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ON THE YOUNG THEOREM FOR AMALGAMS AND BESOV SPACES

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Abstract: In this paper, we obtain a refinement of the Young Theorem. The Young Theorem tells us that the Fourier transform $\mathcal{F}$ sends the L^p functions to the $L^{p'}$ functions, if $1 \leq p \leq 2$. This theorem has a refinement. For example, $\mathcal{F} : L^1 \to B^0_{\infty 1}$, where B^s_{pq} is the Besov space. In this present paper we shall consider the more refined version of this theorem by using the amalgams and the Besov spaces.

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1. Introduction

The aim of this paper is to refine the Young Theorem. The Young Theorem, as is well-known, asserts that the range of the L^p space by the Fourier transform is the $L^{p'}$ space, whenever $1 \leq p \leq 2$:

$$\mathcal{F} : L^p \to L^{p'}.$$

Here and below, for definiteness, we define the Fourier transform of $f \in L^1 \cap L^p$ to be

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)\, e^{-i\xi\cdot x}\, dx.$$

The above well-known theorem has a following refinement

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$$\mathcal{F} : L^p \to B^0_{p'p},$$

where B^s_{pq} is the Besov space whose definition will be recalled later. It is known that $B^0_{\infty 1}$ is a function space contained in L^∞. For some related facts we refer to [1, p. 164]. Let C denote the space of all bounded continuous functions. Since $B^0_{\infty 1} \subset C$, the Besov space $B^0_{\infty 1}$ describes the situation more precisely than the Lebesgue space L^∞. However, once the Besov spaces comes into the play, we hit upon a natural question. How can we make use of the information of q in the Besov spaces B^s_{pq}?

In this present paper we give a more refined version of this theorem and show the sharpness of our result. In Section 2 we give the definition of function spaces to formulate our theorem. The proof of the theorem is presented in Section 3. Finally in Section 4 we exhibit examples showing the sharpness of our result.

2. Function Spaces

In this section we present the definition of function spaces we work on.

2.1. Besov Space

Following [4] and [5] we give a definition of Besov spaces. The definition is somehow different from those in [4] and [5]. However the resulting norms will be equivalent. We use $\mathbb{N}_0$ to denote $\{0, 1, 2, \ldots\}$.

First, given a complex sequence $\{a_j\}_{j \in \mathbb{N}_0}$, we set

$$\|a_j \,:\, l^q\| := \left(\sum_{j \in \mathbb{N}_0} |a_j|^q \right)^{\frac{1}{q}}, \; 0 < q \leq \infty.$$

We also define $\|a_z \,:\, l^q\| := \left(\sum_{z \in \mathbb{Z}^n} |a_z|^q \right)^{\frac{1}{q}}, \; 0 < q \leq \infty$ for $\{a_z\}_{z \in \mathbb{Z}^n}$. If possible confusion can occur, we write

$$\|\{a_j\}_j \,:\, l^q\|, \; \|\{a_z\}_z \,:\, l^q\|$$

instead of $\|a_j \,:\, l^q\|$, $\|a_z \,:\, l^q\|$. Similarly, the notation $\|\{a_{j,z}\}_{j,z} \,:\, l^q\|$ means the l^q-norm of $\{a_{j,z}\}_{j \in \mathbb{N}_0, z \in \mathbb{Z}^n}$. Next, for a sequence of complex valued measurable functions $\{f_j\}_{j \in \mathbb{N}_0}$, we set

$$\|f_j \,:\, l^q(L^p)\| := \| \, \|f_j \,:\, L^p\| \,:\, l^q\|, \; 0 < p, q \leq \infty.$$

If $p = \infty$ and / or $q = \infty$, we make a natural modification in the above formulae.

Definition 1. Let $\phi_0, \phi_1 \in \mathcal{S}$ be even functions satisfying the following conditions.

$$\chi_{[-2,2]^n} \leq \phi_0 \leq \chi_{[-4,4]^n}, \ \chi_{[-4,4]^n \setminus [-2,2]^n} \leq \phi_1 \leq \chi_{[-8,8]^n \setminus [-1,1]^n}.$$

We set $\phi_j(x) := \phi_1(2^{-j+1}x)$ for $j \geq 2$. For $f \in \mathcal{S}'$, we denote $\phi_j(D)f := \mathcal{F}^{-1}(\phi_j \cdot \mathcal{F}f)$.

