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THE INTEGRITY OF DOUBLE VERTEX GRAPH OF BINOMIAL TREES

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Abstract: The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. The integrity of a graph G, $I(G)$, was introduced as a useful measure of the stability of a graph G and is defined as $I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\}$, where $m(G-S)$ denotes the order of a largest component of $G-S$, see [4]. In this paper we calculate the integrity of double vertex graph of binomial trees B_2, B_3, and B_4.

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1. Introduction

The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In an analysis of the stability of a communication network to disruption, two

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questions that come to mind are: (i) How many vertices can still communicate? (ii) How difficult is it to reconnect the graph? The concept of integrity was introduced as a measure of graph stability by Barefoot, Entringer and Swart [4]. Formally, the integrity is

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order of a largest component of $G - S$. The integrity is a measure which deals with the first question stated above, namely how many vertices can still communicate? If the set S achieves the integrity, then it is called an I-set of G. That is, if $|S| + m(G - S) = I(G)$ for any set S, then S is called an I-set.

In Section 2, we review some of the known results on the subject. In Section 3, we compute the integrity of double vertex of binomial trees B_2, B_3, and B_4 .

2. Basic Results

In this section we will review some of the known results.

Theorem 1. (see [5]) *The integrity of:*
(a) the complete graph K_n *is* n*;*
(b) the star graph $K_{1,n}$ *is* 2*;*
(c) the path P_n *is* $\lceil 2\sqrt{n+1} \rceil - 2$*;*
(d) the cycle C_n *is* $\lceil 2\sqrt{n} \rceil - 1$*;*
(e) the complete bipartite graph $K_{m,n}$ *is* $1 + \min\{m, n\}$*.*

Theorem 2. (see [5]) *Let* $2 \leq m \leq n$*. Then* $I(K_m \times K_n) = nm - \max_{1 \leq j < m} \lfloor \frac{n(m-j)}{m} \rfloor$.

The following theorems are about Cartesian product of some special graphs.

Theorem 3. (see [3]) *Let* $n \geq 3$ *be an integer and let* $a = \lfloor \sqrt{n+1} \rfloor$ *and* $b = \lceil 2\sqrt{n+1} \rceil$ *be two integers. Then*

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1, & \text{if } n + 1 \leq a(b - a - \frac{1}{2}), \\ 2I(P_n), & \text{otherwise.} \end{cases}$$

Theorem 4. (see [3]) *Let* $n \geq 3$ *be an integer and let* $a = \lfloor \sqrt{n} \rfloor$ *and* $b = \lceil 2\sqrt{n} \rceil$ *be two integers. Then:*
(i) for $n = 3$ *or* $n = 4$*, then* $I(K_2 \times C_n) = 2I(C_n) - 1 = 5$*;*

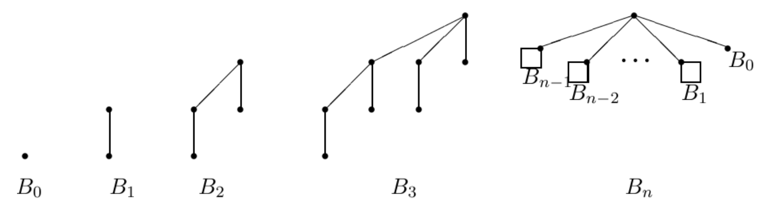

Figure 1:

(ii) for $n \geq 5$,

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1, & \text{if } n+1 \leq a(b-a-\frac{1}{2}), \\ 2I(C_n), & \text{otherwise.} \end{cases}$$

3. The Integrity of Double Vertex Graph of Binomial Trees

In this section we first give the definition of binomial tree B_n .

Definition 5. (see [6]) The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other (Figure 1).

Now we give the definition of double vertex graph of G.

Definition 6. (see [1]) Let G be a graph of order $n \geq 2$. The *double vertex graph* $U_2(G)$ of G is the graph whose vertex set consists of all 2-element subsets of V such that two distinct vertices x, y and u, v are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G.

Figure 2 shows a graph G and its double vertex graph.

Firstly we give the integrity of double vertex graph of binomial tree B_2.

Theorem 7. *Let* $U_2(B_2)$ *be a double vertex graph of binomial tree* B_2. *Then* $I(U_2(B_2)) = 3$.

