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ALMOST b-CONTINUOUS FUNCTIONS

N. Rajesh
Department of Mathematics
Kongu Engineering College
Perundurai, Erode-638 052, Tamil Nadu, INDIA
e-mail: nrajesh_topology@yahoo.co.in

Abstract: The concept of b-open sets was introduced by Andrijevic. The aim of this paper is to introduce and characterize almost b-continuous functions by using b-open sets.

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1. Introduction

As generalization of open sets, b-open sets were introduced and studied by Andrijevic. This notions was further studied by Ekici [3, 4, 5], Park [7] and Caldas et al [2]. In this paper, we will continue the study of related functions with b-open sets. We introduce and characterize the concepts of almost b-continuous functions.

Throughout this paper, X and Y always represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $\text{cl}(A)$ and $\text{int}(A)$ denote the closure of A and interior of A in X respectively. A subset A of X is said to be b-open (see [1]) if $A \subset \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$. The complement of b-open set is called b-closed. The intersection of all b-closed sets of X containing A is called b-closure (see [1]) of A and is denoted by $b\,\text{cl}(A)$. A set A is b-closed if and only if $b\,\text{cl}(A)=A$. The union of all b-sets of X contained in A is called the b-interior of A and is denoted by $b\,\text{int}(A)$. A set A is said to be b-regular (see [7]) if it is b-open and b-closed. The family of all b-open (resp. b-regular) sets of X is denoted by $BO(X)$ (resp. $BR(X)$). We set $BO(X,x) = \{V \in BO(X) | x \in V\}$ for $x \in X$.

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A point x of X is called a b-θ-cluster [7] points of $S \subset X$ if $b\,\mathrm{cl}(U) \cap S \neq \emptyset$ for every $U \in BO(X, x)$. The set of all b-θ-cluster points of S is called the b-θ-closure of S and is denoted by $b\,\mathrm{cl}_\theta(S)$. A subset S is said to be b-θ-closed if and only if $S = b\,\mathrm{cl}_\theta(S)$. The complement of a b-θ-closed set is said to be b-θ-open.

Lemma 1. (see [7]) *Let A be a subset of a topological space X. Then:*
(i) $A \in BO(X)$ if and only if $b\,\mathrm{cl}(A) \in BR(X)$.
(ii) $A \in BO(X)$ if and only if $b\,\mathrm{int}(A) \in BR(X)$.

2. Almost b-Continuous Functions

We have introduce the following definition.

Definition 2. A function $f : X \to Y$ is said to be almost b-continuous if for each point $x \in X$ and each $V \in BO(Y, f(x))$, there exists an open set U containing x such that $f(U) \subset b\,\mathrm{cl}(V)$.

In this section, we obtain several characterizations of almost b-continuous functions.

Theorem 3. *The following are equivalent for a function $f : X \to Y$:*
(i) f is almost b-continuous;
(ii) for each $x \in X$ and each $V \in BR(Y, f(x))$, there exists an open set U containing x such that $f(U) \subset V$;
(iii) $f^{-1}(V)$ is cl-open in X for every $V \in BR(X)$;
(iv) $f^{-1}(V) \subset \mathrm{int}(f^{-1}(b\,\mathrm{cl}(V)))$ for every $V \in BO(Y)$;
(v) $\mathrm{cl}(f^{-1}(b\,\mathrm{int}(V))) \subset f^{-1}(V)$ for every b-closed set V of Y;
(vi) $\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(b\,\mathrm{cl}(V))$ for every $V \in BO(Y)$.

Proof. (i)⇒(ii) For each $x \in X$ and $V \in BR(Y, f(x))$. There exists an open set U containing x such that $f(U) \subset b\,\mathrm{cl}(V) = V$.

(ii)⇒(iii) Let $V \in BR(Y)$ and $x \in f^{-1}(V)$. Then $f(U) \subset V$ for some open set U of X containing x and hence $x \in U \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X. Since $Y - V \in BR(Y)$, $f^{-1}(Y - V)$ is also open and hence $f^{-1}(V)$ is cl-open in X.