What counts about this definition is to adopt cubes instead of balls. We prefer to use cubes because we will consider the amalgam spaces. With this preparation in mind, we shall define the Besov norms.

Definition 2. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Under the notations in Definition 1, we define B^s_{pq} to be the set of the Schwartz distributions $f \in \mathcal{S}'$ for which the quasi-norm

$$\|f \,:\, B^s_{pq}\| := \|2^{js}\phi_j(D)f \,:\, l^q(L^p)\| \tag{1}$$

is finite.

It can be easily shown that the definition of the Besov space B^s_{pq} is independent of the choice of ϕ_0, ϕ_1. For details we refer to [4].

2.2. (Weighted) Amalgam Space

Now we will follow [2] and [3] for the definitions. Given a measurable set A, we set

$$\|f \,:\, L^p(A)\| := \|\chi_A f \,:\, L^p\|.$$

Definition 3. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Set $Q_z := z + [0,1]^n$ for $z \in \mathbb{Z}^n$, the translation of the unit cube. For a Lebesgue locally integrable function f we define

$$\|f \,:\, (L^p, l^q(\langle z\rangle^s))\| := \|\, \langle z\rangle^s \cdot \|f \,:\, L^p(Q_z)\| \,:\, l^q\|,$$

where $\langle a\rangle := \sqrt{|a|^2+1}$ for $a \in \mathbb{R}^n$. $(L^p, l^q(\langle z\rangle^s))$ is a set of all locally integrable functions f for which the quasi-norm $\|f \,:\, (L^p, l^q(\langle z\rangle^s))\| < \infty$. For brevity we write $(L^p, l^q) := (L^p, l^q(1))$.

It can be seen that $(L^p, l^p) = L^p$ with norm coincidence. By definition of the norm, the following multiplication operator is an isomorphism.

$$f \in (L^p, l^q(\langle z\rangle^s)) \mapsto \langle\cdot\rangle^t \cdot f \in (L^p, l^q(\langle z\rangle^{s-t})). \tag{2}$$

Note that $(L^p, l^q(\langle z\rangle^s)) \subset \mathcal{S}'$, if $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. It can be easily seen that

$$(L^{p_1}, l^{q_1}(\langle z\rangle^{s_1})) \subset (L^{p_2}, l^{q_2}(\langle z\rangle^{s_2}))$$

for $p_1 \geq p_2$, $q_1 \leq q_2$ and $s_1 \geq s_2$.

2.3. Main Theorem

With these definitions in mind, we formulate our main theorem.

Theorem 4. *1. Let $1 \le p \le 2,\ 0 < q \le \infty$ and $s \in \mathbb{R}$. Then*

$$\mathcal{F} : (L^p, l^q(\langle z \rangle^s)) \to B_{p'q}^{s-n\left(\frac{1}{p}-\frac{1}{q}\right)_+}. \tag{3}$$

2. Let $1 \le p \le 2,\ 0 < q \le \infty$ and $s \in \mathbb{R}$. Then

$$\mathcal{F} : B_{pq}^s \to \left(L^{p'}, l^q(\langle z \rangle^{s-n\left(\frac{1}{q}-\frac{1}{p'}\right)_+})\right). \tag{4}$$

Here and below, for $a \in \mathbb{R}$ we write $a_+ := \max(a, 0)$.

Before we come to the proof, we state one more corollary.

Corollary 5. *Suppose that $2 \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then*

$$\mathcal{F} : (L^p, l^q(\langle z \rangle^s)) \to B_{2q}^{s-n\left(\frac{1}{2}-\frac{1}{q}\right)_+}.$$

In particular,

$$\mathcal{F} : L^\infty \to B_{2\infty}^{-\frac{n}{2}}.$$

Once we obtain Theorem 4, Corollary 5 is easy to prove: All we have to note is

$$(L^p, l^q(\langle z \rangle^s)) \subset (L^2, l^q(\langle z \rangle^s))$$

for $2 \le p \le \infty$.

The rest of this paper is devoted to the proof of Theorem 4 and to investigating the sharpness of these results.

3. Proof of Theorem 4

First, we recall the boundedness of the lift operator.

$$(1-\Delta)^{\frac{t}{2}} : B_{pq}^s \to B_{pq}^{s-t}. \tag{5}$$

For the proof of (3), (2) and (5) let us to assume $s = 0$.

