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GENERIC STABILITY OF THE SET OF WEAKLY PARETO-NASH EQUILIBRIUM POINTS

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Abstract: In this paper, we study the existence of additive weight Nash-equilibrium and weakly Pareto-Nash equilibrium for multiobjective games. We prove that, for every multiobjective game has at least one additive weight weakly Pareto-Nash equilibrium with respect to the weight vector. As results, we show that in the sense of Baire categories, most multiobjective games have at least one essential weakly Pareto-Nash equilibrium points.

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1. Introduction and Preliminaries

Blackwell [3] firstly investigated zero-sum games with vector payoffs as a gen-

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eralization of the scalar criterion games. In 1959, Shapley [12] introduced the concepts of equilibrium points in games with vector payoff. In the game theory, the stability and perfection of Nash equilibrium points have become important topics. Recently, many authors $[1, 5, 6, 9, 10, 11, 13-20]$ are interested in studying the existence and stability of equilibrium. The existences of weakly Pareto-Nash equilibrium is one of the fundamental problem in the game theory. In this paper, our purpose is to present some existence results of weakly Pareto-Nash equilibria by fixed point, and establish the existence of generic stability of the set of weakly Pareto-Nash equilibrium points for multiobjective games. Our results could be regarded as an unified improvement of corresponding existence results of weakly Pareto-Nash equilibria given in the currently existing literature.

Now we introduce some notations and definitions. We shall consider a finite-players games with multicriteria in its strategic form $G := (X_i, F^i)_i \in N$, where $N := \{1, 2, \ldots, n\}$. For each $i \in N, X_i$ is the set of strategies in R^{k_i} for the player i, each F^i is a mapping from $X := \prod_{i\in N} X_i$ into R^{k_i}, which is called the payoff function of the i-th player, here k_i is a positive integer. If a choice $x := (x^1, x^2, \ldots, x^n) \in X$ is played, each player i is trying to get his/her payoff function $F^i := (f_1^i(x), f_2^i(x), \ldots, f_{k_i}^i(x))$.

For each positive integer m, denote

$$R_+^m := \{(u^1, \ldots, u^m) \in R^m : u^j \geq 0, \forall j = 1, 2 \ldots, m\},$$

and

$$\operatorname{int} R_+^m := \{(u^1, \ldots, u^m) \in R^m : u^j > 0, \forall j = 1, 2 \ldots, m\}.$$

For R^m, we take the norm $\|r\| = \sum_{i=1}^m |r_i|$, where $r = (r_1, \ldots, r_m) \in R^m$. For each $i \in N$, denote $\hat{i} = N\backslash\{i\}, X_{\hat{i}} = \prod_{j\in N\backslash\{i\}} X_j$, $x_{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{\hat{i}}$ and $x := (x_i, x_{\hat{i}}) \in X$. In this paper, we need the following definitions.

Definition 1. A strategy profile $x^* \in X$ is called a weakly Pareto-Nash equilibrium point of a multiobjective game $G := (X_i, F^i)_{i\in N}$ if for each $i \in N$

$$F^i(y_i, x_{\hat{i}}^*) - F^i(x_i^*, x_{\hat{i}}^*) \notin \operatorname{int} R_+^{k_i}, \quad \forall y_i \in X_i.$$

If $k_i = 1$ for all $i = 1, 2, \ldots, n$, then the noncooperative game in the literature and the weakly Pareto-Nash equilibrium points reduce to Nash equilibrium points of usual noncooperative games.

Definition 2. A strategy profile $x^* \in X$ is said to be a additive weight Nash-equilibrium respect to weight vector $W = (W^1, W^2, \ldots, W^n)$ of a game $G := (X_i, F^i)_{i\in N}$ for each $i \in N$, we have that:

(1) $W^i \in R_+^{k_i} \backslash \{0\}$;

(2) $W^i \cdot F^i(x^*) \leq W^i \cdot F^i(x^*_{\hat{i}}, x_i)$ for each $x_i \in X_i$, where " $\cdot$ " denotes the inner production.

Let $f_{(F,W)}(x) = \sum W^i \cdot F^i(x)$.

