International Journal of Pure and Applied Mathematics

Volume 36 No. 2 2007, 267-269

PLANE CURVES WITH ORDINARY SINGULARITIES AND MANY NODES

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Abstract: Here we use a paper of T. Mignon to extend Severi's theory of nodal plane curves to the case some (but not too many) of the prescribed singularities are ordinary multiple points with arbitrary multiplicity.

AMS Subject Classification: 14H50, 14H10
Key Words: plane curve, ordinary multiple point, singular plane curve

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Here we use [3], Theorem 1, and the Severi's theory of nodal plane curves as presented in [5] to prove the following result.

Theorem 1. *Fix an integer $m \geq 3$. There is an integer $a_0(m) \gg 0$ such that for all integers $d \geq a \geq a_0(m)$, the following properties is satisfied. Fix integers $s > 0$ and m_i, $1 \leq i \leq s$, $2 \leq m_i \leq m$, such that $(a^2 + 3a)/2 \geq \sum_{i=1}^{s} m_i(m_i + 1)/2$ and set $\alpha := (a-1)(a-2)/2 - \sum_{i=1}^{s} m_i(m_i - 1)/2$. Fix any integer x such that $0 \leq x \leq (d-1)(d-2)/2 - (a-1)(a-2)/2$. Then there is an irreducible family Γ of integral degree d plane curves such that each $Y \in \Gamma$ has exactly $s + x$ singular points, say $P_1, \dots, P_s, Q_1, \dots, Q_x$, each P_i is an ordinary multiple point with multiplicity m_i and each Q_j is an ordinary node of Y. Varying $Y \in \Gamma$ the s-ples $(P_1, \dots, P_s)$ cover an open subset of $(\mathbf{P}^2)^s$. We have $\dim(\Gamma) = (d^2 + 3d)/2 - \sum_{i=1}^{s} m_i(m_i - 1)/2 + 2s - x$, i.e. Γ has the expected dimension, and it is not contained in a larger family of plane curves with the same or worst singularities.*

Received: December 11, 2006

Proof. Let $a_0(m)$ be the integer $d'(m)$ used in [3], Theorem 1.If $a = d$, then the statement is [3], Theorem 1. Hence we may assume $d > 0$. Fix a, s, m_i, $1 \le i \le s$, and s general points $P_1, \dots, P_s \in \mathbf{P}^2$. Let W be the set of all integral degree a plane curves C such that $\mathrm{Sing}(C) = \{P_1, \dots, P_s\}$ and each P_i is an ordinary point with multiplicity m_i of P. By [3], Theorem 1, $W \neq \emptyset$, W is integral, $\dim(W) = (a^2 + 3m)/2 - \sum_{i=1}^{s} m_i(m_i + 1)/2$, and W is not contained in a larger family of plane curves with the same or worst singularities. Fix any $C \in W$ and let D bethe union of C and $d - a$ general lines Let $u : S \to \mathbf{P}^2$ be the blowing-up of the points $P_1, \dots, P_s$. Let C' (resp. D') be the strict transform of C (resp. D) in S. Thus C' is smooth, D' is nodal and connected and the counterimages of the lines of D do not intersect the exceptional divisors of u. Notice that $p_a(C') = \alpha$. By [5], 2.11 and 2.14, we may smooth all nodes of C' except exactly x ones, obtaining a family Δ of integral nodal curves with exactly x nodes and with the expected dimension. For any $E \in \Delta$, the curve $u(E)$ is integral, with exactly $x + s$ singular points, each P_i is an ordinary point with multiplicity m_i of $u(E)$, while the other singular points of $u(E)$ are ordinary nodes. Moving the points $(P_1, \dots, P_s)$ in $(\mathbf{P}^2)^s$ we get the formula for $\dim(\Gamma)$. □

We may take as $a_0(m)$ the integer $d'(m)$ used in [3] (e.g. we may take $a_0(m) = 2((38(m+2))^{2^{m-1}}$ ([3], line 4 of p. 218).

Let $V(d; s, m_1, \dots, m_s)$ denote the set of all integral degree d plane curves Y such that $\sharp(\mathrm{Sing}(Y)) = s$, say $\mathrm{Sing}(Y) = \{P_1, \dots, P_s\}$, and each P_i is an ordinary point with multiplicity m_i. In general, $V(d; s, m_1, \dots, m_s)$ is not irreducible and it may even have an irreducible component with the expected dimension and another irreducible component with higher dimension ([2], [1]).

Question 1. Is $V(d; s, m_1, \dots, m_s)$ irreducible, when $m_i = 2$ for "many" integers i? Here if $m_1 \ge \dots \ge m_s \ge 2$ we only require $d \ge m_1 + m_2$ (i.e. the restriction coming from Bezout) and $(d-1)(d-2/2 \ge \sum m_i(m_i - 1)/2$ (i.e. the restrixtion coming from the genus formula for integral plane curves. By [4] or [6] this is true if $m_i = 2$ for all $i \ge 2$.)

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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