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ON THE COADJOINT REPRESENTATION OF THE AFFINE GROUP BY THE METHOD OF VECTOR ENVELOPES

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Abstract: The existence for any affine space A over a field $\mathbb{K}$ of a pair $(\hat{A}, \varphi)$, termed the vector envelope of $\vec{A}$, and consisting of a vector space $\vec{A}$ and an affine transformation φ from A to $\hat{A}$, can be used to identify A (respectively the associated vector space $\vec{A}$) with an affine hyperplane of $\hat{A}$ (respectively with a vector hyperplane of $\hat{A}$). In this article, an attempt is made, using the concept of vector envelope of an affine space, to characterize the elements of the Lie algebra dual of the Lie group of the open-orbit affine transformations for the coadjoint representation.

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1. Introduction

Let A be an affine space over field $\mathbb{K}$ ($\mathbb{K}$ can either be the field of real or complex numbers) et let $\vec{A}$ be its direction. Let $G = GA(A)$ denote the Lie group of affine transformations of A, and $\mathcal{G}^* = \mathcal{G}a^*(A)$ the dual of Lie algebra $\mathcal{G} = \mathcal{G}a(A)$ of G.

The representation of an element of $\mathcal{G}^*$ as a pair (v, g), where, $v \in \vec{A}^* = \mathcal{L}(\vec{A}, \mathbb{K})$, and, $g \in \operatorname{End}(\vec{A})$ (the algebra of all endomorphisms of $\vec{A}$), depends upon the choice made for the origin p of A. More precisely if one replaces p by $p + x$, where, $x \in \vec{A}$; then (v, g) have to be replaced by $(v, g - (x \otimes v))$, where, $(x \otimes v)$ defines the following endomorphism over $\vec{A}$: $(x \otimes v)(y) = \langle v, y \rangle x$ for every $y \in \vec{A}$. Thus, v is an element of $\vec{A}^*$ intrinsically attached to the pair (v, g) of $\mathcal{G}^*$, and if $v \neq 0$, then g, on the contrary, is not intrinsically attached

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to the pair. However, $g|_{\mathrm{Ker}\,(v)}$, the restriction of g to the hyperplane $\mathrm{Ker}\,(v)$ is intrinsic. The fact that (v, g) is open orbited is directly impacted on $g|_{\mathrm{Ker}\,(v)}$ when $v \neq 0$.

This work aims at characterizing the elements of $\mathcal{G}a^*(A)$ having an open orbit for the coadjoint representation. To this effect, the notion of vector envelope for an affine space shall be utilized (cf. [2], [4]). Also, the choice of an origin in A, makes it possible to retrieve known results concerning the characterization of elements with coadjoint open orbits in the dual of the Lie algebra of the affine group $A(n, \mathbb{K}) = \left\{ \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix} ; A \in GL(\mathbb{K}^n, x \in \mathbb{K}^n \right\}$ (cf. [1], [3], [5]).

2. Vector Envelope of an Affine Space

Throughout the article, A will always be an affine space over a field $\mathbb{K}$, and $\vec{A}$ the associated vector space. The following basic operations link the two spaces together: $A \times \vec{A} \to A$, $(p, \vec{u}) \mapsto p + \vec{u}$: Maps the pair $(p, \vec{u})$ into the translated of point p by vector $\vec{u}$ and, $A \times A \to \vec{A}$, $(p, q) \mapsto \vec{pq}$: Maps the pair of points (p, q) into the vector whose ends are p and q.

Universal Property of the Vector Envelope. *The pair $(\hat{A}, \varphi)$ consisting of a vector space $\hat{A}$ and an affine transformation φ from A to $\hat{A}$ is called a vector envelope of A if for every vector space over $\mathbb{K}$, V, there exists a unique linear transformation $\hat{f}\colon \hat{A} \to v$ such that, $f = \hat{f} \circ \varphi$.*

Remark. If the pair $(\hat{A}, \varphi)$ exists, then, it is unique ("modulo" some arbitrary isomorphism).

