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GONALITY OF THE NORMALIZATION OF CERTAIN PLANE CURVES

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Abstract: Here we compute the gonality (always computed by a pencil of lines) of the normalization of certain plane curves.

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Here we prove the following results.

Theorem 1. *Let $Y \subset \mathbf{P}^2$ be an integral degree $d \geq 3$ curve, $f : X \to Y$ its normalization and $W \subset \mathbf{P}^2$ the conductor of f. Set $g := p_a(X)$ and $\eta := p_a(Y) - g = (d-1)(d-2)/2 - g = \text{length}(W)$. Let $i(W)$ denote the minimal non-negative integer t such that $h^1(\mathbf{P}^2, \mathcal{I}_W(t)) = 0$. Assume $\eta > 0$ and $i(W) > 0$. Then X has gonality at least $d - i(W) - 2$ and equality holds if and only if $\eta = (i(W)+2)(i(W)+1)/2$, Y has a unique singular point, say P, and Y has an ordinary singularity with multiplicity $i(W)+1$ at P.*

Theorem 2. *Fix an integer $d, s \geq 2$ s distinct points $P_i \in \mathbf{P}^2$, $1 \leq i \leq s$, and integers $m_i \geq 2$, $1 \leq i \leq s$, such that no 3 of the points P_i are collinear, $m_1 > \sum_{j=2}^{s} m_j$ and $d > \sum_{i=1}^{s} m_i$. There is an integral degree d plane curve Y such that $\text{Sing}(Y) = \{P_1, \dots, P_s\}$ and each P_i is an ordinary points of Y with multiplicity m_i. Fix any such Y and let $f : X \to Y$ be the normalization. Then X has gonality $d - m_1$ and a unique $g^1_{d-m_1}$ which is induced by the pencil of all lines through P_1*

Proof of Theorem 1. The "if" part is true by [1] applied to the blowing-up

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of $\mathbf{P}^2$ at P (here we use that $\eta > 0$). Now we will prove the other assertions. Let k be the gonality of X. Let $c(W)$ be the first integer t such that $\eta \le (c(W)+2)(c(W)+1)/2$. Adjunction theory gives $c(W) \le i(W)$. Fix $R \in \mathrm{Pic}^k(X)$ computing its gonality and take a general $D \in |R|$. Hence D is reduced and mapped bijectively by f into Y_{reg}. Set $A := f(D)$. Since $g \ge 2$, then $h^1(X,R) > 0$. Hence $h^0(\mathbf{P}^2, \mathcal{I}_{W\cup A}(d-3)) \neq 0$. Since $h^1(\mathbf{P}^2, \mathcal{I}_W(c(W)) = 0$, we easily get (e.g. using $\sharp(A)$ times Horace Lemma) $\sharp(A) \ge d-2+c(W)$ and that if $\sharp(A) = d-2+c(W)$, then A is contained in a line. Since D is general in $|R|$, we get that $|R|$ is induced by a 1-dimensional linear subseries of $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$. Hence there is $P \in \mathbf{P}^2$ such that $|R|$ is induced by the lines through P. Let $\mu \ge 0$ be the multiplicity of C at P. Since μ is the intersection multiplicity of C at P with a general line through P, we get $\mu = d-k$. Hence Y has multiplicity at least $c(W)+1$ at P. Since $\eta \le (c(W)+2)(c(W)+1)/2$ by the definition of the integer $c(W)$, we also get that P must be an ordinary singular point with multiplicity $c(W)+1$ of Y and that $\mathrm{Sing}(Y) = \{P\}$. □

Proof of Theorem 2. Set $Z := \sum_{i=1}^{s} m_i P_i$ and $W := \sum_{i=1}^{s}(m_i-1)P_i$. It is easy to check using Horace Lemma that $h^1(\mathbf{P}^2, \mathcal{I}_Z(d-2)) = 0$. Hence the existence of some Y's easily follows from Castelnuovo-Mumford's Lemma and Bertini's Theorem. Fix Y and hence X, f. Hence W is the conductor of f. Fix $L \in \mathrm{Pic}^x(X)$ computing the gonality of X. Hence $H^0(X,L) = 2$ and L is spanned. Fix a general $D \in |L|$. The generality of D implies that D is reduced, $D \cap f^{-1}(\mathrm{Sing}(Y))$ and that none of the x points of $A := f(D)$ is contained in one of the $(s-1)$ lines $\langle\{P_1, P_j\}\rangle$, $2 \le j \le s$. By adjunction theory we have $h^1(\mathbf{P}^2, \mathcal{I}_{W\cup A}(d-3)) \neq 0$. Applying $\sum_{j=2}^{s} m_j$ times Horace Lemma, using exactly m_j times the line $\langle\{P_1, P_j\}\rangle$, we get $h^1(\mathbf{P}^2, \mathcal{I}_{W\cup A}(d-3)) \le h^1(\mathbf{P}^2, \mathcal{I}_{aP_1\cup A}(d-3))$, where $a := m_1 - \sum_{j=1}^{s} m_j$. Then use [1] to the blowing-up of $\mathbf{P}^2$ at P_1. □

We work over an algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$.

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