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ANOTHER CHARACTERIZATION FOR A CERTAIN INVARIANT FOR A GROUP PAIR

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Abstract: In this paper we show some examples about the relations of the quotient $\frac{H^1(G,S;M)}{P(G,S,M)}$ with splittings of a split extension and we give another characterization for the invariant $\widetilde{E}(G,S) := 1 + \dim_{\mathbb{Z}_2} \mathrm{Ker}\,(\mathrm{res}\,^G_{S,\mathcal{F}_S G})$ studied in [3] for $(G : S) = \infty$. With this characterization we can extend the definition of $\widetilde{E}(G,S)$ for the case $(G : S) < \infty$.

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1. Introduction

Let (G,S) be a group pair, where G is a group and S is a subgroup of G, and M a $\mathbb{Z}_2G$-module. Consider the group $\mathrm{Der}\,(G,S,M) = \{d : G \to M \mid d(gg') = d(g) + g \cdot d(g'),\ \forall\, g, g' \in G$ and $d_{|S} = 0\}$ and the subgroup $P(G,S,M) = \{d_m \in \mathrm{Der}\,(G,S,M), m \in M \mid d_m(g) = g \cdot m - m,\ \forall\, g \in G\}$.

In [2] the authors presented an interpretation for the quotient

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$\frac{H^1(G,S;M)}{P(G,S,M)} = \frac{\mathrm{Der}\,(G,S,M)}{P(G,S,M)}$ in terms of splittings of a split extension, generalizing the interpretation given for $H^1(G;M) = \frac{\mathrm{Der}\,(G,M)}{P(G,M)}$ in [4], IV, §2. More precisely, consider the extension $0 \to M \xrightarrow{i'} M \rtimes G \xrightarrow{\pi'} G \to 1$, where $M \rtimes G$ is the semi-directed product of G and M relative to the given action, and i' and π' are the canonical inclusion and projection maps. If $\mathcal{M}(G,M)$ denotes the set of the M-conjugacy classes of splittings $l : G \to M \rtimes G$ of the early extension there exists a bijection between $\mathcal{M}(G,M)$ and $H^1(G,M)$ (see [4], Proposition IV.2.3). Moreover, if $\mathcal{M}(G,S,M) := \{l \in \mathcal{M}(G,M) \mid l_{|S} = (0, id_S)\}$, then there exists a bijection between $\mathcal{M}(G,S,M)$ and $\frac{H^1(G,S,M)}{P(G,S,M)}$ (see [2], Theorem 4.1).

In this work, we first show some examples for the last result. In the sequel, we give another characterization for the invariant $\widetilde{E}(G,S) := 1 + \dim_{\mathbb{Z}_2} \mathrm{Ker}\,(\mathrm{res}\,^G_{S,\mathcal{F}_S G})$ defined in [3] for $(G:S) = \infty$. With this characterization we can extend the definition of $\widetilde{E}(G,S)$ for the case $(G:S) < \infty$. Also, we see a relation between this invariant and $\mathcal{M}(G,S,\mathcal{F}_S G)$.

2. Preliminaries and Some Examples

We begin with some properties about the set $\mathcal{M}(G,S,M)$, which can be easily proved using the results shown in [2], Section 4.

Proposition 1. *Let (G,S) be a group pair and M a $\mathbb{Z}_2 G$-module.*

(i) There exists a bijection between $\mathcal{M}(G,S,M)$ and $\ker(\mathrm{res}\,^G_{S,M})$, where $\mathrm{res}\,^G_{S,M} : H^1(G,M) \to H^1(S,M)$ is the restriction map.

(ii) If M is a trivial $\mathbb{Z}_2 G$-module then $\mathcal{M}(G,S,M) = \mathrm{Hom}\,_S(G,M) \times \{id_G\}$, with $\mathrm{Hom}\,_S(G,M) = \{f : G \to M \mid f$ is a homomorphim and $f_{|S} = 0\}$.

