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ON THE MINIMAL FREE RESOLUTION FOR FAT POINTS SCHEMES IN $\mathbf{P}^3$

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Abstract: Let $Z \subset \mathbf{P}^3$ be a general union of fat points. Here we prove that Z has the expected minimal free resolutions if many connected components of Z have multiplicity ones.

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1. Introduction

For any connected smooth scheme A, any $P \in A$ and any integer $m > 0$ let $\{mP, A\}$ denote the fat point with multiplicity m of A with P as its support, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as ideal sheaf. Hence $\{mP, A\}$ is a zero-dimensional scheme, $\{mP, A\}_{red} = \{P\}$ and $\text{length}(A) = \binom{m+\dim(A)-1}{\dim(A)}$. If $A = \mathbf{P}^3$ we will often write mP instead of $\{mP, \mathbf{P}^3\}$. Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^3}$ and $\Omega := \Omega^1{}_{\mathbf{P}^3}$. Let $Z \subset \mathbf{P}^3$ be a non-empty zero-dimensional scheme. Set $z := \text{length}(Z)$. The critical value of Z is the minimal integer $k > 0$ such that $z \leq \binom{k+3}{3}$. We will also say that k is the critical value of the integer $z > 0$. Z is said to have maximal rank if for every integer t the restriction map $\rho_{Z,t} : H^0(\mathcal{O}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ has maximal rank. If Z has critical value k, then Z has maximal rank if and only if $h^0(\mathcal{I}_Z(k-1)) = h^1(\mathcal{I}_Z(k)) = 0$. Z has the expected minimal free resolution if it has maximal rank and no line bundle appears in two different steps of the minimal free resolution of Z. It is well-known that this is equivalent to check that Z has maximal rank, that either $h^0(\mathcal{I}_Z \otimes \Omega(k+1)) = 0$ (case

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$3z \ge k(k+3)(k+2)/2$ or $h^1(\mathcal{I}_Z \otimes \Omega(k+1)) = 0$ (case $3z \le k(k+3)(k+2)/2$ and that either $h^0(\mathcal{I}_Z \otimes T\mathbf{P}^3(k-2)) = 0$ (case $3z \ge (k+3)(k+1)k/2)$ or $h^1(\mathcal{I}_Z \otimes T\mathbf{P}^3(k-2)) = 0$ (case $3z \le (k+3)(k+1)k/2$) (see [2], [3], [4]).

Theorem 1. *Fix integers $s > 0$ and $m_1 \ge \cdots \ge m_s > 0$. Let $Z \subset \mathbf{P}^3$ be a general union of s fat points of multiplicity $m_1, \cdots, m_s$. Let k be the critical value of Z. Assume $m_i = 1$ for at least $(m_1-1)k^2/2 + k\cdot(2m_1+5)^3/12$ integers $i \in \{1, \dots, s\}$. Then Z has the expected minimal free resolution.*

2. The proofs

For any smooth variety A, any finite $E \subset A$ and any integer $m > 0$ set $\{mE, A\} := \cup_{P\in E}\{mP, A\}$.

Remark 1. Fix an integer t. We have $h^0(\Omega(t)) = 0$ if $t \le 1$, $h^0(\Omega(t)) = (t-1)(t+1)(t+2)/2$ if $t \ge 2$, $h^2(\Omega(t)) = 0$, $h^1(\Omega(t)) = 0$ if $t \ne 0$, $h^1(\Omega) = 1$, $h^0(T\mathbf{P}^3(t)) = 0$ if $t \le -2$, $h^0(T\mathbf{P}^3(t)) = (t+3)(t+2)(t+5)/2$ if $t \ge 1$, $h^1(\Omega(t)) = 0$ if $t \ne -4$, and $h^1(T\mathbf{P}^3(t)) = 0$ (use the Euler sequence or see [2], p. 4 and p. 6).

Remark 2. Let $D \subset \mathbf{P}^3$ be a rational normal curve. $\Omega|D$ is a direct sum of 3 line bundles of degree -4. Hence $\Omega(t)|D$ is a direct sum of 3 line bundles of degree $3t-4$ and $T\mathbf{P}^3(t)|D$ is a direct sum of 3 line bundles of degree $3t+4$.

Remark 3. We will follow [1], §2, and [2], §3. Let $S \subset \mathbf{P}^3$ be a smooth cubic surface. Obviously, $h^0(S, \mathcal{O}_S(t)) = \binom{t+3}{3} - \binom{t}{3}$ for every integer $t \ge 0$ and $h^1(S, \mathcal{O}_S(t)) = 0$ for all $t \in \mathbb{Z}$. From the exact sequences

$$0 \to \Omega(t-3) \to \Omega(t) \to \Omega(t)|S \to 0\,, \tag{1}$$

$$0 \to T\mathbf{P}^3(t-3) \to T\mathbf{P}^3(t) \to T\mathbf{P}^3(t)|S \to 0 \tag{2}$$