Construction of the Vector Envelope. *Let $\hat{A} = \vec{A} \cup (\mathbb{K}^* \times A)$, then, A, identified with $1 \times A$, is contained in $\hat{A}$. The latter is a vector space over $\mathbb{K}$ for the following laws: For every (k, h), pair of elements of $\mathbb{K}$, for every x, element of A, and for every $\vec{\xi}$, element of $\vec{A}$:*

(i) $k.(h, x) = (kh, x)$, $0.(k, x) = 0$, in particular $(k, x) = kx$;

(ii) If $k + k' \neq 0$, then $kx + k'x' = (k + k')x''$ where $x'' = x + \frac{k'}{k+k'}x$;

(iii) If $k + k' = 0$, then $kx + k'x' = k\vec{x'x}$;

(iv) $kx + \vec{\xi} = k(x + k^{-1}\vec{\xi}$,

where the laws of the vector space $\vec{A}$ have been applied in items: (iii) and (iv).

Remarks. (1) As part of $\hat{A}$, $\vec{A}$ is a vector hyperplane.

(2) Let $\theta\colon \hat{A} \to \mathbb{K}$, be a map such that for every k, element of $\mathbb{K}$, every x, element of A, and every $\vec{\xi}$, element of $\vec{A}$, $\theta(kx) = k$, and, $\theta(\vec{\xi}) = 0$, then, θ is a linear form over $\hat{A}$ and $\theta^{-1}(1) = A$ does not contain 0. Moreover, the affine

hyperplane $\theta^{-1}(1)$ is isomorphic (as an affine space) to A.

(3) For every a, element of A, we have, $\hat{A} = \vec{A} \oplus \mathbb{K}a$, and, if $q\colon \hat{A} \to \vec{A}$ is the first projection with respect to this direct sum, then, the restriction of q to A coincides with the linear transformation from A to $\vec{A}$ that maps a point $x \in A$ into the vector $\vec{ax}$.

(4) If $\dim A = n$, then, $\dim \hat{A} = n + 1$.

Theorem 1. *The pair $(\hat{A}, \varphi)$ is a vector envelope of A if and only if φ is injective, and $\varphi(A)$ is an affine hyperplane of $\hat{A}$ not containing the zero element of $\hat{A}$.*

Proof. Let p be an arbitrary point in A, and $\vec{A_p}$ the vectorized affine space A at p. Let $f\colon A \to \vec{A_p}$ be the affine transformation defined by $f(m) = \vec{pm}$ and $f(m) = 0$. By definition of the vector envelope, there exists a unique linear transformation $\hat{f}\colon \hat{A} \to \vec{A_p}$ such that $f = \hat{f} \circ \varphi$. Hence, if for an arbitrary point m in A, $\varphi(m) = \varphi(p)$, then $\hat{f} \circ \varphi(m) = \hat{f} \circ \varphi(p)$, i.e., $\vec{pm} = f(p) = \vec{0}$, and, $m = p$ which shows that φ is injective.

Noting that $\varphi(m)$ is an affine subspace of $\hat{A}$, let $V = \mathbb{K}$ and $f\colon A \to \mathbb{K}$ be the affine transformation such that for every m, element of A, $f(m) = 1$, then, there exists a unique linear form $\hat{f}\colon \hat{A} \to \mathbb{K}$ such that for every m, element of A, $\hat{f}(\varphi(m)) = 1$, thus showing that $\varphi(A)$ coincides with $H = \{\hat{m} \in \hat{A}; \hat{f}(\hat{m}) = 1\}$: an affine hyperplane of $\hat{A}$. Indeed, having $\vec{\varphi}(\vec{A}) \subset \operatorname{Ker} \hat{f}$ and $\dim \vec{\varphi}(\vec{A}) = \dim \vec{A} = \dim \hat{A} - 1 = \dim \operatorname{Ker} \hat{f}$ shows that $\vec{\varphi}(\vec{A}) = \operatorname{Ker}(\hat{f})$. In addition, H does not contain the zero element of $\hat{A}$

Inversely, if $\hat{A}$ is a $\mathbb{K}$-vector space, φ a injective affine transformation from A to $\hat{A}$, and $\varphi(A)$ an affine hyperplane of $\hat{A}$ not containing the zero element of $\hat{A}$. Then, there exists a linear form θ over $\hat{A}$ such that: $\varphi(A) = \{\hat{m} \in \hat{A}; \theta(\hat{m}) = 1\}$ and for every m, element of A, $\theta \circ \varphi(m) = 1$.