With this preparation our present task is to estimate

$$\left\| 2^{-jn\left(\frac{1}{p}-\frac{1}{q}\right)_+} \phi_j(D)\mathcal{F}f \ :\ l^q(L^{p'}) \right\| \tag{6}$$

for $f \in (L^p, l^q(\langle z\rangle^s))$. Note that the Young Theorem gives

$$\|\phi_j(D)\mathcal{F}f \,:\, L^{p'}\| \le \|\phi_j \cdot \mathcal{F}(\mathcal{F}f) \,:\, L^p\|.$$

Set $A_j := \mathrm{supp}\,(\phi_j)$ for $j \in \mathbb{N}_0$. Then (6) can be majorized by

$$\left\| 2^{-jn\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|\mathcal{F}(\mathcal{F}f) \,:\, L^p(A_j)\| \,:\, l^q \right\|.$$

Since $\mathcal{F}(\mathcal{F}f)(x) = c\,f(-x)$ and the ϕ_j are even, we have

$$\|\mathcal{F}(\mathcal{F}f) \,:\, L^p(A_j)\| = c\,\|f \,:\, L^p(A_j)\|. \tag{7}$$

The inequality $\left(\sum_{j=1}^N |a_j|\right)^{\frac{q}{p}} \le N^{\left(\frac{q}{p}-1\right)_+} \sum_{j=1}^N |a_j|^{\frac{q}{p}}$ gives us

$$\|f \,:\, L^p(A_j)\|^q \le c\,2^{jqn\left(\frac{1}{p}-\frac{1}{q}\right)_+} \sum_{z\in\mathbb{Z}^n} \|f \,:\, L^p(A_j\cap Q_z)\|^q. \tag{8}$$

If we put (7) and (8) together, then we have

$$\begin{aligned} &\left\| 2^{-jn\left(\frac{1}{p}-\frac{1}{q}\right)_+} \phi_j(D)\mathcal{F}f \,:\, l^q(L^{p'}) \right\| \\ &\qquad \le c \left\| \{\, \|f \,:\, L^p(A_j\cap Q_z)\| \,\}_{j,z} \,:\, l^q \right\|. \end{aligned} \tag{9}$$

Given $z \in \mathbb{Z}^n$, from the definition of the A_j, there are at most three j such that $A_j \cap Q_z \neq \emptyset$, and hence,

$$\sum_{j\in\mathbb{N}_0} \|f \,:\, L^p(A_j\cap Q_z)\|^q \le 3\,\|f \,:\, L^p(Q_z)\|^q. \tag{10}$$

Combining (9) and (10), we obtain

$$\begin{aligned} &\left\| 2^{-jn\left(\frac{1}{p}-\frac{1}{q}\right)_+} \phi_j(D)\mathcal{F}f \,:\, l^q(L^{p'}) \right\| \\ &\qquad \le c\,\|\,\|f \,:\, L^p(Q_z)\| \,:\, l^q\| = \|f \,:\, (L^p, l^q)\|. \end{aligned}$$

This is the desired result.

Next, we prove (4). As before, we assume $s = 0$. We set

$$|z|_\infty = \max(|z_1|, |z_2|, \ldots, |z_n|).$$

First, we observe by the definition of the norm,

$$\begin{aligned}
&\left\| \mathcal{F}f \,:\, \left(L^{p'}, l^q \left(\langle z \rangle^{-n\left(\frac{1}{q}-\frac{1}{p'}\right)_+} \right) \right) \right\| \\
&\quad \sim \left\| \left\{ 2^{-jn\left(\frac{1}{q}-\frac{1}{p'}\right)_+} \sum_{\substack{z \in \mathbb{Z}^n \\ [2^{j-1}] \le |z|_\infty < 2^j}} \|\mathcal{F}f \,:\, L^{p'}(Q_z)\| \right\}_j \,:\, l^q \right\|,
\end{aligned}$$

where $[\cdot]$ denotes the Gauss sign.

