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EXAMPLES OF UNIQUELY SOLVABLE POLYNOMIAL MULTIVARIATE INTERPOLATION PROBLEMS (LAGRANGE, PARTIAL HERMITE OR BIRKHOFF)

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Abstract: Here we give several examples of uniquely solvable polynomial multivariate interpolation problems (Lagrange, partial Hermite or partial Birkhoff).

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1. Examples

Fix integers $n > 0$ and $d > 0$. We will first study some problems on Lagrange and Birkhoff multivariate interpolation on the cube $[0,1]^n \subset \mathbf{R}^n$ or the cube $[0,1]^n \subset \mathbf{R}^n$ with respect to the set $A_{n,d}$ of all polynomials in n real variables with total degree at most d. For all integers $n > 0$, $d \geq 0$, let $\wp_{n,d}$ the set of all real polynomials in n variables $x_1, \dots, x_n$ and $\Gamma(n,d) := \{(i_1, \dots, i_n) \in \mathbb{Z}^n : i_j \geq 0$ for all j and $i_1 + \cdots + i_n \leq d\}$. Notice that $\wp_{n,d}$ is an $\binom{n+d}{n}$-dimensional real vector space and that $\sharp(\Gamma(n,d)) = \binom{n+d}{n}$.

We will first give the set-up for the Lagrange interpolation in the bivariate case.

Example 1. Fix an integer $d > 0$. For all integers i, j such that $0 \leq i \leq d$ and $0 \leq j \leq d - i$, set $P(i,j) := (i/d, j/d) \in [0,1]^2$. Set $S_{2,d} := \{P(i,j)\}$. Notice that $\sharp(S_{2,d}) = (d+2)(d+1)/2$.

Claim. *$S_{2,d}$ is a unique Lagrange interpolation set for $\wp_{2,d}$.*

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Proof of Claim. Since $\sharp(S_{2,d}) = \sharp(\Gamma_{2,d}) = \dim_{\mathbb{R}}(\wp(2,n))$, it is sufficient to show that if $f \in \wp(2,d)$ and $f(P) = 0$ for all $P \in S_{2,d}$, the $f \equiv 0$. Since the case $d = 0$, is obvious, we may assume $d > 0$. For all $c \in \mathbb{R}$ set $D_c := \{x_2 = c/(d+1)\}$. Each D_c is an affine line of $\mathbb{R}^2$. Notice that $S_{2,d} \subset \cup_{c\in\{0,\dots,d\}} D_c$ and that $\sharp(S_{2,d} \cap S_j) = d + 2 - j$ for all $j \in \{0, \dots, d\}$. On an affine line $D_c \cong \mathbb{R}$ every finite $S \subset D_c$, $S \neq \emptyset$ is a unique interpolation set for the $\sharp(S)$-dimensional $\mathbb{R}$-vector space of all polynomials of degree at most $\sharp(S) - 1$. Apply this one-variable fact to D_0. We get that f is divisible by the equation $x_2 = 0$ of D_0. Apply this one-variable fact to to D_1. We get that f/x_2 is divisible by the equation $x_2 - 1/(d + 1)$ of D_1. After $d - 1$ similar steps using $D_2, \dots, D_d$ we get that f is divisible by $h := \prod_{j=0}^{d}(x_2 - j/(d + 1))$. Since $\deg(h) = d + 1$, we get $f \equiv 0$ proving the claim. □

Now we will give the general set-up for the lagrange interpolation in the multivariate case.

Example 2. Fix integers $n > 0$ and $d > 0$. For all $(i_1, \dots, i_n) \in \Gamma(n, d)$, set

$$P(i_1, \dots, i_n) := (i_1/d, \dots i_n/d) \in [0, 1]^n.$$

Set $S_{n,d} := \{P(i_1, \dots, i_n)\}_{(i_1,\dots,i_n)\in\Gamma(n,d)}$. Notice that $\sharp(S_{n,d}) = \binom{n+d}{n}$.

