International Journal of Pure and Applied Mathematics

Volume 35 No. 3 2007, 359-363

ON FP-FLAT MODULES

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Abstract: This paper constructs a new kind of modules – FP-flat modules by means of the research methods of flat modules – and also gives the properties and equivalent propositions of FP-flat modules.

AMS Subject Classification: 13C11
Key Words: FP-flat modules; flat modules; FP-injective modules

1. Introduction

In homological algebra, modules are essential tools for describing rings, hence the research on them will be done in this paper by means of Hom functor, $\otimes$ functor and their derived functors Ext, Tor. Anderson and Fuller originally introduced the concept of $M - flat$ modules in [1]in 1974. Fuchang Cheng elaborated FP-injective modules and their dimensions in [2]. Works [3], [5], [7] also discussed rings with FP-injective modules or flat modules. At the same time, flat module is an important kind of modules in this field.

Enlightened by the research above-mentioned, this paper will mainly discuss the properties of FP-flat modules, compared with flat modules.

2. Main Results on FP-Flat Module

Let R be a ring, A be a right R-module.

Definition 1. A is called an FP-flat module if $\mathrm{Tor}_1^R(A,B) = 0$ holds for any f.p. left R-module B.

Obviously, flat modules are FP-flat modules.

Received: January 1, 2007

Theoreom 2. *The following statements are equivalent:*
(I) A is FP-flat.
(II) For the exact sequence of left R-module

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0, \tag{1}$$

where L is f.p., there is an exact sequence

$$0 \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow A \otimes L \longrightarrow 0\,. \tag{2}$$

(III) For the exact sequence (1) of left R-modules, there exists an exact sequence (2), where L is f.p., N is a projective module.
(IV) For each exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0, \tag{3}$$

there is an exact sequence

$$0 \longrightarrow C \otimes Q \longrightarrow B \otimes Q \longrightarrow A \otimes Q \longrightarrow 0\,,$$

where Q is an arbitrary f.p. left R-module.
(V) There exists an exact sequence satisfying (4)

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0\,, \tag{4}$$

where F is FP-flat.

Proof. (I)$\Longrightarrow$(II) Perform the functor $A\otimes\_$ on the short exact sequence (1), then an sequence can be obtained as follows

$$\cdots \longrightarrow \operatorname{Tor}_1^R(A, L) \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow A \otimes L \longrightarrow 0\,.$$

Thus the right R-module A is an FP-flat module by definition 1, i.e. $\operatorname{Tor}_1^R(A, L) = 0$, so (2) is an exact sequence.

(II)$\Longrightarrow$(III) It is clear.

(III)$\Longrightarrow$ (I) Perform the functor $A\otimes\_$ on the short sequence (1), then an sequence is obtained

$$\cdots \longrightarrow \operatorname{Tor}_1^R(A, N) \longrightarrow \operatorname{Tor}_1^R(A, L) \longrightarrow A \otimes M \longrightarrow A \otimes N \longrightarrow A \otimes L \longrightarrow 0\,,$$

so $\operatorname{Tor}_1^R(A, N) = 0$ by Proposition 3.2 in [4]. Since (2) is an exact sequence, there is $\operatorname{Tor}_1^R(A, L) = 0$, i.e. A is an FP-flat module.

(I)$\Longrightarrow$ (IV) Suppose Q is an arbitrary f.p. left R-module, then $\operatorname{Tor}_1^R(A, Q)=0$ by (I). Perform the functor $\_\otimes Q$ on the exact sequence (3), then the following sequence is exact,

$$0=\operatorname{Tor}_1^R(A,Q)\longrightarrow C\otimes Q\longrightarrow B\otimes Q\longrightarrow A\otimes Q\longrightarrow 0\,.$$

(IV)$\Longrightarrow$ (V) It is clear.

(V)$\Longrightarrow$ (I) Suppose S is an f.p. left R-module, and perform the functor $\_\otimes S$ on the exact sequence (4), then

$$\cdots\longrightarrow \operatorname{Tor}_1^R(F,S)\longrightarrow \operatorname{Tor}_1^R(A,S)\longrightarrow K\otimes S\longrightarrow F\otimes S\longrightarrow A\otimes S\longrightarrow 0\,,$$

where F is FP-flat, so $\operatorname{Tor}_1^R(F,S)=0$. It follows that $K\otimes S\longrightarrow F\otimes S$ is monic from the given condition that the exact sequence (4) satisfies (IV), therefore $\operatorname{Tor}_1^R(A,S)=0$, i.e. A is an FP-flat module. □

Theorem 3. *FP-flat modules keep direct sum, direct summand and module extensions.*

Proof. Suppose $\{M_i|i\in\Omega\}$ is a family of right R-modules. There is an isomorphism $\operatorname{Tor}_n^R(\coprod M_i,B)\cong\coprod\operatorname{Tor}_n^R(M_i,B)$ for $n\geq 1$ by Theorem 8.10 in [6], thus each M_i is an FP-flat module $\Longleftrightarrow$ $\coprod M_i$ is also an FP-flat module, where B is an f.p. left R-module.