(iii) If $S = G$ then $\mathcal{M}(G,G,M)$ has only the class of the trivial splitting and $\frac{H^1(G,G,M)}{P(G,G,M)} \simeq \frac{\mathrm{Der}\,(G,G,M)}{P(G,G,M)} = 0$.

(iv) If G is a group and S is a normal subgroup of G then there is a bijection between $\mathcal{M}(G,S,\mathbb{Z}_2(G/S))$ and $\mathcal{M}((G/S),\mathbb{Z}_2(G/S))$.

Example 2. (1) Let (G,S) be a group pair and M a trivial $\mathbb{Z}G$-module.

(a) If $G = \langle t\rangle \simeq \mathbb{Z}$, $S = \langle t^3\rangle \simeq 3\mathbb{Z}$ and $M = \mathbb{Z}_3$ then $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}}_{3\mathbb{Z},\mathbb{Z}_3}) \simeq \mathrm{Der}\,(\mathbb{Z},3\mathbb{Z},\mathbb{Z}_3) \simeq H^1(\mathbb{Z},3\mathbb{Z},\mathbb{Z}_3) \simeq \mathbb{Z}_3$ and $\mathcal{M}(\mathbb{Z},3\mathbb{Z},\mathbb{Z}_3) = \mathcal{M}(\mathbb{Z},\mathbb{Z}_3)$. Now if $S = \langle t^2\rangle \simeq 2\mathbb{Z}$ we have $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}}_{2\mathbb{Z},\mathbb{Z}_3}) \simeq \mathrm{Der}\,(\mathbb{Z},2\mathbb{Z},\mathbb{Z}_3) \simeq H^1(\mathbb{Z},2\mathbb{Z},\mathbb{Z}_3) = 0$ and $\mathcal{M}(\mathbb{Z},2\mathbb{Z},\mathbb{Z}_3)$ has the only one splitting $l_0(t^{2k}) = (\overline{0}, t^{2k})$. Hence, $\mathcal{M}(\mathbb{Z},2\mathbb{Z},\mathbb{Z}_3) \neq \mathcal{M}(\mathbb{Z},\mathbb{Z}_3)$.

(b) If $G = \mathbb{Z}_n$ and $M = \mathbb{Z}$ then for all subgroup S of G, we have that $\mathcal{M}(\mathbb{Z}_n, S, \mathbb{Z}) = \mathcal{M}(\mathbb{Z}_n, \mathbb{Z})$ has the only one conjugacy class of the trivial splitting and $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}_n}_{S,\mathbb{Z}}) \simeq \mathrm{Der}\,(\mathbb{Z}_n, S, \mathbb{Z}) \simeq H^1(\mathbb{Z}_n, S, \mathbb{Z}) = 0$.

(c) Considering $G = \langle t \rangle \simeq \mathbb{Z}_4$, $S = \langle t^2 \rangle \simeq \mathbb{Z}_2$ and $M = \mathbb{Z}_2$ we have $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}_4}_{\mathbb{Z}_2,\mathbb{Z}_2}) \simeq \mathrm{Der}\,(\mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2) \simeq H^1(\mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $\mathcal{M}(\mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_2) = \mathcal{M}(\mathbb{Z}_4, \mathbb{Z}_2)$

Example 3. Let $G =< t >\simeq \mathbb{Z}$ be and $M = RG$ with $R = \mathbb{Z}$ or $\mathbb{Z}_2$. For all subgroup S of $G \simeq \mathbb{Z}$ with $S \neq \{1\}$, we can conclude, through the splittings, that $\mathcal{M}(G, S, M)$ has the only one conjugacy class of the trivial splitting and we have that $\mathrm{Ker}\,(\mathrm{res}^G_{S,M}) \simeq \frac{\mathrm{Der}\,(G,S,M)}{P(G,S,M)} \simeq \frac{H^1(G,S,M)}{P(G,S,M)} = 0$. Hence $\mathcal{M}(G, S, M) \neq \mathcal{M}(G, M)$ if $S \neq \{1\}$. Now, $\mathrm{Ker}\,(\mathrm{res}^G_{\{1\},M}) \simeq H^1(G, M) \simeq R$ and $\mathcal{M}(G, \{1\}, M) = \mathcal{M}(G, M)$.