Claim. *$S_{n,d}$ is a unique Lagrange interpolation set for $\wp_{n,d}$.*

Proof of Claim. We repeat with only minor modifications the proof of Claim in Example 1. Since $\sharp(S_{n,d}) = \sharp(\Gamma_{n,d}) = \dim_{\mathbb{R}}(\wp(n,n))$, it is sufficient to show that if $f \in \wp(n,d)$ and $f(P) = 0$ for all $P \in S_{n,d}$, the $f \equiv 0$. By Example 1 the case $n = 2$ is true for all d. Hence we may assume $n \geq 3$ and that the result is true for all integers n', d' with $n' < n$. Since the case $d = 0$ is obvious, we may assume $d > 0$. For all $c \in \mathbb{R}$ set $D_c := \{x_n = c/(d + 1)\}$. Each D_c is an affine line of $\mathbb{R}^n$. Notice that $S_{n,d} \subset \cup_{c\in\{0,\dots,d\}} D_c$ and that $\sharp(S_{n,d} \cap S_j) = \binom{n+d-j-1}{n-1}$ for all $j \in \{0, \dots, d\}$. Furthermore after a translation the set $S_{n,d} \cap S_j$ is just the set $S_{n-1,d-j}$. Apply this observation and the inductive assumption to D_0. We get that f is divisible by the equation $x_n = 0$ of D_0. Apply this observation and the inductive assumption for the integer $n - 1$ to D_1. We get that f/x_n is divisible by the equation $x_n - 1/(d + 1)$ of D_1. After $d - 1$ similar steps using $D_2, \dots, D_d$ we get that f is divisible by $h := \prod_{j=0}^{d}(x_n - j/(d+1))$. Since $\deg(h) = d + 1$, we get $f \equiv 0$ proving the claim. □

Now we will consider a few partial Hermite and Birkhoff interpolation problems, i.e. at each interpolation node we prescribe the values of some (but not all) partial derivatives. Set $\partial_i := \partial/\partial_{x_i}$, $1 \leq i \leq n$. Again, we will first do the bivariate case. Use the notations and properties of the set $S_{n,d}$ introduced in Examples 1 and 2. As in those examples D_c, $c \in \mathbb{R}$, denote the affine hyperplane $\{x_n = c/(d+1)\}$ of $\mathbb{R}^n$. We will first consider a bivariate partial Hermite interpolation problem.

Example 3. Let $\epsilon : S_{2,d} \to \{0, \ldots, d\}$ be any function such that $\sum_{P \in S_{2,d} \cap D_j} \epsilon(P) = d + 1 - j$ for all $j \in \{0, \ldots, d\}$. We will say that ϵ is an admissible function for $S_{2,d}$. Set $S(2, d, \epsilon) := \{P \in S_{2,d} : \epsilon(P) \neq 0\}$. The set $S(2, d, \epsilon)$ will be the set of all nodes for the folowing partial bivariate Hermite interpolation on the set of all polynomials of degree at most d in two real variables. For every $P \in S(2, d, \epsilon)$ prescribe the value and all partial derivatives with respect to the variable x_1 up to order $\epsilon(P) - 1$. Since each line D_j, $j \in \{0, \ldots, d\}$, support exactly $d + 1 - j$ condition for a one-variable Hermite interpolation problem for polynomials of degree at most $d - j$, the proof of Claim in Example 1 gives that these data give a uniquely solvable partial Hermite bivariate interpolation problem.

Now we will do the general multivariate case.

Example 4. We use the set-up of Example 2. Let $\epsilon : S_{n,d} \to \{0, \ldots, d\}$ be any function such that for every line D parallel to the x_1-axis and containing at least one point of $S_{n,d}$ we have $\sum_{P \in S_{n,d} \cap D} \epsilon(P) = \sharp(S_{n,d} \cap D)$. Set $S(n, d, \epsilon) := \{P \in S_{n,d} : \epsilon(P) \neq 0\}$. At each point $P \in S(n, d, \epsilon)$ we prescribe the value and all partial derivatives $(\partial_1)^a$ with $a \leq \epsilon(P) - 1$. Then we repeat the inductive proof of Example 2 to check that in this way we get a uniquely solvable partial Hermite multivariate interpolation problem.

Now we may describe a uniquely solvable Birkhoff bivariate problem.

Example 5. Fix an integer $d > 0$ and an admissible function $\epsilon : S_{2,d} \to \{0, \ldots, d\}$. Recall that $D_0 := \{x_2 = 0\}$. Let η denote a degree d real univariate Birkhoff problem with associated incidence matrix E satisfying the Pólya condition and with no odd degree supported sequence (see [2], p. 6). Equivalently, η is order-regular (see [1] or [2], Theorem 2.2.3). We distribute η among the $d+1$ points $D_0 \cap S_{2,d} = \{(j/d, 0)\}_{0 \leq j \leq d}$ in this order. At the other $d(d+1)/2$ points of $S_{2,d} \backslash S_{2,d} \cap D_0$ we impose the Hermite data given by ϵ. We claim that the Birkhoff bivariate problem on $S_{2,d}$ is uniquely solvable. Indeed, its restriction to D_0 is a uniquely solvable Birkhoff problem, because E is order-regular. Then apply the proofs of Examples 1 and 2 to control the remaining conditions: roughly speaking on $S_{2,d} \backslash S_{2,d} \cap D_0$ we have a partial Hermite bivariate problem for polynomial of degree at most $d - 1$ and the proofs of Examples 1 and 3 show that this partial Hermite bivariate problem is uniquely solvable.