Let V be an arbitrary $\mathbb{K}$-vector space and $f\colon A \to V$ an affine transformation. Since $\varphi\colon A \to \hat{A}$ is injective, so is the associated linear transformation $\vec{\varphi}\colon \vec{A} \to \hat{A}$. The affine hyperplane $\varphi(A) = \theta^{-1}(1)$ of $\hat{A}$ has the vector hyperplane $\vec{\varphi}(\vec{A}) = \operatorname{Ker}\theta$ as its direction. Let $\hat{\xi} \in \hat{A}$ such that $\hat{A} = \vec{\varphi}(\vec{A}) \oplus \mathbb{K}\hat{\xi}$, then, $\varphi(A) \cap \mathbb{K}\hat{\xi} = \{\xi\}$; and hence there exists $(\xi \in A)$ such that $\varphi(\xi) = \hat{\xi}$. As a result, the linear transformation $\hat{f}\colon \hat{A} \to v$ defined by: $\hat{f}(\hat{\xi}) = f(\xi)$ and $\hat{f}(\vec{\varphi}(u)) = \vec{f}(\vec{\varphi}(u))$ for every u, element of $\vec{A}$, satisfies $\hat{f} \circ \varphi = f$. ☐

Consequence. If $(\hat{A}, \varphi)$ is a vector envelope of A, then we could identify A (respectively, $\vec{A}$) with the affine hyperplane $\varphi(A)$ (respectively, the vector hyperplane $\vec{\varphi}(\vec{A})$). There exists then a unique linear form θ over $\hat{A}$ such that $A = \theta^{-1}(1)$ and $\vec{A} = \theta^{-1}(0) = \operatorname{Ker}(\theta)$.

Theorem 2. *Let A_1 and A_2 be two $\mathbb{K}$-affine spaces whose vector envelopes* *are* $(\hat{A}_1, \varphi_1)$ *and* $(\hat{A}_2, \varphi_2)$*, respectively. Then, for any affine transformation* $f\colon A_1 \to A_2$*, there exists a unique linear transformation* $\hat{f}\colon A_1 \to \hat{A}_2$ *satisfying* $\varphi_2 \circ f = \hat{f} \circ \varphi_1$.

Proof. The theorem is a direct application of the universal property of the vector envelope $(\hat{A}_1, \varphi_1)$ to the linear transformation $\varphi_2 \circ f\colon A_1 \to \hat{A}_2$. □

Consequence. The last theorem establishes a bijection between the set of linear transformations $f\colon A_1 \to A_2$ and the set of linear transformations $\hat{f}\colon \hat{A}_1 \to \hat{A}_2$ with $\hat{f}(A_1) \subset A_2$. If $\hat{f}$ is given, then, f and its vector part $\vec{:}\vec{A}_1\vec{A}_2$ are retrieved by restriction of $\hat{f}$ to A_1 and $\vec{A}_1$ respectively ($f = \hat{f}|_{A_1}, \vec{f} = \hat{f}|_{\vec{A}_1}$).

3. The Group of Affine Transformations of *A*

Let A be an n-dimensional $\mathbb{R}$-affine space, and, $GA(A)$ the group of affine transformations of A. $GA(A)$ is isomorphic to the subgroup of $GL(\hat{A})$, G, consisting of the linear transformations $\hat{g}\colon \hat{A} \to \hat{A}$ such that $\hat{g}(A) = A$. The corresponding isomorphism $\psi\colon GA(A) \to G$ is defined by $\psi(g) = \hat{g}$. If $g \in GA(A)$ and $\hat{g} = \psi(g)$, then the linear part of g, $\vec{g}\colon \vec{A} \to \vec{A}$, is obtained as $\hat{g}|_{\vec{A}}$, the restriction of $\hat{g}$ to $\vec{A}$. Therefore, the structure of a real Lie group over $GA(A)$ is inherited from the open linear group of $GL(\hat{A})$ of $\mathbb{R}^{(n+1)^2}$.