The inequality $\sum_{j=1}^N |a_j|^{\frac{q}{p'}} \le N^{\left(1-\frac{q}{p'}\right)_+} \left(\sum_{j=1}^N |a_j| \right)^{\frac{q}{p'}}$ and Young's Theorem yield

$$\begin{aligned}
&\left\| \mathcal{F}f \,:\, \left(L^{p'}, l^q \left(\langle z \rangle^{-n\left(\frac{1}{q}-\frac{1}{p'}\right)_+} \right) \right) \right\| \\
&\quad \le \ c \left\| \left\{ \left(\sum_{\substack{z \in \mathbb{Z}^n \\ [2^{j-1}] \le |z|_\infty < 2^j}} \|\mathcal{F}f \,:\, L^{p'}(Q_z)\|^{p'} \right)^{\frac{1}{p'}} \right\}_j \,:\, l^q \right\| \\
&\quad \sim \ \left\| \phi_j \cdot \mathcal{F}f \,:\, l^q(L^{p'}) \right\| = c \left\| \mathcal{F}\phi_j(D)f \,:\, l^q(L^{p'}) \right\| \\
&\quad \le \ c \, \|\phi_j(D)f \,:\, l^q(L^p)\| = c \, \|f \,:\, B^0_{pq}\|.
\end{aligned}$$

This proves (4).

4. Sharpness of the Results

In this section we deduce some necessary conditions. We consider the following problem.

Problem 6. Let $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < q_1, q_2 \le \infty$ and $s_1, s_2 \in \mathbb{R}$. Under what condition does the Fourier transform $\mathcal{F}$ send $(L^{p_1}, l^{q_1}(\langle z \rangle^{s_1}))$ continuously to $B^{s_2}_{p_2 q_2}$? That is, when is the estimate

$$\|\mathcal{F}f \,:\, B^{s_2}_{p_2 q_2}\| \le c \, \|f \,:\, (L^{p_1}, l^{q_1}(\langle z \rangle^{s_1}))\| \tag{11}$$

is true? Find necessary conditions of (11).

First, we prove that the smoothness parameter s cannot be improved.

Proposition 7. *If* (11) *is true, then* $s_2 \le s_1$.

Proof. Let $\tau \in \mathcal{S}$ be an even function with $\chi_{B(1/4)} \le \tau \le \chi_{B(1/2)}$, where $B(r)$ denotes the open ball centered at the origin of radius $r > 0$. We set $e_1 = (1, 0, 0, \dots, 0)$, the elementary vector in $\mathbb{R}^n$, and define $\tau_j(x) := \tau(x - 2^j e_1)$. Then we obtain

$$\|\tau_j \,:\, (L^{p_1}, l^{q_1}(\langle z\rangle^{s_1}))\| \sim 2^{js_1}$$

and

$$\|\mathcal{F}\tau_j \,:\, B^{s_2}_{p_2q_2}\| \sim \|2^{js_2}\mathcal{F}\tau_j \,:\, L^{p_2}\| \sim \|2^{js_2}\mathcal{F}\tau \,:\, L^{p_2}\| \sim 2^{js_2}.$$

Since by assumption we have

$$\|\mathcal{F}f \,:\, B^{s_2}_{p_2q_2}\| \le c\,\|f \,:\, (L^{p_1}, l^{q_1}(\langle z\rangle^{s_1}))\|$$

for all $f \in (L^{p_1}, l^{q_1}(\langle z\rangle^{s_1}))$, it follows that $s_2 \le s_1$. □

Next, we discuss how the integrability parameter changes by the Fourier transform.

Proposition 8. *If* (11) *is true, then* $p_2 \ge p_1'$.

Proof. We use τ in the proof in Proposition 7 again. Consider $f(x) := |x|^{-\alpha}\tau(x)$. Set $\delta = 1 - \tau$ and $g(x) := |x|^{-\alpha}\delta(x)$. It is well-known that

$$\mathcal{F}f(\xi) + \mathcal{F}g(\xi) = c\,|\xi|^{\alpha - n}.$$

Since $|\xi|^{2N}\mathcal{F}g(\xi) = \mathcal{F}[(-\Delta)^N g](\xi)$ and $(-\Delta)^N g \in L^1$ for $N \gg 1$, it follows that $|\mathcal{F}g(\xi)| \le c\,|\xi|^{-2N}$. From this we deduce

$$|\mathcal{F}f(\xi)| \sim |\xi|^{\alpha - n} \text{ as } \xi \to \infty.$$

$f \in (L^{p_1}, l^{q_1}(\langle z\rangle^{s_1}))$ if and only if $p_1\alpha < n$. Meanwhile, from the definition of the norm, $\mathcal{F}f \in B^{s_2}_{p_2q_2}$ forces $p_2(n - \alpha) > n$. Thus it is necessary that $\alpha < n/p_1$ implies $\alpha < n - n/p_2$. From this it follows that $n/p_1 \le n - n/p_2$, which is equivalent to $p_2 \ge p_1'$. □