Proof. We label the vertices of the binomial tree B_2 by $1, 2, 3, 4$ as in Figure 3. Also the double vertex graph of B_2 is shown in Figure 3. To obtain the I-set of $U_2(B_2)$ we must separate the graph to components which have as possible as equal vertex numbers. Hence we have two cases:

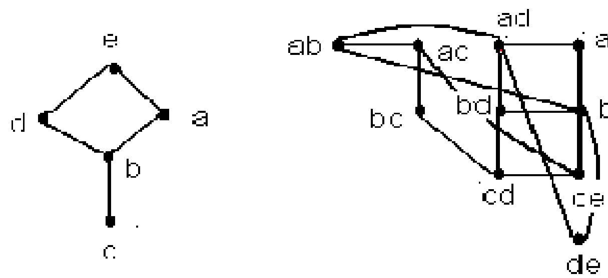

Figure 2:

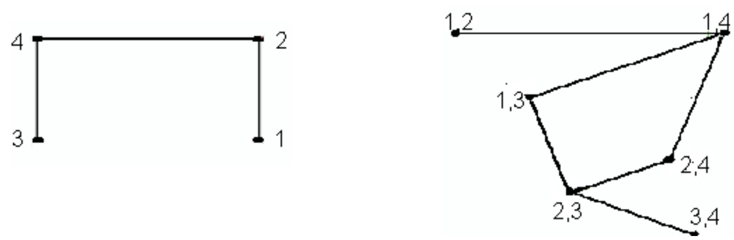

Figure 3.

Case 1. If $S = \{(1,4),(2,3)\}$, then $m(U_2(B_2) - S) = 1$ and $I_1(U_2(B_2)) = 3$.

Case 2. Whatever we select as the set S (except $S = \{(1,4),(2,3)\}$), we obtain $m(U_2(B_2) - S) \geq 1$ and $I_2(U_2(B_2) \geq 3$.

Consequently, $I(U_2(B_2)) = \min\{I_1(U_2(B_2)), I_2(U_2(B_2))\} = 3$. □

Theorem 8. *Let $U_2(B_3)$ be a double vertex graph of binomial tree B_3. Then $I(U_2(B_3)) = 12$.*

Proof. We label the vertices of the binomial tree B_3 by $1, 2, 3, ..., 8$ as in Figure 4. Also the double vertex graph of B_3 is shown in Figure 4. Let S be a subset of $V(U_2(B_3))$. Then we have two cases:

Case 1. Let $S = \{(1,6),(3,6),(4,6),(2,5),(2,7),(2,8)\}$. If we remove the vertices of S from graph $U_2(B_3)$, then we have the disconnected graph in Figure 5. So we have $m(U_2(B_3) - S) = 6$ and $I_1(U_2(B_3) = 12$.

Case 2. Whatever we select as the set S (except $S = \{(1,6),(3,6),(4,6),(2,5),(2,7),(2,8)\}$), we obtain $m(U_2(B_3) - S) \geq 6$ and $I_2(U_2(B_3)) \geq 12$.

Consequently, $I(U_2(B_3)) = \min\{I_1(U_2(B_3)), I_2(U_2(B_3))\} = 12$. □

Theorem 9. *Let $U_2(B_4)$ be a double vertex graph of binomial tree B_4.*

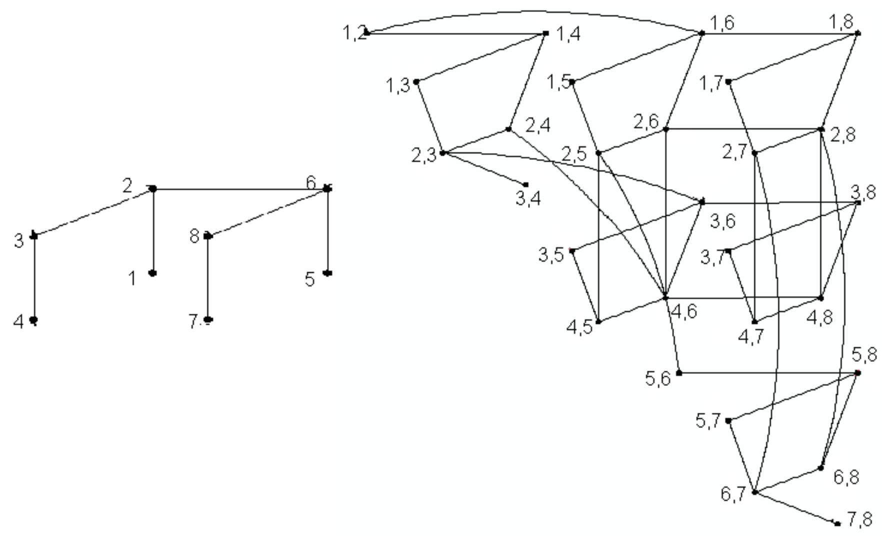

Figure 4.