Definition 3. Let X and Y are the Hausdorff spaces, $F : Y \to 2^X$ is a set-valued mapping, then:

(1) F is said to be upper semicontinuous (u.s.c) at $y \in Y$, if for each open set $G \supset F(y)$, there exists an open neighborhood $O(y)$ of y such that $G \supset F(y^{'})$ for any $y^{'} \in O(y)$. If F is upper semicontinuous on Y, and for each $y \in Y, F(y)$ is compact, we say that F is an usco mapping.

(2) F is said to be lower semicontinuous (l.s.c) at $y \in Y$, if for each open set $G \cap F(y) \neq \emptyset$, there exists an open neighborhood $O(y)$ of y such that $G \cap F(y^{'}) \neq \emptyset$ for any $y^{'} \in O(y)$.

(3) F is said to be almost lower semicontinuous (a.l.s.c) at $y \in Y$, if there exists $x \in F(y)$ such that for each open neighborhood $N(x)$ of x, there exists an open neighborhood $O(y)$ of y such that $N(x) \cap F(y^{'}) \neq \emptyset$ for any $y^{'} \in O(y)$.

2. The Existence of Weakly Pareto-Nash Equilibrium

In order to study the existence of weakly Pareto-Nash equilibria, we shall first study the existence of additive weight Nash equilibria. As weakly Pareto-Nash equilibria problems can be reduces to the study of additive weight Nash-equilibria under certain conditions. We will apply fixed point theory to investigate the existence of additive weight Nash-equilibria.

Lemma 1. (see [4, 7]) *Let X be a nonempty compact and convex subset of a Hausdorff topological vector space E. Suppose a set-valued mapping F : $X \to 2^X$ has the following properties:*

(1) $F(x)$ is nonempty and convex for each $x \in X$;

(2) F has open inverse valued, i.e. $F^{-1}(y) = \{x \in X : y \in F(x)\}$, for each $y \in X$ is open in X. Then F has at least one fixed point.

Lemma 2. *Let X and Y be two Hausdorff topological spaces and X be compact. Suppose $f : X \times Y \to R$ is a real-valued function and*

(1) f is u.s.c on $X \times Y$;

(2) For each fixed $x \in X$, the mapping $y \to f(x, y)$ is l.s.c on Y.

Then the function $\Psi : Y \to R$ defined by $\Psi(y) = \max_{u \in X} f(u, y)$ is continuous on Y.

Proof. As X is compact, Ψ is u.s.c on Y by (1) and Theorem 1 of Aubin (see [2], p. 67). By condition (2), Ψ is also l.s.c on y. □

Theorem 1. Let $G = (X_i, F^i)_{i\in N}$ be a given multiobjective game. For each $i \in N, X_i$ is a nonempty compact and convex subset of a Hausdorff topological vector space E_i. If there is a weight vector $W = (W^1, \ldots, W^n)$ with $W^i \in R_+^{k_i}\backslash\{0\}$ and the following are satisfied for each $i \in N$:

(1) The function $(x, y) \to W^i \cdot F^i(x_{\hat{i}}, y_i)$ is l.s.c on $X \times X$;

(2) For each fixed $y \in X$, the mapping $x \to W^i \cdot F^i(x_{\hat{i}}, y_i)$ is u.s.c on X;

(3) For each fixed $x \in X, W^i \cdot F^i(x_{\hat{i}}, y_i)$ is convex on X.

Then, G has at least one additive weight Nash -equilibrium respect to the weight vector W.

Proof. For each $k \in N$, defined a set-valued mapping $A_k : X \to 2^X$ by

$$A_k(x) = \prod_{i=1}^{n}\{y_i \in X_i : W^i \cdot F^i(x_{\hat{i}}, y_i) < \min_{u_i\in X_i} W^i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k}\}$$