and Remark 2 we get $h^0(S, \Omega(3)|S) = 6$, $h^0(S, \Omega(t)|S) = 3(t-1)(3t-2)/2$, for all $t \ge 4$, $h^1(S, \Omega(t)|S) = 0$ for all $t \ge 4$, $h^0(S, T\mathbf{P}^3(t)|S) = 3(t+2)(3t+5)/2$ for every $t \ge 0$ and $h^1(S, T\mathbf{P}^3(t)|S) = 0$ for every $t \ge 0$. There are rational normal curves $C, D \subset S$ such that $C \cup D$ is a complete intersection of S with a smooth quadric surface, $C \cup D$ is nodal $C^2 = D^2 = 1$ and $\sharp(C \cap D) = 5$. Hence $(T\mathbf{P}^3(t-2)|S)(D)|D$ (resp. $(\Omega(t+1)|S)(D)|D$) is a direct sum of 3 line bundles of degree $3t-1$ (resp. $3t$). Consider the exact sequence on S:

$$0 \to \mathcal{O}_S(t) \to \mathcal{O}_S(t)(D) \to \mathcal{O}_S(t)(D)|D \to 0\,. \tag{3}$$

Since $D^2 = 1$, $\mathcal{O}_S(t)(D)|D$ is a line bundle of degree $3t+1$. From (3) we get $h^0(S, \mathcal{O}_S(t)(D)) = \binom{t+3}{3} - \binom{t}{3} + 3t + 2$ and $h^1(S, \mathcal{O}_S(t)(D)) = 0$ for all integers $t \geq 0$. From the exact sequences on S:

$$0 \to \Omega(t+1)|S \to (\Omega(t+1)|S)(D) \to (\Omega(t+1)|S)(D)|D \to 0\,, \qquad (4)$$

$$0 \to T\mathbf{P}^3(t-2)|S \to (T\mathbf{P}^3(t-2)|S)(D) \to (T\mathbf{P}^3(t-2)|S)(D)|D \to 0\,, \quad (5)$$

we get $h^0(S, (\Omega(t+1)|S)(D)) = (9t^2 - 87t + 204)/2 + 9t + 3$ for $t \geq 0$, $h^1(S, \Omega(t+1)|S)(D)) = 0$ for $t \geq 0$, $h^0(S, (T\mathbf{P}^3(t-2)|S)(D)) = 3t(3t-1)/2 + 9t$ for $t \geq 2$ and $h^1(S, (T\mathbf{P}^3(t-2)|S)(D)) = 3t(3t-1)/2 + 9t$ for $t \geq 1$. We will also use the following exact sequences on S:

$$0 \to \mathcal{O}_S(t)(D) \to \mathcal{O}_S(t+2) \to \mathcal{O}_C(t+2) \to 0\,, \qquad (6)$$

$$0 \to (\Omega(t+1)|S)(D) \to \Omega(t+3)|S \to \Omega(t+3)|C \to 0\,, \qquad (7)$$

$$0 \to (T\mathbf{P}^3(t-2)|S)(D) \to T\mathbf{P}^3(t)|S \to T\mathbf{P}^3(t)|C \to 0\,. \qquad (8)$$

Remark 4. Let X be an integral projective scheme, E a vector bundle on X, $Z \subset X$ a closed subscheme and $D \subset X$ an effective Cartier divisor. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of X with $Hom(\mathcal{I}_D, \mathcal{I}_Z)$ as its ideal sheaf. For instance, if $P \in D_{reg}$ and $m > 0$, then $\{mP, X\}|D = \{mP, D\}$ and $\mathrm{Res}_D(\{mP, X\}) = \{(m-1)P, X\}$, with the convention $\{0P, X\} := \emptyset$. We have an exact sequence on X:

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(X)}}(-D) \to \mathcal{I}_Z \to \mathcal{I}_{Z\cap D, D} \to 0\,. \qquad (9)$$

By tensoring the exact sequence (9) with E we obtain the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D) \to \mathcal{I}_Z \otimes E \to \mathcal{I}_{Z\cap D, D} \otimes (E|D) \to 0\,. \qquad (10)$$

Hence:

$$h^0(X, \mathcal{I}_Z \otimes E) \leq h^0(X, \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D)) + h^0(D, \mathcal{I}_{Z\cap D, D} \otimes (E|D))\,,$$

$$h^1(X, \mathcal{I}_Z \otimes E) \leq h^1(X, \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D)) + h^1(D, \mathcal{I}_{Z\cap D, D} \otimes (E|D))\,.$$

Lemma 1. *Let $S \subset \mathbf{P}^3$ be a smooth cubic surface. Fix integers $s > 0$ and $m_1 \geq \cdots \geq m_s > 0$. Let δ be the number of integers $i \in \{1, \ldots, s\}$ such that $m_i = 1$. Set $z := \sum_{i=1}^{s} \binom{m_i+1}{2}$. Let $Z \subset S$ be a general union of s fat points of S with multiplicities $m_1, \ldots, m_s$. Let x be the minimal positive integer such that $z \leq \binom{t+3}{3} - \binom{t}{3}$. Assume $\delta \geq (m_1 - 1)k + \binom{2m_1+5}{3}$. Then $h^1(S, \mathcal{I}_{Z,S}(t)) = 0$ for every integer $t \geq x$ and $h^0(S, \mathcal{I}_{Z,S}(t)) = 0$ for every integer $t < x$.*