The Lie algebra $\mathcal{G}a(A)$ of the Lie group $GA(A)$ is the vector space of the affine transformations $f\colon A \to \vec{A}$. This space is isomorphic to the vector space of affine transformations $\hat{f}\colon \hat{A} \to \vec{A}$. Every linear transformation $f\colon A \to \vec{A}$, being an element of $\mathcal{G}a(A)$, can be uniquely extended to a linear transformation $\hat{f}\colon \hat{A} \to \vec{A}$, also viewed as an endomorphism of $\hat{A}$. Now, the endomorphisms of $\hat{A}$ whose range is contained in $\vec{A}$ are the elements of an associative sub-algebra of the algebra $\operatorname{End}(\hat{A})$ of all endomorphisms of $\hat{A}$. The associative character of the sub-algebra is the basis for the "bracket" property of the Lie algebra $\mathcal{G}a(A)$. Recall that the Lie bracket of two elements f and f' of $\mathcal{G}a(A)$ is defined by $[f, f'] = \vec{f} \circ f' - \vec{f'} \circ f$, where, $\vec{f}\colon \hat{A} \to \vec{A}$ is the linear part of the affine transformation $f\colon A \to \vec{A}$. The "bracket" property says that if $\hat{f}\colon \hat{a} \to \vec{a}$ is the unique linear transformation that extends f, then, $\vec{f} = \hat{f}|_{\vec{A}}$ and $\widehat{[f, f']} = \vec{f} \circ \hat{f}' - \vec{f}' \circ \hat{f}$.

Proposition 1. *The dual* $\mathcal{G}a^*(A)$ *of* $\mathcal{G}a(A)$ *can be identified with* $\operatorname{Hom}(\vec{A}, \hat{A})$*: The space of linear transformations from* $\vec{A}$ *to* $\hat{A}$*. If* $h \in \operatorname{Hom}(\vec{A}, \hat{A})$ *and* $f \in \mathcal{G}a(A)$*, the above identification is a consequence of the following identity:*

$$\langle h, f \rangle = \operatorname{tr}(\hat{f} \circ h) = tr(h \circ \hat{f}) .$$

Proof. Let p be an origin in A. Linear isomorphisms from $\mathcal{G}a(A)$ to $\vec{A}\oplus \operatorname{End}(\vec{A})$, and from $\mathcal{G}a^*(A)$ to $\vec{A}^*\oplus\operatorname{End}(\vec{A})$ can be constructed as follows: First, every element of $\hat{A}$ can be written as $\lambda.p+u$ with $\lambda\in\mathbb{R}$ and $u\in\vec{A}$, and every element h of $\operatorname{Hom}(\vec{A},\hat{A})$ can be written as: $h(u)=h^0(u).p+h_p(u)$, for every u, element of $\vec{A}$, where, h^0 is a linear form over $\vec{A}(h^0\colon \vec{A}\xrightarrow{h}\hat{A}\xrightarrow{pr}\hat{A}/\vec{A}\cong\mathbb{R})$, and, h_p is an endomorphism over $\vec{A}$ (both h^0 and h_p depend upon p). As a result, a linear isomorphism can be obtained by associating h with the pair (h^0,h_p):

$$\operatorname{Hom}(\vec{A},\hat{A})\xrightarrow{\sim}\vec{A}^*\oplus\operatorname{End}(\vec{A}),\quad h\mapsto(h^0,h_p)=h^0\oplus h_p\,.$$

Next, if $f\in\mathcal{G}a(A)$ and $\lambda.p+u\in\hat{A}$, one can write $\hat{f}(\lambda.p+u)=\lambda f(p)+\vec{f}(u)$. Hence, for every u, element of $\vec{A}$,

$$\begin{aligned}(\hat{f}\circ h)(u)=\hat{f}(h^0(u).p+h_p(u))&=h^0(u).f(p)+(\vec{f}\circ h_p)(u)\\&=(f(p)\otimes h^0+\vec{f}\circ h_p)(u)\,.\end{aligned}$$

This shows that $\operatorname{tr}(\hat{f}\circ h)=\langle h^0,f(p)\rangle+\operatorname{tr}(\vec{f}\circ h_p)$, which is consistent with the duality between $\vec{A}\oplus\operatorname{End}(\vec{A})\cong\mathcal{G}a(A)$ and $\vec{A}^*\oplus\operatorname{End}(\vec{A})\cong\mathcal{G}a^*(A)\cong\operatorname{Hom}(\vec{A},\hat{A})$. □