Then $A_k(x)$ is nonempty and convex by the condition (3) for each $x \in X$. Note that $A_k^{-1}(y) = \cap_{i=1}^{n}\{x \in X : W^i \cdot F^i(x_{\hat{i}}, y_i) < \min_{u_i\in X_i} W^i \cdot F^i(x_{\hat{i}}, u_i) + \frac{1}{k}\}$, which is open in X by conditions (1) and (2) for each $y \in X$. Thus, by Fan-Browder Theorem(Lemma 1), the mapping A_k has a fixed point $x(k)$. From the definition of A_k, it follow that

$$W^i \cdot F^i(x_{\hat{i}}(k), x_i(k)) < \min_{u_i\in X_i} W^i \cdot F^i(x_{\hat{i}}(k), u_i) + \frac{1}{k},$$

without loss of generality, we may assume that $x(k) \to x^* \in X$ as X is compact. By condition (1) and lemma 2, we have that $W^i \cdot F^i(x_{\hat{i}}, x_i) \le \inf_{k\to\infty} W^i \cdot F^i(x_{\hat{i}}(k), x_i(k)) \le \inf_{k\to\infty} \min_{u_i\in X_i} W^i \cdot F^i(x_{\hat{i}}(k), u_i)$
$= min_{u_i\in X_i} W^i \cdot F^i(x_{\hat{i}}, u_i)$. Thus, x^* is a additive weight equilibrium of G respect to the weight vector W given above. The proof is complete. □

Now, let X be a nonempty convex compact subset of a Hausdorff space E. Let $C(X)$ be the collection of all $F = (F^1, F^2, \cdots, F^n)$ which is defined on X such that:

(1) The function $(x, y) \to W^i \cdot F^i(x_{\hat{i}}, y_i)$ is l.s.c on $X \times X$;

(2) for each fixed $y \in X$, The mapping $x \to F^i(x_{\hat{i}}, y_i)$ is u.s.c on X;

(3) for each fixed $x \in X, F^i(x_{\hat{i}}, y_i)$ is convex on x.

Let $T(F, W)$ denote the set of all additive weight solution of F subject to $W \in R_+^{k_i}\backslash\{0\}$. Then, $T(F, W) \neq \emptyset$ for each $W \in R_+^{k_i}\backslash\{0\}$ and $F \subset C(X)$. Then, T is a set-valued mapping from $C(X)$ to 2^X.

Lemma 3. *Each additive weight Nash-equilibrium x^* with a weight $W = (W^1, W^2, \cdots, W^n)$ is a weakly Pareto-Nash equilibrium of the game $(X_i, F^i)_{i\in N}$.*

Proof. Let x^* is a additive Nash equilibrium of the game $(X_i, F^i)_{i\in N}$, Then for each $i \in N$ there exists $W^i \in R_+^{k_i}$ such that $W^i \cdot F^i(x^*_{\hat{i}}, x^*_i) \le W^i \cdot F^i(x^*_{\hat{i}}, x_i)$

for all $x_i \in X_i$ (I). Suppose x^* is not a weakly Pareto-Nash equilibrium of the game $(X_i, F^i)_{i\in N}$. Then there exist $i \in N$ and $u_i \in X_i$ such that $F^i(x^*_{\hat{i}}, x^*_i) - F^i(x^*_{\hat{i}}, u_i) \in \text{int}R^{k_i}_+$ (II). Therefore, by (II) and $W^i \in R^{k_i}_+$, we obtain that $W^i \cdot F^i(x^*_{\hat{i}}, x^*_i) > W^i \cdot F^i(x_{\hat{i}}, u_i)$, which is a contradiction with (I). Therefore, x^* is a weakly Pareto-Nash equilibrium. The proof is complete. □

Theorem 2. Let $G = (X_i, F^i)_{i\in N}$ be a given multiobjective game. X_i is a nonempty compact and convex subset of a Hausdorff topological vector space E_i. If there is a weight vector $W = (W^1, \ldots, W^n)$ with $W^i \in R^{k_i}_+\backslash\{0\}$ such that the following are satisfied for each $i \in N$: (1) The function $(x, y) \to W^i \cdot F^i(x_{\hat{i}}, y_i)$ is l.s.c on $X \times X$;

(2) For each fixed $y \in X$, the mapping $x \to W^i \cdot F^i(x_{\hat{i}}, y_i)$ is u.s.c on X;

(3) For each fixed $x \in X, W^i \cdot F^i(x_{\hat{i}}, y_i)$ is convex on X.