Example 4. Considering $G = \{1, t\} \simeq \mathbb{Z}_2$ and $M = \mathbb{Z}$ with the $\mathbb{Z}G-module$ structure given by the action $t.r = -r, \forall r \in \mathbb{Z}$, we have that the only subgroup of G is the trivial. Thus, $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}_2}_{\{1\},\mathbb{Z}}) \simeq H^1(\mathbb{Z}_2, \mathbb{Z}) \simeq \mathbb{Z}_2$ and $\mathcal{M}(\mathbb{Z}_2, \{1\}, \mathbb{Z}) = \mathcal{M}(\mathbb{Z}_2, \mathbb{Z})$. Moreover, $\mathrm{Ker}\,(\mathrm{res}^{\mathbb{Z}_2}_{\mathbb{Z}_2,\mathbb{Z}}) \simeq 0$ and $\mathcal{M}(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}) = 0$.

Example 5. Let $G = \langle a, b \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$, $S = \langle b \rangle \simeq \{1\} \oplus \mathbb{Z}$ and $M = \mathbb{Z}_2(G/S) \simeq \mathbb{Z}_2(\mathbb{Z})$ with $\mathbb{Z}_2G$-module structure given by the left multiplication of G. We have that $\mathrm{Ker}\,(\mathrm{res}^G_{S,M}) \simeq \mathbb{Z}_2$ and $\mathcal{M}(G, S, M) \neq \mathcal{M}(G, M)$, because $\mathcal{M}(G, M)$ has 4 conjugacy classes of splittings.

3. A Characterization for the Invariant $\widetilde{E}(G, S)$

Let (G, S) be a group pair and consider the $\mathbb{Z}_2G$-submodules of $\mathcal{P}G$, the power set of G, $FG := \{F \subset G \mid F \text{ is finite }\}$, $QG := \{A \subset G \mid A + g \cdot A \in FG,\ \forall\ g \in G\}$, $\mathcal{F}_SG := \{B \subset G \mid B \subset F.S$ for some finite set F of $G\}$ and $\mathcal{A}_SG := \{A \subset G \mid A + g \cdot A \in \mathcal{F}_SG,\ \forall\ g \in G\}$, where " $+$ " denotes the symmetric difference operation in $\mathcal{P}G$ and $g \cdot A := \{ga \mid a \in A\}$. We denote $B + \mathcal{F}_SG \in \mathcal{P}G/\mathcal{F}_SG$ by $\overline{B}$. We observe that $FG \subset \mathcal{F}_SG$, $QG \subset \mathcal{A}_SG$ and $\mathcal{P}G/\mathcal{F}_SG$ is a $\mathbb{Z}_2G$-module with the induced G-action, i.e., $g \cdot \overline{B} = \overline{g \cdot B}$.

Proposition 6. *Let G be a group, S and T subgroups of G with $S \leq T \leq G$. Then:*

(i) $\mathcal{F}_SG \subset \mathcal{A}_SG$;

(ii) S finite $\Longrightarrow \mathcal{F}_SG = FG$;

(iii) $[G : S] < \infty \Longrightarrow \mathcal{F}_SG = \mathcal{P}G$;

(iv) $[T : S] < \infty \Longrightarrow \mathcal{F}_SG = \mathcal{F}_TG$;

(v) $\left(\frac{\mathcal{P}G}{\mathcal{F}_S G}\right)^G = \frac{\mathcal{A}_S G}{\mathcal{F}_S G}$ *and, in particular,* $\frac{QG}{FG} = \frac{\mathcal{A}_{\{1\}}G}{\mathcal{F}_{\{1\}}G} = \left(\frac{\mathcal{P}G}{FG}\right)^G$.

