International Journal of Pure and Applied Mathematics
Volume 35 No. 3 2007, 329-334

ON SUMMING MAPS

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Abstract: In the present paper, we show that a bounded linear map T from a Hilbert space H into a reflexive Banach space F is an absolutely 2-summing map iff it is a 2-quasi-nuclear map. Also, for a nuclear G_∞-spaces $\lambda(P)$ and $\lambda(Q)$, we show that a bounded linear map T between normed spaces is 2-quasi-$\lambda(P) * \lambda(Q)$ iff it is quasi-$\lambda(P) * \lambda(Q)$-nuclear. Then we introduce an example to show that the nuclearity of $\lambda(P)$ and $\lambda(Q)$ are necessary.

AMS Subject Classification: 46A45, 47B10, 47L20
Key Words: nuclear maps, absolutely 2-summing maps, quasi λ-nuclear maps

1. Basic Concepts

The concept of 2-quasi-nuclear map is introduced by Pitsich in 1967. While Dubinsiky introduced quasi-λ nuclear map. In 2005, Shatanawi introduced the concept of 2-quasi-λ-nuclear map. In this paper, we study the relation between 2-quasi-nuclear maps and absolutely 2-summing maps from a Hilbert space H into a reflexive Banach space F. Also we study some properties of 2-quasi-λ-nuclear map.

Received: November 21, 2006

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In the sequel E' denotes to the set of all bounded linear functional. For two sequences of real numbers $x = (x_n)$ and $y = (y_n)$ we write $x_n = O(y_n)$ if there is a positive real number ρ such that $x_n \leq \rho y_n$ for all $n \in \mathbf{N}$.

A set A of sequences of non-negative real numbers is called a *Köthe set*, if it satisfies the following conditions:

1. For each pair of elements $a = (a_n), b = (b_n) \in A$ there is $c = (c_n) \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.

2. For every $r \in N$ there exists $a = (a_n) \in A$ with $a_r > 0$.

The space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n| \, a_n < +\infty$$

for all $a \in A$, is called the *Köthe space*, $\lambda(A)$, generated by A, see [6].

A Köthe set P will be called a *power set of infinite type* if it satisfies the following conditions:

1. For each $a = (a_n) \in P$, $1 \leq a_n \leq a_{n+1}$ for all n.

2. For each $a = (a_n) \in P$, there exists $b = (b_n) \in P$ such that $a_n^2 = O(b_n)$.

A Köthe space of the form $\lambda(P)$, where P is a power set of infinite type is called a G_∞*-space* or a *smooth sequence space of infinite type*, see [6].

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then $P_\infty = \{(k^{\alpha_n}) : \ k \in \mathbf{N}\}$ is a countable Köthe set. The corresponding Köthe space $\Lambda_\infty(\alpha) = \lambda(P_\infty)$ is called the *power series of infinite type*.

A linear map T from a normed space E into a normed space F is called a *2-quasi-nuclear map*, if there is a sequence (a_n) in E' with

$$\sum\nolimits_n ||a_n||^2 < +\infty \qquad \text{such that} \qquad ||Tx|| \leq \left(\sum\nolimits_n |\langle x, a_n\rangle|^2\right)^{1/2}, \text{ see [3]},$$

and *absolutely 2-summing*, if there is a positive constant c such that for an arbitrary sequence (x_n) in E we have

$$\left(\sum_n ||Tx_n||^2\right)^{1/2} \leq c \sup\left\{\left(\sum\nolimits_n |\langle x_n, a\rangle|^2\right)^{1/2} : \ a \in E', \ ||a|| \leq 1\right\}, \text{ see [2].}$$

A linear map T of a normed space E into a normed space F is called a *quasi-λ-nuclear map* if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that $||Tx|| \leq \sum_n |\alpha_n| \, |\langle x, a_n\rangle|$, for all $x \in E$, see [1], [4], and a *2-quasi-λ-nuclear map*, if there exist a sequence (α_n) in λ and a bounded sequence (a_n) in E' such that

$$||Tx|| \leq \left(\sum_n |\alpha_n| \, |\langle x, a_n\rangle|^2\right)^{1/2},$$

for all x in E, see [5].

In this paper, we study the relationship between absolutely 2-summing map and 2-quasi-nuclear map T from a Hilbert space H into a reflexive Banach space F. Also, for nuclear G_∞-spaces $\lambda(P)$ and $\lambda(Q)$, we study the relation between a 2-quasi-$\lambda(P) * \lambda(Q)$ map and a quasi-$\lambda(P) * \lambda(Q)$-nuclear map.

2. Main Results

We start our work by giving a characterization of an absolutely 2-summing map between a Hilbert space H and a reflexive Banach space F.

Theorem 2.1. *Suppose that H is a Hilbert space and F is a reflexive Banach space. A bounded linear map T from H into F is 2-quasi nuclear iff it is absolutely 2-summing.*

Proof. From the definitions of a 2-quasi-nuclear map and absolutely 2-summing we can show that if T is 2-quasi-nuclear, then T is absolutely 2-summing. For the other reverse, assume that T is absolutely 2-summing. Then there exists a compact Hausdorff space K and a positive radon measure μ on K with $||\mu|| = 1$ such that T can be factored through the identity map $I : C(K) \to L_2(K, \mu)$ such that I is absolutely 2-summing; that is, there is a bounded linear maps $P : H \to C(K)$ and $Q : L_2(K, \mu) \to F$ such that $T = QIP$. As I is an absolutely 2-summing map, then IP is absolutely 2-summing. Since H and $L_2(K, \mu)$ are Hilbert spaces, we have IP is 2-quasi-nuclear, and therefore $(IP)'$ is 2-quasi-nuclear. Since $T' = (IP)'Q'$, it follows that T' is 2-quasi-nuclear. Since F is reflexive, we get the 2-quasi-λ-nuclearity of T. □