(iii)⇒(iv) Let $V \in BO(Y)$. Then $V \subset b\,\mathrm{cl}(V)$ and $b\,\mathrm{cl}(V) \in BR(Y)$ by Lemma 1. By (iii), we have $f^{-1}(V) \subset f^{-1}(b\,\mathrm{cl}(V))$ and $f^{-1}(b\,\mathrm{cl}(V))$ is open in X. Therefore, we obtain $f^{-1}(V) \subset \mathrm{int}(f^{-1}(b\,\mathrm{cl}(V)))$.

(iv)⇒(v) Let V be a b-closed set of Y. By (iv), we have $f^{-1}(Y-V) = \operatorname{int}(f^{-1}(b\operatorname{cl}(Y-V))) = \operatorname{int}(f^{-1}(Y-b\operatorname{int}(V))) = X-\operatorname{cl}(f^{-1}(b\operatorname{int}(V)))$. Therefore, we obtain $\operatorname{cl}(f^{-1}(b\operatorname{int}(V))) \subset f^{-1}(V)$.

(v)⇒(vi) Let $V \in BO(Y)$. Then $b\operatorname{cl}(V) \in BR(Y)$ by Lemma 1 and by (v), we obtain $\operatorname{cl}(f^{-1}(V)) \subset \operatorname{cl}(f^{-1}(b\operatorname{cl}(V))) \subset f^{-1}(b\operatorname{cl}(V))$.

(vi)⇒(i) Let $x \in X$ and $V \in BO(Y, f(x))$. By Lemma 1, we have $b\operatorname{cl}(V) \in BR(X)$ and $f(x) \notin Y - b\operatorname{cl}(V) = b\operatorname{cl}(Y - b\operatorname{cl}(V))$. Thus, by (vi) we obtain $x \notin \operatorname{cl}(f^{-1}(Y-b\operatorname{cl}(V)))$. There exists an open neighbourhood U of x such that $U \cap f^{-1}(Y - b\operatorname{cl}(V)) = \emptyset$. Therefore, we have $f(U) \cap (Y - b\operatorname{cl}(V)) = \emptyset$ and hence $f(U) \subset b\operatorname{cl}(V)$. This shows that f is almost b-continuous. □

Theorem 4. *The following are equivalent for a function $f : X \to Y$:*

(i) f is almost b-continuous.

(ii) for each $x \in X$ and each $V \in BR(Y, f(x))$, there exists a cl-*open set U containing x such that $f(U) \subset V$.*

(iii) for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists an open set U containing x such that $f(\operatorname{cl}(U)) \subset b\operatorname{cl}(V)$.

Proof. (i)⇒(ii) Let $x \in X$ and $V \in BR(Y, f(x))$. By Theorem 3, $f^{-1}(V)$ is cl-open in Y. Put $U = f^{-1}(V)$, then $x \in U$ and $f(U) \subset V$.

The proofs of the other implications are obvious. □

Recall that a point $x \in X$ is called θ-cluster point [8] of a set A in X if and only if $\operatorname{cl}(U) \cap A \neq \emptyset$, for each open neighbourhood U of x. The collection of all θ-cluster points of A is called the θ-closure of A, denoted by $\operatorname{cl}_\theta(A)$. A set A is called θ-closed [8] if and only if $A = \operatorname{cl}_\theta(A)$. The complement of θ-closed set is called θ-open [8].

Theorem 5. *The following are equivalent for a function $f : X \to Y$:*

(i) f is almost b-continuous;

(ii) $\operatorname{cl}_\theta(f^{-1}(B)) \subset f^{-1}(b\operatorname{cl}_\theta(B))$ for every subset B of Y;

(iii) $f(\operatorname{cl}_\theta(A)) \subset b\operatorname{cl}_\theta(f(A))$ for every subset A of X;

(iv) $f^{-1}(F)$ is θ-closed in X for every b-θ closed set F of Y;

(v) $f^{-1}(V)$ is θ-open in X for every b-θ-open set V of Y.

Proof. By using Theorems 3 and 4, the proof of this theorem is similar to the proof of Theorem 3. □

Theorem 6. *Let $f : X \to Y$, $g : Y \to Z$ be functions and $g \circ f$: $X \to Z$ the composition. Then the following hold:*

(i) If f is an open surjection and $g \circ f$ is almost b-continuous, then g is almost b-continuous;

(ii) If f is continuous and g is almost b-continuous, then $g \circ f$ is almost b-continuous.