Proof. (i) If $B \in \mathcal{F}_S G$ then $B \subset F.S$ for some finite subset F of G and so $g \cdot B \subset F'.S$, where $F' = g \cdot F$ which is finite. Thus $B + g \cdot B \subset (F \cup F').S \in \mathcal{F}_S G$ and therefore $B \in \mathcal{A}_S G$.

(ii) If S is finite and $B \in \mathcal{F}_S G$ then $B \in FG$. Hence, $\mathcal{F}_S G \subset FG$ and, since $FG \subset \mathcal{F}_S G$, we obtain $\mathcal{F}_S G = FG$.

(iii) By definition $\mathcal{F}_S G \subset \mathcal{P}G$. Now, since $[G : S] < \infty$, it follows that G/S has a finite number of disjoint cosets $g_1S, \ldots, g_kS \in G/S$. Thus, $G = g_1S \cup \ldots \cup g_kS$ and so every $B \in \mathcal{P}G$ is such that $B \subset F.S$, for $F = \{g_1, \ldots, g_k\}$. Hence, $B \in \mathcal{F}_S G$ and therefore $\mathcal{P}G \subset \mathcal{F}_S G$.

(iv) Since $S \leq T$, it is easy to see that $\mathcal{F}_S G \subset \mathcal{F}_T G$. Now, if $B \in \mathcal{F}_T G$ then $B \subset F'.T$ for some finite subset F' de G. But $[T : S] < \infty$ implies $T = (t_1S \cup \ldots \cup t_kS) \subset F''.S$ with $F'' = \{t_1, \ldots, t_k\}$. Hence, $B \subset F'.T \subset F'.(F''.S) \subset F.S$ with $F = F'.F''$ and so $B \in \mathcal{F}_S G$. Therefore $\mathcal{F}_T G \subset \mathcal{F}_S G$.

(v) It is enough to note that $g.\overline{B} = \overline{B}$, for all $g \in G$ if, and only if, $B + g.B \in \mathcal{F}_S G$. □

The next result shows that $\mathcal{F}_S G$ is a induced module. This fact, which will be useful in many situations, is known, but we have not seen it proved in the literature.

We recall that, given a group G, a subgroup S of G and a $\mathbb{Z}_2G$-module M, we define $\mathrm{Ind}_S^G M = \mathbb{Z}_2G \otimes_{\mathbb{Z}_2S} M$ and $\mathrm{Coind}_S^G M = \mathrm{Hom}_{\mathbb{Z}_2S}(\mathbb{Z}_2G, M)$.

We denote $\mathrm{Coind}_{\{1\}}^G \mathbb{Z}_2$ by $\overline{\mathbb{Z}_2G}$ and we can see $\mathbb{Z}_2G$ as $\mathrm{Ind}_{\{1\}}^G \mathbb{Z}_2$.

The map $\rho : \overline{\mathbb{Z}_2G} \to PG$, defined by $\rho(f) = \{g \in G : f(g^{-1}) = 1\}$ is a $\mathbb{Z}_2G$-isomorphism and the submodule $\mathbb{Z}_2G$ of $\overline{\mathbb{Z}_2G}$ is mapped by ρ onto FG.

Proposition 7. *If (G, S) is a group pair then $\mathcal{F}_S G \simeq \mathrm{Ind}_S^G \overline{\mathbb{Z}_2S}$ as $\mathbb{Z}_2G$-modules.*

Proof. Consider the following maps $\mathrm{Ind}_S^G \overline{\mathbb{Z}_2S} \xrightarrow{\vartheta} \mathrm{Coind}_S^G \overline{\mathbb{Z}_2S} \xrightarrow{\nu} \overline{\mathbb{Z}_2G} \xrightarrow{\rho} \mathcal{P}G$ with ϑ and ν the $\mathbb{Z}_2G$-maps given by

$$\vartheta(g' \otimes \widetilde{f})(g) = \begin{cases} (gg') \cdot \widetilde{f}, & \text{if } gg' \in S\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

$\forall\ g' \otimes \widetilde{f} \in \mathrm{Ind}_S^G \overline{\mathbb{Z}_2S}$, $\forall\ g \in G$; $\nu(f')(g) = f'(g)(1)$, $1 \in \mathbb{Z}_2S$, $\forall\ g \in G$, $\forall\ f' \in \mathrm{Coind}_S^G \overline{\mathbb{Z}_2S}$.