Now by induction on n we may do the case $n > 2$ of Example 5.

Example 6. Fix integers $d > 0$, $n > 2$ and take ϵ as in Example 4. Set $D := \{x_2 = \cdots = x_n = 0\}$. Let η denote a degree d real univariate Birkhoff problem with associated incidence matrix E satisfying the Pólya condition and with no odd degree supported sequence ([2], p. 6). We distribute η among the $d+1$ points $D \cap S_{n,d} = \{(j/d, 0, \dots, 0)\}_{0 \le j \le d}$ in this order. At the other $\binom{n+d}{n} - d - 1$ points of $S_{2,d} \backslash S_{2,d} \cap D_0$ we impose the Hermite data given by ϵ. We claim that the Birkhoff multivariate problem on $S_{n,d}$ is uniquely solvable. Indeed, the case $n = 2$ is proved in Example 5, while in the inductive step needed to copy the proof of Example 2 and 4 to conclude by induction on n we only use a partial Hermite multivariate problem (not a general partial Birkhoff problem) and this partial Hermite multivariate problem is uniquely solvable by Example 4.

In the next two examples we will consider bivariant interpolation in which the nodes are contained in $[-1,1]^2$. We will simultaneously consider Lagrange, Hermite and Birkhoff interpolation.

Example 7. Fix an even integer $d \ge 2$. For every $j \in \{0, \dots, d\}$ let $L_j \subset \mathbb{R}^2$ denote the half-line $\{t(\cos(2\pi j/(d+1))), t(\sin(2\pi j/(d+1)))\}_{t>0}$. For $1 \le i \le d+1-j$ set $P_{j,i} := ((i/(d+1)(\cos(2\pi j/(d+1)), ((i/(d+1)\sin(2\pi j)(d+1))) \in L_j \cap [-1,1]^2$. We use the nodes $P_{j,i}$ for the Lagrange interpolation. To see that the corresponding interpolation problem is uniquely solvable we use the lines associared to the half-lines $L_0, \dots, L_d$ (in this order) and the inductive proof of Example 1. For the Hermite interpolation we fix the function ϵ. If $P \in L_j$ and $\epsilon(P) \ge 2$, then we prescribe the values at P of the first $\epsilon(P) - 1$ partial derivatives in the direction of the line associated to L_j. For Birkhoff interpolation we use the function η on the half-line L_0. Here we use that d is even and hence that for all $i \ne j$ the lines associated to the half-lines L_i and L_j are different.

Example 8. Fix the even integer d and the half-lines L_j as in Example 7. Now we use as nodes the points $Q_{j,i} := ((i/(d+1-j)(\cos(2\pi j/(d+1-j)), ((i/(d+1-j)\sin(2\pi j)(d+1-j))) \in L_j \cap [-1,1]^2$.

When $n > 2$ for Lagrange and partial Hermite interpolation we may mix Examples 7 and 8 with all other examples (plus the clasical case on the real line) and get the following result.

Example 9. Fix an even integer $d \ge 2$ and an integer $n > 23$. For every $j \in \{0, \dots, d\}$ let $L_j \subset \mathbb{R}^2$ denote the half-line $\{t(\cos(2\pi j/(d+1))), t(\sin(2\pi j/(d+1)))\}_{t>0}$ and D_j the associated line. See $\mathbb{R}^2$ as the 2-dimensional linear subspace $\{x_3 = \cdots = x_n = 0\}$ of $\mathbb{R}^n$. Set $H_j = D_j \times \mathbb{R}^{n-2}$. Set $n' := n - 2$. Use the set up of Examples 1 and 3 (resp. Examples 2 and 4) for the integer n' when $n' = 2$ (resp. ($n' > 2$)). Use the clasical set-up on the real line if $n' = 1$. Taking the product on the factor $\mathbb{R}^2$ with either the Example 7 or the Example 8 we get other Lagrange or partial Hermite uniquely solvable interpolation problem for polynomials of degree at most d in n real variables.