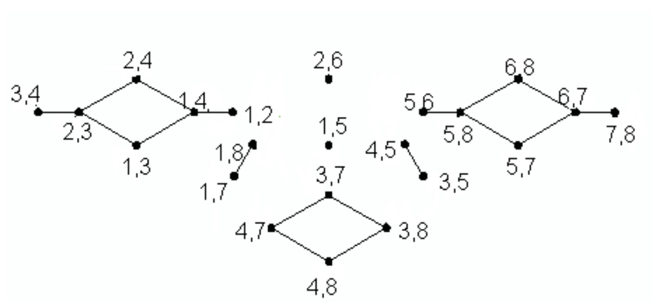

Figure 5.

Then $I(U_2(B_4)) = 38$.

Proof. We label the vertices of the binomial tree B_4 by $1, 2, 3, ..., 16$ as in Figure 6. Moreover the double vertex graph of B_4 is shown in Figure 7. Let S be a subset of $V(U_2(B_4))$ such that $|S| + m((U_2(B_4) - S) = I(U_2(B_4))$. To obtain the I-set of $V(U_2(B_4))$, then we have the following steps:

Step 1. Consider the graph B_4 and the edge $(6, 10)$ in the Figure 6. Firstly we remove the vertices in such way that these vertices are incident to the

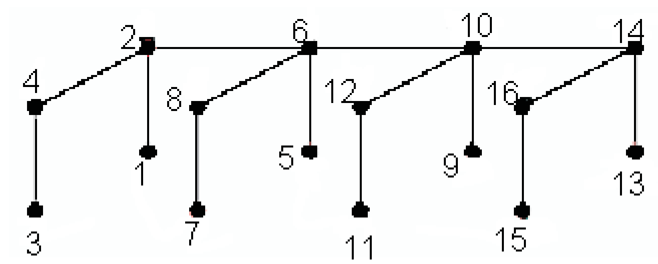

Figure 6.

n	2	3	4
$I(B_n)$	3	4	6
$I(U_2(B_n))$	3	12	38

Table 1:

edges in $U_2(B_4)$ which are obtained by using the edge (6,10). Let S_1 be a set of these vertices. Then $S_1 = \{(1,10),(2,10),(3,10),(4,10),(5,10),(7,10),(8,10),(6,9),(6,11),(6,12),(6,13),(6,14),(6,15),(6,16)\}$. If we remove the vertices of S_1 from graph $U_2(B_4)$, then we have the disconnected graph in Figure 8.

Step 2. The remaining graph has two subgraphs $U_2(B_3)$ and other components. By Theorem 3.2, we can easily determine the removing vertices from each one of subgraphs $U_2(B_3)$. These vertices are $S_2 = \{(1,6),(3,6),(4,6),(2,5),(2,7),(2,8),(9,14),(10,13),(10,15),(10,16),(11,14),(12,14)\}$. Now we consider the component with order 16 in Figure 8. Hence we must remove the vertices $S_3 = \{(2,13),(2,14),(4,15),(4,16)\}$ from this component. Consequently we have $S = S_1 \bigcup S_2 \bigcup S_3$. If we remove the vertices of S, then we obtain the graph in Figure 9.

Consequently we have $m(U_2(B_4) - S) = 8$ and $I(U_2(B_4)) = 38$. □

4. Conclusion

The binomial trees are known as an important kind of trees. Now suppose that we want to design a communication network. We know that the double vertex graph of a binomial tree with order 2^n has $2^n(2^n - 1)$ vertices, see [1]. Now we consider the following Table 1 (see [3] for $I(B_2)$, $I(B_3)$, and $I(B_4)$). Although