Since ϑ is a monomorphism and ν is a isomorphism (see [4], Proposition III.5.9) we have that $\mathrm{Ind}_S^G \overline{\mathbb{Z}_2S}$ is a $\mathbb{Z}_2G$-submodule of $\overline{\mathbb{Z}_2G}$ which is $\mathbb{Z}_2G$-isomorphic (by ρ) to the $\mathbb{Z}_2G$-submodule $(\rho \circ \nu \circ \vartheta)(\mathrm{Ind}_S^G \overline{\mathbb{Z}_2S})$ of $\mathcal{P}G$. Hence, to conclude the proposition, it is enough to show that $(\rho\circ\nu\circ\vartheta)(\mathrm{Ind}_S^G\overline{\mathbb{Z}_2S}) = \mathcal{F}_SG$. This equality follows from the following facts:

(I) $B \in \mathcal{F}_SG \Longleftrightarrow B = g_0S_0 \cup \ldots \cup g_kS_k$, with $g_i \in G$, $S_i \subset S$ and $k \in \mathbb{N}$.

(II) Consider the $\mathbb{Z}_2S$-isomorphism $\overline{\rho} : \overline{\mathbb{Z}_2S} \to \mathcal{P}S$ and the injective map $j : \overline{\mathbb{Z}_2S} \to \overline{\mathbb{Z}_2G}$ such that $j(\widetilde{f}) = f$ with $f(g) = \begin{cases} \widetilde{f}(g), & \text{if } g \in S\,, \\ 0, & \text{otherwise}\,. \end{cases}$ We have the commutative diagram:

$$\begin{array}{ccc} \overline{\mathbb{Z}_2S} & \xrightarrow{\ j\ } & \overline{\mathbb{Z}_2G} \\ {\scriptstyle\overline{\rho}}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathcal{P}S & \hookrightarrow & \mathcal{P}G \end{array}$$

(III) If $g_0 \otimes \widetilde{f}_0$ is a basic element of $\mathrm{Ind}_S^G\overline{\mathbb{Z}_2S}$, and $g \in G$,

$$(\nu\circ\vartheta)(g_0\otimes\widetilde{f}_0)(g^{-1}) = \begin{cases} [(g^{-1}g_0)\cdot\widetilde{f}_0](1) = \widetilde{f}_0(g^{-1}g_0), & \text{if } g^{-1}g_0 \in S\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Hence, $(\rho\circ\nu\circ\vartheta)(g_0\otimes\widetilde{f}_0)) = \{g \in G \mid (\nu\circ\vartheta)(g_0\otimes\widetilde{f}_0)(g^{-1}) = 1\} = \{g \in G \mid g^{-1}g_0 \in S$ and $\widetilde{f}_0((g_0^{-1}g)^{-1}) = 1\} = \{g \in G \mid g_0^{-1}g \in \overline{\rho}(\widetilde{f}_0)\} = \{g \in G \mid g_0^{-1}g = s \in \overline{\rho}(\widetilde{f}_0)\} = \{g_0s \mid s \in \overline{\rho}(\widetilde{f}_0)\}$, that is, $(\rho\circ\nu\circ\vartheta)(g_0\otimes\widetilde{f}_0)) = g_0S_0 \in \mathcal{F}_SG$ with $S_0 = \overline{\rho}(\widetilde{f}_0) \subset S$. More generally, if $x = g_0\otimes\widetilde{f}_0 + g_1\otimes\widetilde{f}_1 + \ldots + g_k\otimes\widetilde{f}_k \in \mathrm{Ind}_S^G\overline{\mathbb{Z}_2S}$ then $(\rho\circ\nu\circ\vartheta)(x) = g_0S_0\cup\ldots\cup g_kS_k \in \mathcal{F}_SG$ with $S_i = \overline{\rho}(\widetilde{f}_i) \subset S$ for $i = 0,\ldots,k$.