4. Orbits of the Coadjoint Representation of the Affine Transformations Group

Let A be an n-dimensional $\mathbb{R}$-affine space and $\vec{A}$ the associated $\mathbb{R}$-vector space. Let $\varphi\colon\hat{A}\to\mathbb{R}$ (a linear form over $\hat{A}$) be such that $A=\varphi^{-1}(1)$ and $\vec{A}=\operatorname{Ker}\varphi$. Let $h\colon\vec{A}\to\hat{A}$ be a linear transformation ($h\in\mathcal{G}a^*(A)=\operatorname{Hom}(\vec{A},\hat{A})$).

Let us note first that $GA(A)$ identifies with the Lie group G of the linear transformations $\hat{g}$ of A satisfying the identity $\varphi\circ\hat{g}=\varphi$, and that its Lie algebra $\mathcal{G}a(A)$ identifies with the vector space $\operatorname{Hom}(\hat{A},\vec{A})$ of the linear transformations from $\hat{A}$ to $\vec{A}$.

Let $\vec{A}_0=\vec{A}$, $A_0=A$, $\hat{A}_0=\hat{A}$, $\varphi_0=\varphi$ and $h_0=h$. If we assume that $\varphi_1=\varphi_0\circ h_0$ (a linear form over $\vec{A}_0$) is different than zero, which is equivalent to assuming that $h_0(\vec{A}_0)\not\subset\vec{A}_0$, then, $A_1=\varphi_1^{-1}(1)$ is an affine hyperplane of $\hat{A}_1=\vec{A}_0$, and $\vec{A}_1=\operatorname{Ker}\varphi_1$ is the associated vector space.

Let $h_1=h_0|_{\vec{A}_1}$, then, $h_1\in\operatorname{Hom}(\vec{A}_1,\hat{A}_1)$, indeed, $\varphi_1(x)=0$, for every x, element of $\vec{A}_1$, i.e. $\varphi_0(h_0(x))=0$, hence, $h_0(x)\in\operatorname{Ker}\varphi_0=\vec{A}_0=\hat{A}_1$; and we also have $\vec{A}_1=h_0^{-1}(\vec{A}_0)=\operatorname{Ker}\varphi_1$. Similarly, $A_1=h_0^{-1}(A_0)=\varphi_1^{-1}(1)$. Therefore, under the condition, $h_0(\vec{A}_0)\not\subset\vec{A}_0$, and starting with the given triplet

$(\hat{A}_0, \varphi_0, h_0)$, we have constructed a new triplet $(\hat{A}_1, \varphi_1, h_1)$. It is clear that we can repeat this procedure and obtain:

(i) a sequence of affine spaces, (A_i), and a sequence of associated vector spaces $(\vec{A}_i)$, such that $\vec{A}_i = \vec{A}_{i-1}$,

(ii) a sequence of linear transformations $h_i \in \mathrm{Hom}\,(\vec{A}_i, \hat{A}_{i+1})$, and,

(iii) a sequence of linear forms $\varphi_i \colon \hat{A}_i \to \mathbb{R}$.

We can define the triplet $(\hat{A}_{i+1}, \varphi_{i+1}, h_{i+1})$ if and only if $h_i(\vec{A}_i) \not\subset \vec{A}_i$. We would then have:

$$\hat{A}_{i+1} = \vec{A}_i \xrightarrow{h_i} \hat{A}_{i+1} = \vec{A}_i \xrightarrow{\varphi_i} \mathbb{R} \;\text{ and }\; \varphi_{i+1} = \varphi_i \circ h_i;$$

$$\vec{A}_{i+1} = h_i^{-1}(\vec{A}_i) = \mathrm{Ker}\,\varphi_{i+1}, \quad A_{i+1} = h_i^{-1}(A_i) = \varphi_{i+1}^{-1}(1),$$

$$h_{i+1} = h_i|\vec{A}_{i+1}, \;\text{ and finally, }\; \hat{A}_{i+1} = \vec{A}_i$$

(if necessary, extended to $i = -1$, by setting $\vec{A}_{-1} = \hat{A}_0 = \hat{A}$).

