)=u_{xx}(l)=0$ for $u\in W$, we have

$$||u||_\infty \le \sqrt{l}||u_x||_2,\ \ ||u_x||_\infty \le \sqrt{l}||u_{xx}||_2,\ \ ||u_x||_2 \le l||u_{xx}||_2. \tag{3.8}$$

Noting that

$$I_1 = M(||u_x^m(t+\xi)||_2^2)\int_0^l (u_x^m(t+\xi)-u_x^m(t))(u_{xt}^m(t+\xi)-u_{xt}^m(t))dx \\ + \triangle M\int_0^l u_x^m(t)(u_{xt}^m(t+\xi)-u_{xt}^m(t))dx,$$

where

$$\triangle M = M(||u_x^m(t+\xi)||_2^2)-M(||u_x^m(t)||_2^2),$$

integrating by parts we have that

$$I_1 = M(||u_x^m(t+\xi)||_2^2)(u_x^m(l,t+\xi)-u_x^m(l,t))(u_t^m(l,t+\xi)-u_t^m(l,t)) \\ - M(||u_x^m(t+\xi)||_2^2)\int_0^l (u_{xx}^m(t+\xi)-u_{xx}^m(t))(u_t^m(t+\xi)-u_t^m(t))dx \\ + \triangle M u_x^m(l,t)(u_t^m(l,t+\xi)-u_t^m(l,t)) - \triangle M\int_0^l u_{xx}^m(t)(u_t^m(t+\xi)-u_t^m(t))dx.$$

Since $M\in C^1([0,\infty))$, we have

$$|\triangle M| \le |M'(\eta)|\cdot|\|u_x^m(t+\xi)\|_2^2-\|u_x^m(t)\|_2^2| \\ = |M'(\eta)|\cdot|\int_0^l ((u_x^m(t+\xi))^2-(u_x^m(t))^2)dx| \\ \le |M'(\eta)|\cdot\|u_x^m(t+\xi)+u_x^m(t)\|_2\cdot\|u_x^m(t+\xi)-u_x^m(t)\|_2,\ \ \forall t\in[0,T].$$

From (3.8), we deduce that $\|u_x^m(t)\|_2 \le l\|u_{xx}^m(t)\|_2$. From (3.4) it follows that $\|u_{xx}^m(t)\|_2^2 \le \sqrt{M_1}$. So $\|u_x^m(t)\|_2 \le l\sqrt{M_1}$, hence

$$|\triangle M| \le 2c_0 l\sqrt{M_1}\|u_x^m(t+\xi)-u_x^m(t)\|_2.$$

There exists $c_1 > 0$ such that

$$|\triangle M| \leq c_1 \|u_x^m(t+\xi) - u_x^m(t)\|_2 \leq c_1 l \|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2.$$

Using $M(\|u_x^m(t+\xi)\|_2^2) \leq c$, we conclude that

$$\begin{aligned}
|I_1| \leq c| \int_0^l (u_{xx}^m(s,t+\xi) - u_{xx}^m(s,t))ds| \cdot |u_t^m(l,t+\xi) - u_t^m(l,t)| \\
+ c\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 \cdot \|u_t^m(t+\xi) - u_t^m(t)\|_2 \\
+ c_1 l \|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 \cdot | \int_0^l u_{xx}^m(s,t))ds| \cdot |u_t^m(l,t+\xi) - u_t^m(l,t)| \\
+ c_1 l \|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 \cdot \|u_{xx}^m(t)\|_2 \cdot \|u_t^m(t+\xi) - u_t^m(t)\|_2 \\
\leq D\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 \cdot (|u_t^m(l,t+\xi) - u_t^m(l,t)| + \|u_t^m(t+\xi) - u_t^m(t)\|_2),
\end{aligned}$$

where $D = (\sqrt{l}+1)(c + c_1 l\sqrt{M_1})$. Since

$$\begin{aligned}
&\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 \|u_t^m(t+\xi) - u_t^m(t)\|_2 \\
&\qquad \leq \frac{D}{4}\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2^2 + \frac{1}{D}\|u_t^m(t+\xi) - u_t^m(t)\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2 |u_t^m(l,t+\xi) - u_t^m(l,t)| \\
&\qquad \leq \frac{D}{2\rho}\|u_{xx}^m(t+\xi) - u_{xx}^m(t)\|_2^2 + \frac{\rho}{2D}|u_t^m(l,t+\xi) - u_t^m(l,t)|^2,
\end{aligned}$$

so there exists $c_2 > 0$ such that

$$\begin{aligned}
|I_1| \leq c_2 ||u_{xx}^m(t+\xi) - u_{xx}^m(t)||_2^2 + ||u_t^m(t+\xi) - u_t^m(t)||_2^2 \\
+ \frac{\rho}{2}|u_t^m(l,t+\xi) - u_t^m(l,t)|^2, \quad (3.9)
\end{aligned}$$

where $c_2 = \frac{D^2}{4} + \frac{D^2}{2\rho}$. Since $f : \mathbf{R} \to \mathbf{R}$ is continuous differentiable function, therefore, we conclude that

$$\begin{aligned}
&|f(u^m(l,t+\xi)) - f(u^m(l,t))||u_t^m(l,t+\xi) - u_t^m(l,t)| \\
&\qquad \leq |f'(\eta)||u^m(l,t+\xi) - u^m(l,t)||u_t^m(l,t+\xi) - u_t^m(l,t)|,
\end{aligned}$$