(IV) If $B = g_0S_0 \cup \ldots \cup g_kS_k \in \mathcal{F}_SG$ is easy to see that $B \in (\rho\circ\nu\circ\vartheta)(\mathrm{Ind}_S^G\overline{\mathbb{Z}_2S})$. □

Remark 8. Since $\overline{\mathbb{Z}_2S} \simeq \mathcal{P}S$, we obtain from the former proposition that $\mathcal{F}_SG \simeq \mathrm{Ind}_S^G\mathcal{P}S$.

Now we recall the definition of $\widetilde{E}(G,S)$ (see [3], Section 4).

Definition 9. Let (G,S) be a group pair with $[G : S] = \infty$, then $\widetilde{E}(G,S) := E(G,S,\mathcal{F}_SG) = 1 + \dim_{\mathbb{Z}_2} \mathrm{Ker}\,(\mathrm{res}^G_{S,\mathcal{F}_SG})$, where $\mathrm{res}^G_{S,\mathcal{F}_SG}$ is the restriction map.

Remark 10. We have that $\mathrm{Ker}\,(\mathrm{res}^G_{S,\mathcal{F}_SG}) \simeq \frac{\mathrm{Der}\,(G,S,\mathcal{F}_SG)}{P(G,S,\mathcal{F}_SG)}$ (see [1], Lemma 1.4). Hence, if $[G : S] = \infty$, $\widetilde{E}(G,S) = 1 + \dim_{\mathbb{Z}_2}\frac{\mathrm{Der}\,(G,S,\mathcal{F}_SG)}{P(G,S,\mathcal{F}_SG)} = 1 + \dim_{\mathbb{Z}_2}\frac{H^1(G,S,\mathcal{F}_SG)}{P(G,S,\mathcal{F}_SG)}$.

Proposition 11. *Let (G,S) be a group pair with $[G : S] = \infty$. Then,*

(i) $\widetilde{E}(G,S) = k < \infty$ if and only if $\#\mathcal{M}(G,S,\mathcal{F}_SG) = 2^{k-1}$, where $\#\mathcal{M}(G,S,\mathcal{F}_SG)$ denotes the number of elements of $\mathcal{M}(G,S,\mathcal{F}_SG)$.

(ii) $\widetilde{E}(G,S) = \infty$ if and only if $\mathcal{M}(G,S,\mathcal{F}_SG)$ is an infinite set.

Proof. This follows from the remark above and Proposition 1. □

Corollary 12. *If (G,S) is a duality pair then $\#\mathcal{M}(G,S,\mathcal{F}_SG)=1$, and so the extension $0\longrightarrow \mathcal{F}_SG\longrightarrow \mathcal{F}_SG\rtimes G\longrightarrow G\longrightarrow 1$ has only one class of splittings.*

Proof. It is a consequence of the last result and Proposition 8 in [3]. □

Now we will provide another characterization for the invariant $\widetilde{E}(G,S)$, extending the definition of for the case $(G:S)<\infty$. First we need the following lemma.

Lemma 13. *If $\eta:\mathcal{A}_SG\rightarrow \operatorname{Der}(G,\mathcal{F}_SG)$ is the map defined by $\eta(A)(g):=A+g\cdot A$, $\forall\ g\in G$, then η is an epimorphism. Furthermore, if $A\in\mathcal{A}_SG\cap(\mathcal{P}G)^S$ then $\eta(A)\in\operatorname{Der}(G,S,\mathcal{F}_SG)$.*

Proof. It is clear that $\operatorname{Im}\eta=P(G,\mathcal{A}_SG)$. By Shapiro's Lemma we have that $H^1(G,\mathcal{P}G)=0$ and so $\operatorname{Der}(G,\mathcal{F}_SG)\subset\operatorname{Der}(G,\mathcal{P}G)=P(G,\mathcal{P}G)$. Now it is easy to see that $P(G,\mathcal{A}_SG)=\operatorname{Der}(G,\mathcal{A}_SG)$ and so η is an epimorphism. The second statement follows from the fact that $s\cdot A=A$, $\forall\ s\in S$. □