Proof. Take $C, D \subset S$ as always. We insert on C multiple points with maximal possible multiplicity (if there is room for that scheme), then with the next maximal multiplicity, and so on. We require that the support of the scheme is To be sure to get $3k+1$ independent conditions on C, we need to insert at most $m_1 - 1$ simple points on S. At the nect step on D for the linear system $|\mathcal{O}_S(k-2)(D)|D$ to get a scheme whose intersection with D has length exactly $3k-6$ we need to use at most $m_1 - 1$ simple points. Hence for going from the critical value k to the critical value $k-2$ we need at lost $2m_1 - 2$ simple points. Of course, at the next step we have a residual scheme on S, but its intersection with C and D is lower with respect to the previous one exactly by the number of connected components. For instance, if we inserted a scheme W such that $W_{red} \subset D$, then $\text{length}(\text{Res}_D(W)) - \text{length}(W) = \sharp(W_{red})$. Since $\text{length}(W) = h^0(D, (\mathcal{O}_S(k-2)(D)|D) = 3k-4$ and each connected component of $W \cap D$ has length at most m_1, this is OK because $3k-4 \geq 2m_1$. We continue untill we exaust all the non-simple points of Z. Since $\delta \geq m_1k^2/2 + \binom{2m_1+5}{3}$ we stop at some critical value t with $t \leq k$, $t \equiv k \pmod 2$ and $t \geq 2m_1$. At this point we have a sertain residual scheme Z' such that each connected component of Z' is a fat point of S with multiplicity at most $m_1 - 1$. Then we continue inserting simple points till critical value 5 (case k odd) or 4 (case k). After at most $m_1 - 1$ steps the residual of Z' is done and we may take general points in S. □

Proposition 1. *Fix integers $s > 0$ and $m_1 \geq \cdots \geq m_s > 0$. Let δ be the number of integers $i \in \{1, \ldots, s\}$ such that $m_i = 1$. Set $z := \sum_{i=1}^{s} \binom{m_i+2}{3}$. Let $Z \subset \mathbf{P}^3$ a general union of s fat points with multiplicities $m_1, \ldots, m_s$. Let k be the critical value of Z. Assume $\delta \geq (m_1-1)k^2/2 + k \cdot (2m_1+4)^3/12 + 20$. Then Z has maximal rank.*

Proof. Use Lemma 1 for all integers k' such that $3 \leq k' \leq k$, $k' \equiv k \pmod 2$ and then add $20 = h^0(\mathcal{O}(3))$ general points. □

Lemma 2. *Let $S \subset \mathbf{P}^3$ be a smooth cubic surface. Fix integers $s > 0$ and $m_1 \geq \cdots \geq m_s > 0$. Let δ be the number of integers $i \in \{1, \ldots, s\}$ such that $m_i = 1$. Set $z := \sum_{i=1}^{s} \binom{m_i+1}{2}$. Let $Z \subset S$ be a general union of s fat points of S with multiplicities $m_1, \ldots, m_s$. Let x be the minimal positive integer such that $3z \leq 3t(3t+1)/2$. Assume $\delta \geq (m_1-1)k + \binom{2m_1+5}{3}$. Then $h^1(S, \mathcal{I}_{Z,S} \otimes (\Omega(t+1)|S)) = 0$ for every integer $t \geq x$ and $h^0(S, \mathcal{I}_{Z,S} \otimes (\Omega(t+1)|S)) = 0$ for every integer $t < x$.*

Proof. We just copy the proof of Lemma 1, except that at the very last step we need to handle a general union of simple points of S for the vector space $H^0(S, \Omega(m+1)|S)$, $m \in \{4, 5\}$. This is done in [1], p. 50, or [2], middle of page 8. $\square$

In the same way we prove the following result.

Lemma 3. *Let $S \subset \mathbf{P}^3$ be a smooth cubic surface. Fix integers $s > 0$ and $m_1 \geq \cdots \geq m_s > 0$. Let δ be the number of integers $i \in \{1, \ldots, s\}$ such that $m_i = 1$. Set $z := \sum_{i=1}^{s} \binom{m_i+1}{2}$. Let $Z \subset S$ be a general union of s fat points of S with multiplicities $m_1, \ldots, m_s$. Let x be the minimal positive integer such that $3z \leq (9t^2 - 87t + 204)/2$. Assume $\delta \geq (m_1 - 1)k + \binom{2m_1+5}{3}$. Then $h^1(S, \mathcal{I}_{Z,S} \otimes (T\mathbf{P}^3(t-2)|S) = 0$ for every integer $t \geq x$ and $h^0(S, \mathcal{I}_{Z,S} \otimes (T\mathbf{P}^3(t-2)|S) = 0$ for every integer $t < x$.*

Proof of Proposition 1. Z has maximal rank by Proposition 1. Apply Lemmas 2 and 3 and then use [2] for the critical values $3, 4, 5$. $\square$

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