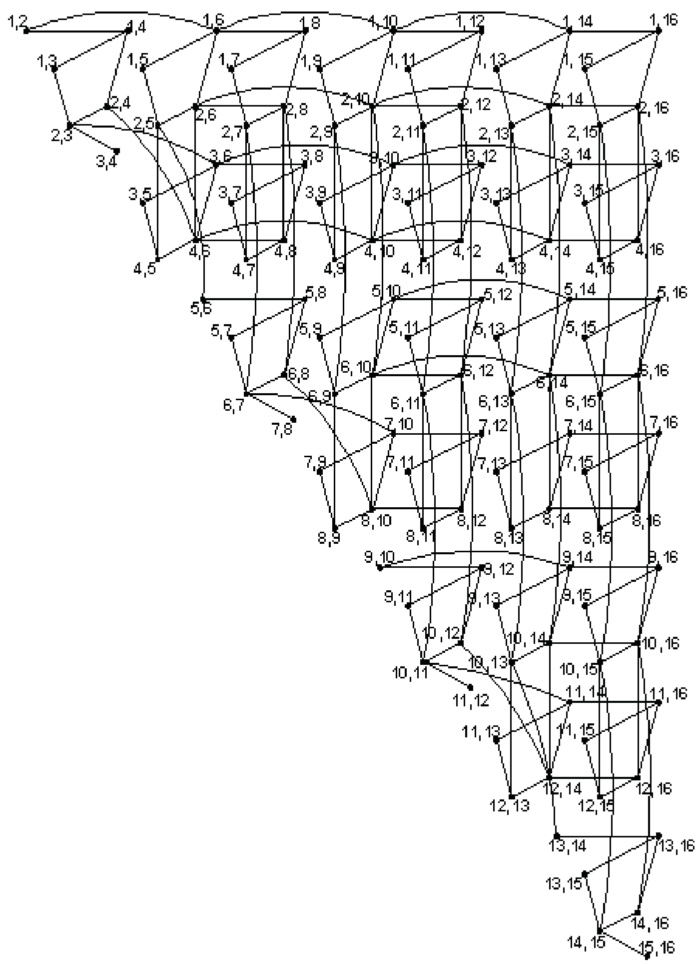

Figure 7.

the number of the vertices of double vertex graph of a binomial tree are increase in high number, the vulnerability values are not. This means that the double vertex graph of a binomial tree can be more strong according to a binomial tree in an analysis of the vulnerability of a communication network to disruption. Consequently if we design a communication network, then we can prefer the double vertex graph of a binomial tree.