Then G has at least one Pareto-Nash equilibrium

Proof. By theorem 1, we derive that G has at least one additive weight Nash equilibrium x^*. From Lemma 3, we derive x^* is a weakly Pareto-Nash equilibrium for G.

Let $S(F)$ denote the set of all weakly Pareto-Nash equilibrium of F. Then, S is a set-valued mapping from $C(X)$ to 2^X.

3. The Generic Stability of the Set of Weakly Pareto-Nash Equilibrium Points

Definition 4. For each $F \in C(X), x^* \in S(F)$ (resp. $x^* \in T(F, W)$) is said to be an essential equilibrium (resp. essential additive weight equilibrium) of F provided that for any neighborhood $N(x^*)$ of x^* in X. There exists an open neighborhood $O(F)$ of F in $C(X)$ such that $N(x^*) \cap S(F') \neq \emptyset$ (resp., $N(x^*) \cap T(F', W) \neq \emptyset$) for all $F' \in O(F)$. Further, F is said to be essential (resp. additive weight essential) if all its equilibrium (resp. additive weight equilibrium) are essential.

Definition 5. For each $F \in C(X)$, let $e(F)$ be a nonempty closes subset of $S(F)$, $e(F)$ is said to be an essential weakly Pareto-Nash equilibrium set of F provided that for any open set $U \supset e(F)$, there exists an open neighborhood $O(F)$ of F in $C(X)$ such that for any $F' \in O(F), U \cap S(F') \neq \emptyset$.

By the Definition 3 and 4, it is easy to obtain the following results.

Lemma 4. Let $F \in C(X)$:

(1) F is essential (resp. additive weight essential) if and only if the mapping $S : C(X) \to 2^X$ (resp. $T(\cdot, W) : C(X) \to 2^X$ is l.s.c on X.

(2) There exists an essential equilibrium $x^* \in S(F)$ if and only if the mapping $S : C(X) \to 2^X$ is a. l.s.c on $C(X)$.

(3) There exists an essential additive weight equilibrium if and only if the mapping $T(\cdot, W) : C(X) \to 2^X$ is l.s.c on $C(X)$.

Lemma 5. (See Theorem 2 from [8]) Let X be a metric space, Y be a Baire space, $F : Y \to 2^X$ is an usco mapping. Then there is a dense G_δ subset Q' of Y such that F is l.s.c on Q'.

Lemma 6. For each fixed $W = (W^1, W^2, \cdots, W^n), W^i \in R_+^{k_i}, T(\cdot, W) : C(X) \to 2^X$ is an usco mapping on $C(X)$.

Proof. X is compact, what we shall do next is to proved $T(\cdot, W)$ is u.s.c.

Suppose $T(\cdot, W)$ were not u.s.c at $F \in C(X)$. Then there exists an open set U of X with $U \supset T(F, W)$ and a sequence $\{F^n\} \subset C(X)$ with $F^n \to F$ for each $n \in N$, one can find $x_n \in T(F^n, W)$ which satisfies $x_n \notin U$. Due to X is compact and $\{x_n\} \subset X$, without loss of generality, we may assume that $x_n \to x_0$. It follows from $x_n \notin U$ that $x_0 \notin U$ and $x_0 \notin T(F, W)$. Then, there exists some $x' \in X$ such that $f_{(F,W)}(x') - f_{(F,W)}(x_0) < 0$. Therefore, for all $x \in X$, we have

$$\begin{aligned}
f_{(F^n,W)}(x') - f_{(F^n,W)}(x)& \\
&= f_{(F^n,W)}(x') - f_{(F,W)}(x') + f_{(F,W)}(x') - f_{(F,W)}(x_0) \\
&\quad + f_{(F,W)}(x_0) - f_{(F,W)}(x) + f_{(F,W)}(x) - f_{(F^n,W)}(x) \\
&= \sum_{i=1}^{k_i} W^i \cdot (F^{i^n}(x') - F^i(x')) + f_{(F,W)}(x' - f_{(F,W)}(x_0) \\
&\quad + f_{(F,W)}(x_0) - f_{(F,W)}(x) + \sum_{i=1}^{k_i} W^i \cdot (F^{i^n}(x) - F^i(x)) \\
&\le 2\|F^n - F\| + f_{(F,W)}(x' - f_{(F,W)}(x_0) + f_{(F,W)}(x_0) - f_{(F,W)}(x)\,.
\end{aligned}$$