The restrictions appearing in Propositions 7 and 8 are natural, if we take into account $\mathcal{F} : L^p(\langle z\rangle^s) \to W^s_{p'}$, where $W^s_{p'}$ denotes the Sobolev space. It is well-known that

$$B^{s+\varepsilon}_{pq} \subset B^s_{pq'},\ 0 < p, q, q' \le \infty,\ \varepsilon > 0$$

and

$$B^{s_1}_{p_1q} \subset B^{s_2}_{p_2q},\ 0 < p_1, p_2, q \le \infty,\ s_1 > s_2,\ s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}. \tag{12}$$

Thus, if $s_1 > s_2$ or $p_1' < p_2$, then the situation can be considered degenerate. Next, we explain the decay of the parameter s in Theorem 4 when $p < q$.

Proposition 9. *If* (11) *is true, then* $s_2 \le s_1 - \frac{n}{p_2'} + \frac{n}{q_1}$.

From this proposition, the restriction $s_2 \le s_1 - \frac{n}{p_2'} + \frac{n}{q_1}$ in Theorem 4 is essential.

Proof. Let ϕ_j be the function in Definition 2.

$$\|\phi_j \,:\, (L^{p_1}, l^{q_1}(\langle z\rangle^{s_1}))\| \sim 2^{j\left(s_1+\frac{n}{q_1}\right)}$$

and

$$\|\mathcal{F}\phi_j \,:\, B^{s_2}_{p_2q_2}\| \sim \|2^{js_2}\mathcal{F}\phi_j \,:\, L^{p_2}\| \sim 2^{j\left(s_2+\frac{n}{p_2'}\right)}.$$

As a result the desired inequality follows. □

We tackle a subtler problem: Can we improve q_2 in (11)? The case when $s_2 < \min(s_1, s_1 - n/p_2' + n/q)$ can be regarded as degenerate and we concentrate on the limit case.

Proposition 10. *Assume* $s_2 = \min\left(s_1, s_1 - \frac{n}{p_2'} + \frac{n}{q_1}\right)$. *If* (11) *is true, then* $q_2 \ge q_1$.

Proof. We consider two cases separately: $s_2 = s_1 - \frac{n}{p_2'} + \frac{n}{q_1}$ and $s_2 = s_1$. By using the lift operators, we may assume $s_1 = 0$.

First, we tackle the case $s_2 = s_1 - \frac{n}{p_2'} + \frac{n}{q_1}$. Let $\{a_j\}_{j\in\mathbb{N}}$ be a complex sequence as before such that $a_j = 0$ if 3 does not divide j. Let $f := \sum_{j\in\mathbb{N}_0} a_j\phi_j$, where the ϕ_j are from Definition 2. Then by the same reasoning as before we obtain

$$\begin{aligned} \|f \,:\, (L^{p_1}, l^{q_1})\| &\sim \|2^{\frac{jn}{q_1}} a_j \,:\, l^{q_1}\|, \\ \|\mathcal{F}f \,:\, B^{s_2}_{p_2q_2}\| &\sim \|2^{\frac{jn}{q_1}} a_j \,:\, l^{q_2}\|. \end{aligned}$$

From this we deduce $q_2 \ge q_1$.

Now we turn to the case when $s_2 = s_1 = 0$. Then we use the τ_j in Proposition 7. Let $f := \sum_{j\in\mathbb{N}_0} a_j\tau_j$, where $\{a_j\}_{j\in\mathbb{N}_0}$ is a complex sequence as before. Then we have

$$\|f \,:\, (L^{p_1}, l^{q_1})\| \sim \|a_j \,:\, l^{q_1}\| \ \text{ and } \|\mathcal{F}f \,:\, B^0_{p_2q_2}\| \sim \|a_j \,:\, l^{q_2}\|.$$

Thus, we obtain $q_2 \ge q_1$. □

Finally we show the last result.

Proposition 11. *Let $0 < q < 2$. Then the mapping $\mathcal{F} : (L^2, l^q) \to B^0_{2q}$ is not a surjection.*

Proof. By interpolation, we may assume that $1 \leq q < 2$. Assume that $\mathcal{F} : (L^2, l^q) \to B^0_{2q}$ is surjective. Then by duality we would have $\mathcal{F} : B^0_{2q'} \to (L^2, l^{q'})$ is bijective. This would imply $\mathcal{F} : (L^2, l^{q'}) \to B^0_{2q'}$ is also bijective. This contradicts to Proposition 9. □

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