For nuclear Köthe spaces $\lambda(A)$ and $\lambda(B)$, it is known that $\lambda(A) * \lambda(B) = \lambda(A) \times \lambda(B)$, where

$$\lambda(A) * \lambda(B) := \{(x_n) : (x_{2n-1}) \in \lambda(A),\ (x_{2n}) \in \lambda(B)\}.$$

For nuclear G_∞ spaces $\lambda(P)$ and $\lambda(Q)$, our next result indicates the relation between a 2-quasi-$\lambda(P) * \lambda(Q)$-nuclear map and a quasi-$\lambda(P) * \lambda(Q)$-nuclear map

Theorem 2.2. *Suppose that $\lambda = \lambda(P) * \lambda(Q)$, where $\lambda(P)$ and $\lambda(Q)$ are nuclear G_∞-spaces, a bounded linear map between normed spaces is 2-quasi-λ-nuclear iff it is quasi-λ-nuclear.*

Proof. The proof follows from [5, Theorem 2.1] and the fact that for nuclear G_∞ space $\lambda(P)$ and $\lambda(Q)$, we have $(\sqrt{\alpha_n}) \in \lambda(P) * \lambda(Q)$ for all $(\alpha_n) \in \lambda(P) * \lambda(Q)$. □

Now let as arise the following question.

Question. Suppose that $\lambda(P)$ and $\lambda(Q)$ are G_∞-spaces. Does Theorem 2.2 still valid for the sequence space $\lambda = \lambda(P) * \lambda(Q)$ in case $\lambda(P)$ or $\lambda(Q)$ is not nuclear?

The answer of this question is given by the following example.

Example 2.1. Define a map $D : c_0 \to \ell_2$ by putting

$$Dx = (\frac{x_1}{1}, \frac{x_2}{2^2}, \frac{x_3}{3}, \frac{x_4}{4^4}, \ldots).$$

Then D is 2-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear which is not quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear.

Proof. Note that

$$||Dx||_2^2 = \sum_{n=1}^{\infty} |\frac{x_{2n-1}}{(2n-1)}|^2 + \sum_{n=1}^{\infty} |\frac{x_{2n}}{(2n)^{2n}}|^2.$$

Let $a_n = e_n$ and

$$(\beta_n) = (1, \frac{1}{2^4}, \frac{1}{3^2}, \frac{1}{4^8}, \frac{1}{5^2}, \frac{1}{6^{12}}, \frac{1}{7^2}, \ldots).$$

As (e_n) is a bounded sequence in c_0, $(\beta_n) \in \ell_1 * \Lambda_\infty(n)$ and

$$||Dx||_2^2 = \sum_n |\beta_n||\langle x, e_n\rangle|^2,$$

we have D is 2-quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear. Suppose that D is quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear. Then there are a sequence (α_n) in $\ell_1 * \Lambda_\infty(n)$ and a bounded sequence $(a_n := (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots))_{n=1}^\infty$ in ℓ_1 such that

$$||Tx|| \leq \sum_{n=1}^{\infty} |\alpha_n||\langle a, a_n\rangle|.$$

For $m \in N$, we have

$$\frac{1}{2m-1} \leq \sum_{n=1}^{\infty} |\alpha_{n=1}||\langle e_{2m-1}, a_n\rangle|,$$

and

$$\frac{1}{2m} \leq \sum_{n=1}^{\infty} |\alpha_n||\langle e_{2m}, a_n\rangle|.$$

Since $\langle e_{2m-1}, a_n\rangle = a_{2m-1}^{(n)}$ and $\langle e_{2m}, a_n\rangle = a_{2m}^{(n)}$, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{2m-1} + \sum_{m=1}^{\infty} \frac{1}{(2m)^{2m}} &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n||a_m^{(n)}| \\
&= \sum_{n=1}^{\infty} |\alpha_n| \sum_{m=1}^{\infty} |a_m^{(n)}| \\
&= \sum_{n=1}^{\infty} |\alpha_n|||a_n||.
\end{aligned}$$

Which is a contradiction, hence D is not quasi-$\ell_1 * \Lambda_\infty(n)$-nuclear, this complete the proof. □

We follow the same arguments in proving Example 2.1 to prove our next example.

Example 2.2. Define a map $D : c_0 \leftarrow \ell_2$ by putting $Dx = (\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots)$. Then D is 2-quasi-ℓ_1-nuclear which is not quasi-ℓ_1-nuclear.

Remark. Since ℓ_1 is a non nuclear G_∞-space, Example 2.2 is a solution for the question appeared in [5].

As a consequence result of Theorem 2.2, we have the following result.

Corollary 2.1. *Suppose that $\lambda = \lambda(P)$ is a nuclear G_∞-space and T is a bounded linear map from a normed space E into a normed space F. Then the following are equivalent:*

1. *T is 2-quasi-$\lambda\lambda^\times$-nuclear.*
2. *T is 2-quasi-λ-nuclear.*
3. *T is quasi-λ-nuclear.*
4. *T is quasi-$\lambda\lambda^\times$-nuclear.*

Proof. Follows from Theorem 2.2, Theorem 2.1, see [5], and noting that for a nuclear G_∞-space $\lambda(P)$, we have $\lambda(P)\lambda(P)^\times = \lambda(P)$, where $\lambda(P)^\times := \{(x_n) : (x_n y_n) \in \ell_1 \ \forall\ y = (y_n) \in \lambda(P)\}$. □

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