Let us now define $\vec{A}_\infty$ to be the largest vector subspace of $\vec{A}_0 = \vec{A}$ with the property, $h_0(\vec{A}_\infty) \subset \vec{A}_\infty$ (this space exists because for any two subspaces w_1 and w_2 of $\vec{A}$, if $h_0(w_1) \subset w_1$ and $h_0(w_2) \subset w_2$ then, $h_0(w_1 + w_2) \subset w_1 + w_2$).

Let $h_\infty = h_0|_{\vec{A}_\infty}$, since $h_0(\vec{A}_\infty) \subset \vec{A}_\infty$, then, h_∞ can be look upon as an endomorphism of $\vec{A}_\infty$.

All of the vector subspaces $\vec{A}_i$ of $\vec{A}$ contain $\vec{A}_\infty$; and if $\vec{A}_i \neq \vec{A}_\infty$, then, $\vec{A}_{i+1}$ is a hyperplane of $\hat{A}_{i+1} = \vec{A}_i$. The sequence (A_i) defined by the recurrence: $\vec{A}_0 = \vec{A}$ and $\vec{A}_{n+1} = h_i^{-1}(\vec{A}_i)$ will ultimately become constant beyond a certain rank. Let $r \in \mathbb{N}^*$ be the smallest rank, i.e., the smallest integer such that $\vec{A}_r = \vec{A}_\infty$, then, we would have, $h_\infty = h_r$ and $h_r(\vec{A}_r) = h_0(\vec{A}_r) \subset \vec{A}_r$. Consequently, $\{0\} \subset \vec{A}_\infty = \vec{A}_r \subset \vec{A}_{r-1} \subset \cdots \subset \vec{A}_1 \subset \vec{A}_0 = \vec{A}$.

Similar sequences can be constructed from any arbitrary $\mathbb{R}$-affine space A' for a given vector envelope $(\hat{A}', \varphi')$ and a given linear transformation $h' \in \mathrm{Hom}\,(\vec{A}', \hat{A}')$, and with assumptions on the triplet $(\hat{A}', \varphi', h')$ identical to those made on the triplet $(\hat{A}, \varphi, h)$.

Let $(\vec{A}'_i)$ be the sequence of vector subspaces of $\vec{A}'$. This sequence will also be constant beyond a certain rank and we could define an integer r' to be the smallest integer such that $\vec{A}'_{r'} = \vec{A}'_\infty$, then, $h'_{r'}(\vec{A}'_{r'}) \subset \vec{A}'_{r'}$ and $h'_{r'} = h'_\infty = h'_0|_{\vec{A}'_\infty}$. We also have: $\{0\} \subset \vec{A}'_\infty = \vec{A}'_{r'} \subset \vec{A}'_{r'-1} \subset \cdots \subset \vec{A}'_1 \subset \vec{A}'_0 = \vec{A}'$. Let us now make the assumption that $\dim \hat{A}_0 = \dim \hat{A}'_0$, $h_0(\vec{A}_0) \not\subset \vec{A}_0$, $h'_0(\vec{A}'_0) \not\subset \vec{A}'_0$, and define the two sets Γ_0 and Γ_1 by:

$$\Gamma_0 = \{\hat{g}_0; \hat{A}_0 \to \hat{A}'_0 \text{ linear bijection } | \varphi'_0 \circ \hat{g}_0 = \varphi_0 \text{ and } h'_0 \circ \vec{g}_0 = \hat{g}_0 \circ h_0\},$$

$$\Gamma_1 = \{\hat{g}_1; \hat{A}_1 \to \hat{A}'_1 \text{ linear bijection } | \varphi'_1 \circ \hat{g}_1 = \varphi_1 \text{ and } h'_1 \circ \vec{g}_1 = \hat{g}_1 \circ h_1\},$$

where, if $\hat{g}_i \in \Gamma_i$ $(i = 0, 1)$, then, $\vec{g}_i = \hat{g}_i|_{\vec{A}_i}$ is a linear bijection from $\vec{A}_i$ to $\vec{A}'_i$. Similarly, $g_i = \hat{g}_i|_{A_i}$ is an affine bijection from $\vec{A}_i$ to $\vec{A}'_i$.