For the exact sequence of left R-modules

$$0\longrightarrow X\longrightarrow Y\longrightarrow Z\longrightarrow 0,$$

where X , Z are FP-flat modules, there is an exact sequence for any f.p. left-module T

$$\cdots\longrightarrow \operatorname{Tor}_1^R(X,T)\longrightarrow \operatorname{Tor}_1^R(Y,T)\longrightarrow \operatorname{Tor}_1^R(Z,T)$$
$$\longrightarrow X\otimes T\longrightarrow Y\otimes T\longrightarrow Z\otimes T\longrightarrow 0\,.$$

Since X, Z are FP-flat modules, $\operatorname{Tor}_1^R(X,T)=\operatorname{Tor}_1^R(Z,T)=0$ is obtained, therefore $\operatorname{Tor}_1^R(Y,T)=0$, i.e. Y is an FP-flat module. □

Theorem 4. *FP-flat modules keep direct limit.*

Proof. Suppose $\{M_i|i\in\Omega\}$ is a family of right R-modules and each M_i is an FP-flat module.

$$0\longrightarrow A'\longrightarrow A$$

is an exact sequence of left R-modules, where A' is an f.g. submodules of the f.g. free modules A, thus the following diagram is commutative.

$$\begin{array}{ccccc}
0 & \longrightarrow & (\varinjlim M_i)\otimes A' & \longrightarrow & (\varinjlim M_i)\otimes A\\
 & & \downarrow & & \downarrow\\
0 & \longrightarrow & \varinjlim(M_i\otimes A') & \longrightarrow & \varinjlim(M_i\otimes A)
\end{array}$$

Tensor functors keep direct limit by Lemma 2.20 in [6], so the columns are isomorphic. For M_i is an FP-flat module, the sequence

$$0 \longrightarrow M_i \otimes A' \longrightarrow M_i \otimes A$$

is exact by Theorem 1. Direct limit keeps exactness by Theorem 2.18 in [6], so the bottom row as well as the top row is exact. By means of Theorem 1, M_i is FP-flat. □

Corollary 5. *The right R-module A is FP-flat if all of the f.g. submodules of A are FP-flat modules.*

Theorem 6. *The right R-module A is FP-flat if and only if the character module A^* of A is FP-injective.*

Proof. $A^*=\mathrm{Hom}_R(A, Q/Z)$ is a left R-module by Theorem 1.15 in [6]. Assume $f : M' \longrightarrow M$ is monic where M' is an f.g. submodule of the f.g. free module M, then investigate the following commutative diagram

$$\begin{array}{ccccc} (A \otimes M)^* & \longrightarrow & (A \otimes M')^* & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}_R(M, A^*) & \longrightarrow & \mathrm{Hom}_R(M', A^*) & \longrightarrow & 0 \end{array}$$

in which the columns are adjoint isomorphic by Theorem 1.15 in [6], hence the top row is exact if and only if the bottom row is exact. The following sequence

$$\mathrm{Hom}_R(M, A^*) \longrightarrow \mathrm{Hom}_R(M', A^*) \longrightarrow 0$$

is exact if and only if A^* is FP-injective (see [1]). It follows that

$$(A \otimes M)^* \longrightarrow (A \otimes M')^* \longrightarrow 0$$

is exact if and only if A^* is FP-injective, and the sequence

$$0 \longrightarrow A \otimes M' \longrightarrow A \otimes M$$

is exact if and only if A^* is FP-injective by Lemma 3.51 in [6], namely, the right R-module A is FP-flat if and only if the character module A^* of A is FP-injective by Theorem 2. □

3. Summary

FP-flat modules discussed here have many different properties from flat modules, and the connection with FP-injective modules has also been shown in Theorem 6. Therefore, this kind of modules is indispensable for further study on describing coherent rings, hereditary rings and so on.

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