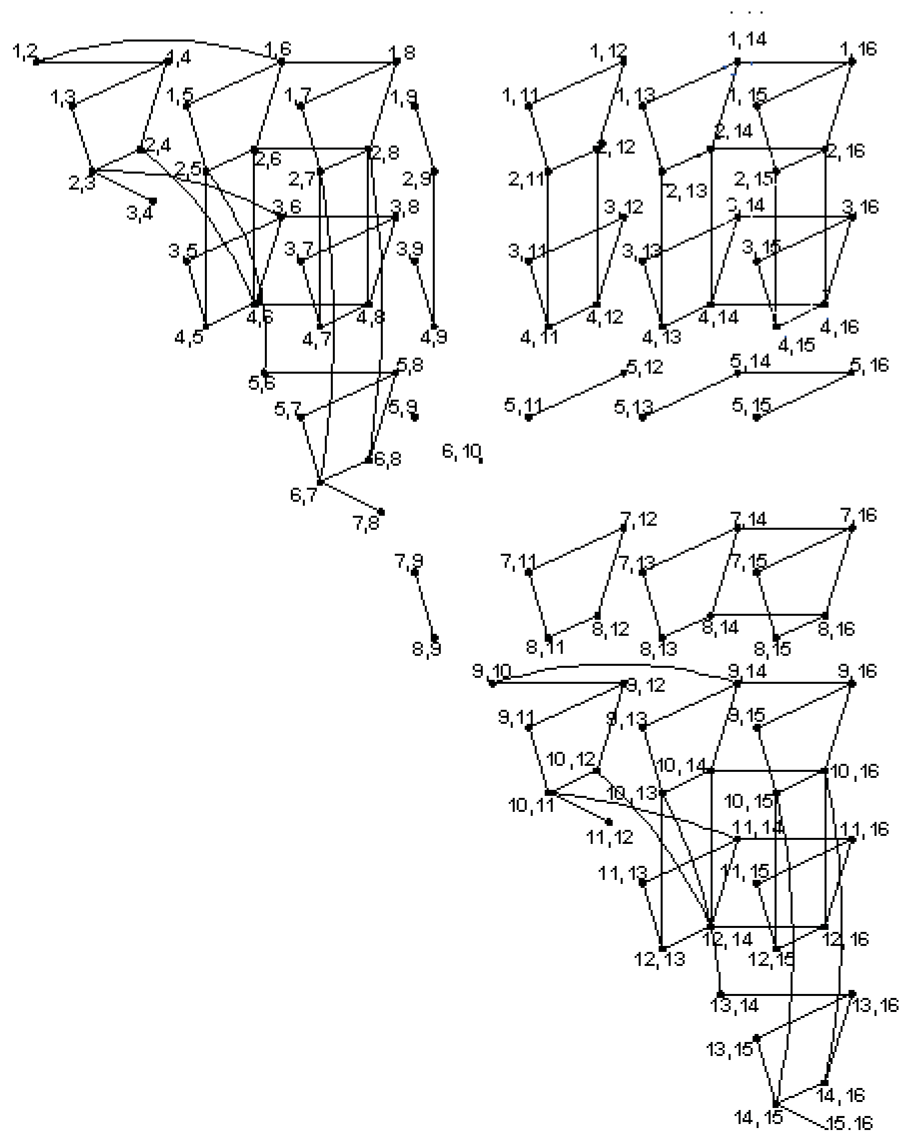

Figure 8.

Acknowledgments

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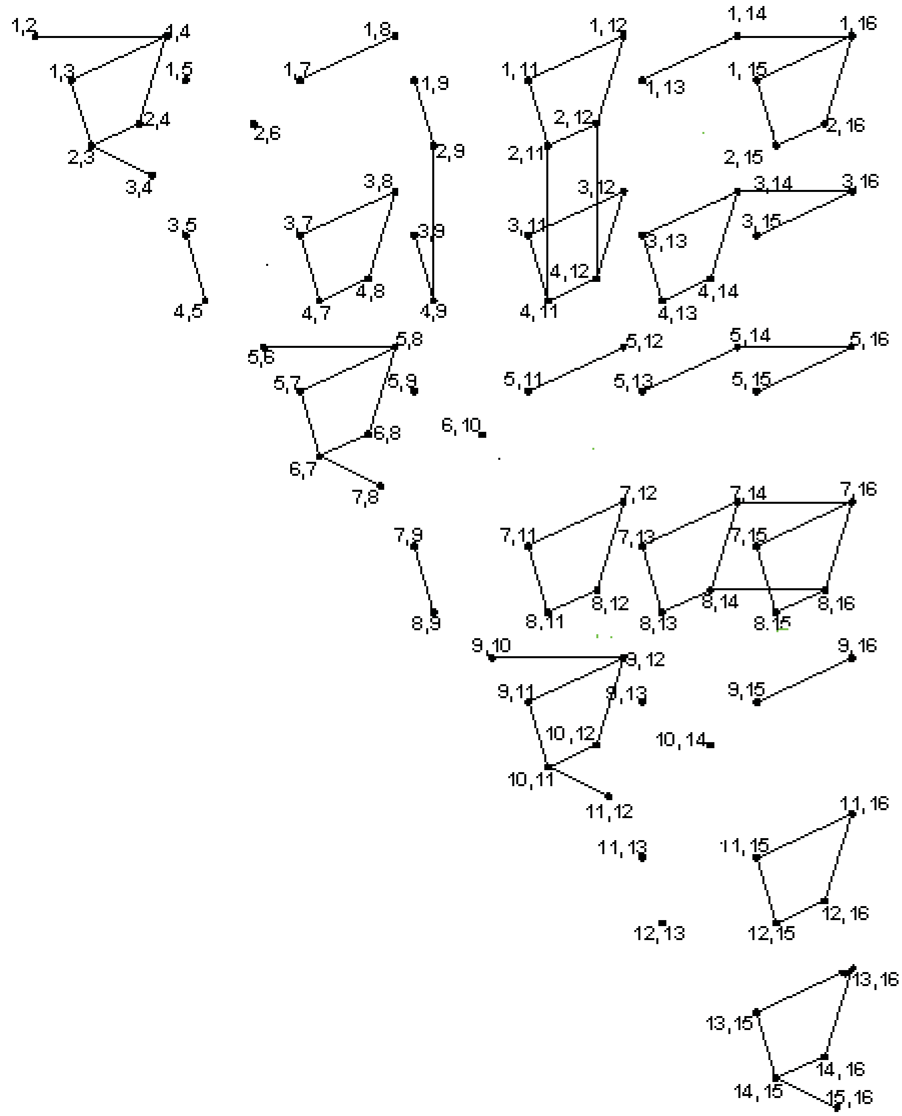

Figure 9.

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