Proof. The proof is obvious and is thus omitted. □

Recall that a space X is said to be b-T_2 (see [7]) if for each pair of distinct points x, y of X, there exist $U, V \in BO(X)$ such that $x \in U$, $Y \in V$ and $U \cap V = \emptyset$.

Lemma 7. (see [7]) *A space X is b-T_2 if and only if for each pair of distinct points x, y of X, there exists $U, V \in BO(X)$ such that $x \in U$, $y \in V$ and $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \emptyset$.*

Theorem 8. *If $f : X \to Y$ is an almost b-continuous injection and Y is b-T_2, then X is Uryson space.*

Proof. Let x and y be a pair of distinct points of X. Since f is injective, $f(x) \neq f(y)$ and by Lemmma 7 there exist $U, V \in BO(Y)$ such that $f(x) \in U$, $f(y) \in V$ and $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \emptyset$. By Lemma 1, $b\,\mathrm{cl}(U)$ and $b\,\mathrm{cl}(V)$ are b-regular and hence $f^{-1}(b\,\mathrm{cl}(U))$ and $f^{-1}(b\,\mathrm{cl}(V))$ are cl-open in X. Moreoever, we have $x \in f^{-1}(b\,\mathrm{cl}(U))$, $y \in f^{-1}(b\,\mathrm{cl}(V))$ and $f^{-1}(b\,\mathrm{cl}(U)) \cap f^{-1}(b\,\mathrm{cl}(V)) = \emptyset$. Therefore X is Uryson. □

Theorem 9. *If $f, g : X \to Y$ is almost b-continuous and Y is b-T_2, then the set $A = \{x \in X : f(x) = g(x)\}$ is θ-closed in X.*

Proof. Let $x \in X - A$. Then $f(x) \neq g(x)$ and by Lemma 7 there exist $U, V \in BO(X)$ such that $f(x) \in U$, $g(x) \in V$ and $b\,\mathrm{cl}(U) \cap b\,\mathrm{cl}(V) = \emptyset$. By Theorem 4, there exist cl-open neighbourhoods G, H of x such that $f(G) \subset b\,\mathrm{cl}(U)$ and $g(H) \subset b\,\mathrm{cl}(V)$. Let $O = G \cap H$, then O is a cl-open neighbourhood of x and $f(O) \cap g(O) = \emptyset$. Therefore, we obtain $O \cap A = \emptyset$ and $x \in X - \mathrm{cl}_\theta(A)$. This shows that A is θ-closed in X. □

Theorem 10. *If $f : X \to Y$ is almost b-continuous and Y is b-T_2, then a set $A = \{(x_1, x_2) | f(x_1) = f(x_2)\}$ is θ-closed in $X \times X$.*

Proof. Let $(x_1, x_2) \in X \times X - A$. Then $f(x_1) \neq f(x_2)$ and by Lemma 7 there exists $V_1, V_2 \in BO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $b\,\mathrm{cl}(V_1) \cap b\,\mathrm{cl}(V_2) = \emptyset$. There exist a cl-open neighbourhood U_i of x_i such that $f(U_i) \subset b\,\mathrm{cl}(V_i)$ for $i = 1, 2$. Let $U = U_1 \times U_2$, then U is a cl-open neighbourhood of (x_1, x_2) and $U \cap A = \emptyset$. Therefore, we obtain $(x_1, x_2) \in X \times X - \mathrm{cl}_\theta(A)$. This shows that A is θ-closed in $X \times X$. □

Recall that a space X is said to be b-connected (see [7]) if it cannot be expressed as the union of two disjoint non-empty b-open sets.

Theorem 11. *If $f : X \to Y$ is an almost b-continuous surjection and X is connected, then Y is b-connected.*

Proof. Suppose that Y is not b-connected. There exist non-empty b-open sets V_1, V_2 of Y such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Let $W_i = b\,\mathrm{cl}(V_i)$ for $i = 1, 2$. Then W_1 and W_2 are non-empty b-regular sets of Y such that $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 = Y$. Therefore, we have $f^{-1}(W_i) \neq \emptyset$, $f^{-1}(W_1) \cap f^{-1}(W_2) = \emptyset$ and $f^{-1}(W_1) \cup f^{-1}(W_2) = X$. Moreover, by Theorem 3, $f^{-1}(W_i)$ is cl-open in X for $i = 1, 2$. This shows that X is not connected. □

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