where η is between $u^m(l,t+\xi)$ and $u^m(l,t)$. Since $u^m(l,t) = \int_0^l u_x^m(s,t)ds$, we have

$$|u^m(l,t)| \leq \sqrt{l}||u_x^m(t)||_2 \leq l\sqrt{l}||u_{xx}^m(t)||_2 \leq l\sqrt{lM_1},$$

hence $|f'(\eta)| \leq c$. Therefore, we conclude that

$$\begin{aligned}
&|f(u^m(l,t+\xi)) - f(u^m(l,t))||u_t^m(l,t+\xi) - u_t^m(l,t)| \\
&\qquad \leq cl\sqrt{l}||u_{xx}^m(t+\xi) - u_{xx}^m(t)||_2|u_t^m(l,t+\xi) - u_t^m(l,t)| \\
&\qquad \leq c_3||u_{xx}^m(t+\xi) - u_{xx}^m(t)||_2^2 + \frac{\rho}{2}|u_t^m(l,t+\xi) - u_t^m(l,t)|^2, \quad (3.10)
\end{aligned}$$

where $c_3 = \frac{c^2 l^3}{2\rho}$. Set

$$\phi^m(t,\xi) = ||u_{xx}^m(t+\xi) - u_{xx}^m(t)||_2^2 + ||u_t^m(t+\xi) - u_t^m(t)||_2^2, \ \ \forall t \in [0,T],$$

without loss of generality, we can take assumption $c_2 + c_3 \geq 1$. Taking into account (3.9) and (3.10), we deduce from (3.6) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\phi^m(t,\xi) + (g(u_t^m(l,t+\xi)) - g(u_t^m(l,t)))(u_t^m(l,t+\xi) - u_t^m(l,t)) \\
&\qquad\qquad \leq (c_2+c_3)\phi^m(t,\xi) + \rho|u_t^m(l,t+\xi) - u_t^m(l,t)|^2.
\end{aligned}$$

Then from (2.2) we get $\frac{d}{dt}\phi^m(t,\xi) \leq 2(c_2+c_3)\phi^m(t,\xi)$ and there

$$\phi^m(t,\xi) \leq \phi^m(0,\xi)\exp((2(c_2+c_3)T)), \ \ \forall t \in [0,T].$$

Dividing the above inequality by ξ^2 and let $\xi \to 0$ gives

$$||u_{txx}^m(t)||_2^2 + ||u_{tt}^m(t)||_2^2 \leq (||u_{xx}^1||_2^2 + ||u_{tt}^m(0)||_2^2)\exp(2(c_2+c_3)T).$$

From Estimate 1 and Estimate 2 we find a constant $M_3 > 0$ depending only in T, such that

$$||u_{txx}^m(t)||_2^2 + ||u_{tt}^m(t)||_2^2 \leq M_3, \quad \forall m \in \mathbf{N}, \ \ \forall t \in [0,T]. \qquad (3.11)$$

It follows from (3.4) and (3.11) that

$$||u_t^m(t)||_2, \ \ ||u_{xx}^m(t)||_2, \ \ ||u_{xx}^m(t)||_2, \ \ ||u_{tt}^m(t)||_2 \leq c, \ \ \forall m \in \mathbf{N}, \ \ \forall t \in [0,T].$$

From (3.8) it follows that

$$||u_{tx}^m(t)||_2, \ \ ||u_x^m(t)||_2, \ \ ||u^m(t)||_2 \leq c, \ \ \forall m \in \mathbf{N}, \ \ \forall t \in [0,T].$$

It follows from Estimates 1-3 that

$$||u_t^m(t)||_2^2 + ||u_{xx}^m(t)||_2^2 + \int_0^T g(u_t^m(l,t))u_t^m(l,t)dt \leq M_1,$$

$$||u_{tt}^m(0)||_2 \leq M_2, \quad \forall m \in \mathbf{N}.$$

Therefore $u^m(t)$ will be extended to the interval $[0, T]$, for any given $T > 0$.

Step 3. Convergence. We can use Lions-Aubin Lemma to get the necessary compactness in order to pass (3.2) to the limit. Then it is a matter of routine to conclude the existence of global solution in $[0, T]$

Step 4. Uniqueness. Let u and v be two solution of (1.1)-(1.4) with the same initial data. Then writing $z = u - v$, we see that $z(0) = z_t(0) = 0$ and from (3.1)

$$\begin{aligned} 0 = \int_0^l z_{tt}(t) w dx + \int_0^l z_{tt}(t) w_{xx} dx + M(||u_x^m(t)||_2^2) \int_0^l u_x(t) w_x dx \\ - M(||v_x(t)||_2^2) \int_0^l (v_x(t) w_x dx + (f(u(l,t)) - f(v(l,t))) w(l,t) \\ + (g(u_t(l,t)) - g(v_t(l,t))) w(l,t). \end{aligned}$$

Putting $w = z_t$ and using Mean Value Theorem, (3.4), (3.7) and Young inequalities as in Estimate 3, *we* deduce that for some constant $c > 0$

$$\frac{d}{dt}(||z_t(t)||_2^2 + ||z_{xt}||_2^2) \leq c(||z_t(t)||_2^2 + ||z_{xt}||_2^2), \quad \forall t \in (0, T).$$

Then from Gronwall's Lemma we see that $u = v$.

The proof is now complete. □

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