Theorem 14. *Let (G,S) be a group pair with $[G:S]=\infty$. Then:*
(1) The sequence

$$0\longrightarrow\{\emptyset,G\}\xrightarrow{\ \chi\ }\frac{\mathcal{A}_SG\cap(\mathcal{P}G)^S}{\mathcal{F}_SG\cap(\mathcal{P}G)^S}\xrightarrow{\ \overline{\eta}\ }\frac{\operatorname{Der}(G,S,\mathcal{F}_SG)}{P(G,S,\mathcal{F}_SG)}\longrightarrow 0\,,$$

where $\chi(C):=\overline{C}=C+\big(\mathcal{F}_SG\cap(\mathcal{P}G)^S\big)$, $\forall\ C\in\{\emptyset,G\}$ and $\overline{\eta}(\overline{A}):=\eta(A)+P(G,S,\mathcal{F}_SG)$, where η is the map defined in Lemma 13, is exact.
(2) $\widetilde{E}(G,S)=\dim_{\mathbb{Z}_2}\frac{\mathcal{A}_SG\cap(\mathcal{P}G)^S}{\mathcal{F}_SG\cap(\mathcal{P}G)^S}=\dim_{\mathbb{Z}_2}\frac{(\mathcal{A}_SG)^S}{(\mathcal{F}_SG)^S}$.

Proof. (1) We have that $\ker\chi=\{\emptyset\}$. In fact, if $\chi(G)=0$, then $G\in\mathcal{F}_SG$, $G\subset g_0S\cup\ldots\cup g_kS$ and therefore $[G:S]<\infty$. It is easy to see that $\operatorname{Im}\chi=\{\overline{\emptyset},\overline{G}\}\subset\operatorname{Ker}\overline{\eta}$. Now consider $\overline{A}\in\operatorname{Ker}\ \overline{\eta}$. Then $\eta(A)\in P(G,S,\mathcal{F}_SG)$ and $\eta(A)\in P(G,S,\mathcal{F}_SG)$. So, there exists $B\in\mathcal{F}_SG$ such that $\eta(A)(g)=d_B(g)$, $\forall\ g\in G$ and $\eta(A)_{|S}=0$. Hence, $\eta(A)(g)=d_B(g)$, $\forall\ g\in G\Rightarrow A+g\cdot A=B+g\cdot B$, $\forall\ g\in G\Rightarrow A+B=g\cdot(A+B)$, $\forall\ g\in G\Rightarrow A+B\in(\mathcal{P}G)^G\Rightarrow A+B\in\{\emptyset,G\}$. If $A+B=\emptyset$ then $A=B\in\mathcal{F}_SG$ and so $A=A+\emptyset\in\mathcal{F}_SG\cap(\mathcal{P}G)^S$. Hence $\overline{A}=\overline{\emptyset}=\chi(\emptyset)\in\operatorname{Im}\chi$. Similarly, if $A+B=G$ then $A+G=B\in\mathcal{F}_SG\cap(\mathcal{P}G)^S$ and $\overline{A}=\overline{G}=\chi(G)\in\operatorname{Im}\chi$. Therefore $\operatorname{Ker}\overline{\eta}\subset\operatorname{Im}\chi$ and so $\ker\ \overline{\eta}=\operatorname{Im}\chi$. Now, in order to see that $\overline{\eta}$ is an epimorphism, consider $d+P(G,S,\mathcal{F}_SG)$ with $d\in\operatorname{Der}(G,S,\mathcal{F}_SG)$. Then $d_{|S}=0$ and $d\in\operatorname{Der}(G,\mathcal{F}_SG)$. Hence, by Lemma

13, there exists $A \in \mathcal{A}_S G$ such that $\eta(A) = d$. But $\eta(A)_{|_S} = d_{|_S} \Rightarrow A + s \cdot A = \emptyset,\ \forall\ s \in S \Rightarrow s \cdot A = A,\ \forall\ s \in S \Rightarrow A \in (\mathcal{P}G)^S$. Therefore, there exists $A \in \mathcal{A}_S G \cap (\mathcal{P}G)^S$ such that $\eta(\overline{A}) = d + P(G, S, \mathcal{F}_S G)$. Thus, the sequence is exact, as desired.