Now we describe a less elementary uniquely solvable bivariate partial Hermitian problem.

Example 10. For any $P \in \mathbb{R}^2$ and for all integers $m \geq b \geq 1$, let $(m, b)P$ and $[m, b]P$ denote the following partial bivariate Hermitian data. For $(m, b)P$ we prescribe the evaluation at P of all partial derivatives of order at most $m-1$ in which δ_2 appears only up to power $b-1$. Notice that $(m, b)P$ imposes $(m+1)m/2 - (m-b+1)(m-b)/2$ independent linear conditions to the set of all polynomial of degree at most $m-1$ on 2 variable. For $[m, b]P$ we prescribe the evaluation at P of all derivatives $(\delta_1)^u(\delta_1)^v$ with $u \leq m-1$ and $v \leq b-1$. Notice that $[m, b]P$ imposes mb independent linear conditions to the set of all polynomial of degree at most $m+b-1$ on 2 variable Fix integers $d \geq 1$ and $s > 0$. Fix s distinct lines D_i, $1 \leq i \leq s$, parallel to the x_1-axis. Fix integers $b_i > 0$, $1 \leq i \leq s$, such that $b_1 + \cdots + b_s = d+1$. On each D_i fix some distinct points, say $P_{i,j}$, $1 \leq j \leq c_i$, and integers $m_{i,j} \geq b_i$, $1 \leq j \leq c_i$, such that $\sum_{j=1}^{c_i} m_i = d + 1 - \sum_{l=1}^{i-1} b_i$. We use as interpolation data the union of all data $(m_{i,1}, b_i)P_{i,1}$, $1 \leq i \leq s$, and all data $[(m_{i,j}, b_i]P_{i,j}$, $1 \leq i \leq s$, $2 \leq j \leq c_i$. To see that the associated interpolation problem is uniquely solvable we use the so-called Horace Lemma first b_1 times with respect to D_1, then b_2 times with respect to D_2, and so on.

Example 11. We use the notations $(m, b)P$ and $[m, b]P$ introduced in Example 10. Fix integers $d \geq 1$ and $s > 0$. Fix s distinct lines D_i, $1 \leq i \leq s$, parallel to the x_1-axis. Fix integers $b_i > 0$, $1 \leq i \leq s$, such that $b_1 + \cdots + b_s = d+1$. On each D_i fix some distinct points, say $P_{i,j}$, $1 \leq j \leq c_i$, and integers $m_{i,j} \geq b_i$, $1 \leq j \leq c_i$. We also fix integers ϵ_i, $1 \leq i \leq s$, such that $1 \leq \epsilon_i \leq \min\{s, c_i\}$. On each D_i we add ϵ_i interpolation data of type $(m_{i,j}, b_i)P_{i,j}$ and $c_i - \epsilon_i$ data of type $[m_{i,j}, b_i]P_{i,j}$. We first apply the Horace Lemma with respect to D_1. If $\epsilon_1 = 1$, we apply again Horace Lemma with respect to D_1 $(b_1 - 1)$ times and then we reduce to a case with $c' := c - 1$. If $\epsilon_1 \geq 2$, we apply the Horace Lemma with respect to D_2. In this case we need to have $\sum_{j=1}^{c_2} m_{2,j} = d$. And so on. We need to assume that the total number of conditions on all $D_1, \ldots, D_s$ is $(d+2)(d+1)/2$. We give all possibilities in the case $s = 2$. The case covered by Example 10 is the case $\epsilon_1 = 1$, which forces $\epsilon_2 = 1$. In all the other cases we have $\epsilon_1 = \epsilon_2 = 2$ and we need to assume

$\sum_{j=1}^{c_1} m_{1,j} = d+1$ and $\sum_{j=1}^{c_2} m_{2,j} = d$.

When $n > 2$ Examples 10 and 11 give by induction on n the following partial Hermite problem in which only derivatives with respect to x_1 and to x_2 appear.

Example 12. First asssume $n = 3$. Take $d+1$ distinct hyperplanes H_j, $0 \le j \le d$, of the form $x_3 = \alpha_j$. On each hyperplane H_j prescribe the data of Examples 10 or 11 with respect to the integer $d-j$. Again applying the proof of Example 2 first to H_0, then to H_1, and so on, we see that this interpolation problem is uniquely solvable. If $n > 3$ take again $d+1$ hyperplanes M_j, $0 \le j \le d$, and apply the case $n' := n-1$ with respect to the integer $d' := d-j$.

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