Since $F^n \to F$ and $f_{(F,W)}$ is continuous at $x_0, \|F^n - F\| \to 0$ when $n \to \infty$ and $f_{(F,W)}(x) - f_{(F,W)}(x_0)$ arbitrarily nears 0 when x sufficiently nears x_0. Hence there exists some open neighborhood $O(x_0)$ of x_0 and $n_1 \in N$ such that $f_{(F^n,W)}(x') - f_{(F^n,W)}(x) < 0$ for all $x \in O(x_0)$ and $n \ge n_1$. Moreover, since $x_n \to x_0$, there exists $n_2 \ge n_1$ such that $x_{n_2} \in O(x_0)$ and by this, $f_{(F^{n^2},W)}(x') < f_{(F^{n^2},W)}(x_{n_2})$. Consequently, $x_{n_2} \notin T(F^{n_2}, W)$, This is a contradiction to the assumption $x_n \in T(F^n, W)$. This lemma is proved.

Theorem 3. Let $S : C(X) \to 2^X$. Then:

(1) There exists an mapping $S_0 : C(X) \to 2^X$ for each $F \in C(X), S_0(F) \subset S(F)$;

(2) For each $F \in C(X)$ and $W \in R_+^{k_i} \backslash \{0\}$, The set of additive weight Nash equilibrium $T(F, W)$ is an essential weakly Pareto-Nash equilibrium set of F.

Proof. (1) For any $W \in R_+^{k_i} \backslash \{0\}$, let $S_0(F) = T(F, W), \forall F \in C(X)$. Then statement (1) is immediate from Lemma 6 and Lemma 3, and the fact that $T(\cdot, W)$ is an usco mapping on $C(X)$.

(2) Let $W \in R_+^{k_i} \backslash \{0\}$, According to Lemma 6 and Lemma 3 $T(\cdot, W)$ is an usco mapping and $T(F, W) \subset S(F)$ for all $F \in C(X)$. Then for any open set $U \supset T(F, W)$, There exists some open neighborhood $O(F)$ of F such that $T(F', W) \subset S(F')$, we conclude that $U \cap S(F') \supset U \cap T(F', W) = T(F', W) \neq \emptyset$. Thus, $T(F, W) \subset S(F)$ is essential weakly Pareto-Nash equilibrium set of F. □

Lemma 7. For each fixed $W \in R_+^{k_i} \backslash \{0\}$, There exists a dense G_δ subset Q of $C(X)$ such that for each $T(\cdot, W)$ is l.s.c at each $F \in Q$.

Proof. According to Lemma 5, Lemma 6, Lemma 3 and Theorem 3, the results can easily be obtain. □

Theorem 4. There exists a dense G_δ subset Q of $C(X)$ such that each $F \in Q$, there is at least one $x \in S(F)$ such that x is an essential weakly Pareto-Nash equilibrium. i. e, S is a.l.s. c at every $F \in Q$.

Proof. Take an arbitrary $W = (W^1, \ldots, W^n)$ with $W^i \in R_+^{k_i} \backslash \{0\}$. It follows from Lemma 7 that there exists a dense G_δ subset Q of $C(X)$ such that $T(\cdot, W)$ is l.s.c at each $F \in Q$. For each $F \in Q$, let $x \in T(F, W) \subset S(F)$, since $T(\cdot, W)$ is l.s.c at F, for any open neighborhood $N(x)$ of x, there exists an open neighborhood $O(F)$ of F such that $N(x) \cap T(F', W) \neq \emptyset$ for all $F' \in F$. To show that x is an essential weakly Pareto-Nash equilibrium, it is clearly enough to show that $N(x) \cap S(F') \neq \emptyset$. Observing that $T(F', W) \subset S(F')$ and $N(x) \cap T(F', W) \neq \emptyset$. This completes the proof. □

Theorem 4 shows that in the sense of Baire categories, most multiobjective games have at least an essential weakly Pareto-Nash equilibrium point.

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