(2) By condition (1) we have $\operatorname{Ker} \overline{\eta} \simeq \mathbb{Z}_2$ and $\overline{\eta}$ is an epimorphism. Thus,

$$\begin{aligned} \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G \cap (\mathcal{P}G)^S}{\mathcal{F}_S G \cap (\mathcal{P}G)^S} &= \dim_{\mathbb{Z}_2} \operatorname{Ker}\ \overline{\eta} + \dim_{\mathbb{Z}_2} \operatorname{Im} \overline{\eta} \\ &= 1 + \dim_{\mathbb{Z}_2} \frac{\operatorname{Der}(G, S, \mathcal{F}_S G)}{P(G, S, \mathcal{F}_S G)} = \widetilde{E}(G, S). \end{aligned}$$

The equality at right is obvious. □

The last result suggest to define $\widetilde{E}(G, S) = \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G \cap (\mathcal{P}G)^S}{\mathcal{F}_S G \cap (\mathcal{P}G)^S}$ for all group pair (G, S).

Corollary 15. *If we define* $\widetilde{E}(G, S) = \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G \cap (\mathcal{P}G)^S}{\mathcal{F}_S G \cap (\mathcal{P}G)^S}$ *for all group pair* (G, S)*, than:*

(1) $\widetilde{E}(G, S) = 0 \Longleftrightarrow [G : S] < \infty$.

(2) $\widetilde{E}(G, S) = 1 + \dim_{\mathbb{Z}_2} \operatorname{Ker}(\operatorname{res}^G_{S, \mathcal{F}_S G}) \Longleftrightarrow [G : S] = \infty$.

(3) $\widetilde{E}(G, \{1\}) = e(G)$ *for all group* G.

Proof. To obtain (1) and (2), we use the last result and the fact that if $[G : S] < \infty$ then $\mathcal{F}_S G = \mathcal{A}_S G = \mathcal{P}G$ (by Proposition 6, (i) and (iii)) and so $\dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G \cap (\mathcal{P}G)^S}{\mathcal{F}_S G \cap (\mathcal{P}G)^S} = 0$. To see (3), we use that $e(G) = \dim_{\mathbb{Z}_2} \left(\frac{\mathcal{P}G}{FG}\right)^G$ (see [7]), and the Proposition 6, (i). □

We finally observe that in [3], Section 4, for $[G : S] = \infty$, we have studied some relations between $\widetilde{E}(G, S)$ and the invariant $\widetilde{e}(G, S)$ defined by Kropholler Holler, [5]. In the next result we compare these invariants by considering characterization for $\widetilde{E}(G, S)$, given in the Corollary 15. We recall that $\widetilde{e}(G, S) = \dim_{\mathbb{Z}_2} \frac{\mathcal{P}G}{\mathcal{F}_S G}$ (see [5], Definition 1.1), and if $[G : S] = \infty$ then $\widetilde{e}(G, S) = 1 + \dim_{\mathbb{Z}_2} H^1(G; \mathcal{F}_S G)$ (see [5], Lemma 1.2).

Proposition 16. *If* (G, S) *is a group pair with* $[G : S]$ *infinity or not, then* $\widetilde{E}(G, S) = \widetilde{e}(G, S) \Longleftrightarrow \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G \cap (\mathcal{P}G)^S}{\mathcal{F}_S G \cap (\mathcal{P}G)^S} = \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G}{\mathcal{F}_S G}$.

Proof. By Proposition 6 (v), we have that $\widetilde{e}(G, S) = \dim_{\mathbb{Z}_2} \frac{\mathcal{A}_S G}{\mathcal{F}_S G}$. Thus the result follows from Theorem 14(2). □

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