If $\hat{g}_0 \in \Gamma_0$, let $\vec{g}_0 = \hat{g}_0|_{\vec{A}_0}$, then, $\hat{g}_0 \in \Gamma_1$, indeed,

— $\hat{g}_0(\vec{A}_0) = \hat{g}_0(\operatorname{Ker}\varphi_0) = \operatorname{Ker}\varphi'_0 = \vec{A}'_0$ (because, $\vec{A}_0 = \operatorname{Ker}\varphi_0$, $\hat{g}_0\colon \hat{A}_0 \tilde{\to} \hat{A}'_0$ is a linear bijection, and $\varphi'_0 \circ \hat{g}_0$) therefore, $\vec{g}_0\colon \vec{A}_0 \tilde{\to} \vec{A}'_0 = \hat{A}'_0$ is a linear bijection.

— For every u, element of $\vec{A}$

$$\varphi'_0 \circ (\vec{g}_0(u)) = \varphi'_0 \circ h'_0 \circ \hat{g}_0(u) = \varphi'_0 \circ \hat{g}_0 \circ h_0(u) = \varphi_0 \circ h_0(u) = \varphi_1(u)\,.$$

Therefore, $\varphi'_1 \circ \hat{g}_0 = \varphi_1$.

— Finally, for every u, element of $\vec{A}_1$, $\vec{g}_0(u) \in \vec{A}'_1$, and,

$$h'_1 \circ \vec{g}_0(u) = h'_1(\vec{g}_0(u)) = h'_0 \circ \vec{g}_0(u) = \hat{g}_0 \circ h_0(u) = \vec{g}_0 \circ h_1(u).$$

Lemma 1. $\varphi_{0,1}\colon \Gamma_0 \to \Gamma_1$, $\hat{g} \mapsto \vec{g} = \hat{g}|_{\vec{A}_0}$ *is a bijection from* Γ_0 *to* Γ_1.

Proof. Existence. Let $\hat{g}_1 \in \Gamma_1$ and let $a \in A_1$, then $h_0(a) \in A_0$ and so $\hat{A}_0$ is spanned by $\vec{A}_0$ and $h_0(a)$; we even have the direct sum $\hat{A}_0 = \vec{A}_0 \oplus \mathbb{R}h_0(a)$. We then obtain the linear bijection $\hat{g}\colon \hat{A}_0 \to \hat{A}'_0$ by setting: $\hat{g}_0(h_0(a)) = h'_0(\hat{g}_1(a))$, and, $\hat{g}_0(u) = \hat{g}_1(u)$ for every u, element of $\hat{A}_1 = \vec{A}_0$.

Hence, $\hat{g}_0|_{\vec{A}_0} = \vec{g}_0 = \hat{g}_1$.

On another side, $\vec{A}_0 = \hat{A}_1$ is spanned by $\vec{A}_1$ and $a \in A_1$; we have the direct sum $\hat{A}_1 = \vec{A}_1 \oplus \mathbb{R}a$. By definition of $\hat{g}_0$ we have: $\hat{g}_0(h_0(a)) = h'_0(\hat{g}_1(a))$, and, for every u, element of $\vec{A}_1$, we can check easily that $\hat{g}_0 \circ h_0(u) = h'_0 \circ \vec{g}_0(u)$. And similarly we show that $\varphi'_0 \circ \hat{g}_0 = \varphi_0$.

Uniqueness. Let $(\hat{g}_0, \hat{g}) \in \Gamma_0 \times \Gamma_0$ and such that: $\varphi_{0,1}(\hat{g}_0) = \varphi_{0,1}(\hat{g})$, then,

First, $\hat{g}_0|_{\vec{A}_0} = \hat{g}|_{\vec{A}_0}$;

Next, by definition of Γ_0, $\hat{g}_0(h_0(a)) = \hat{g}(h_0(a))$.

Hence, $\hat{g}_0 = \hat{g}$ in all $\hat{A}_0$. □

Let us now make the assumption that $\dim \hat{A}_0 = \dim \hat{A}'_0$ and $r = r'$. Then, $\dim \hat{A}_i = \dim \hat{A}'_i$ for all integers i, $1 \le i \le r$. Indeed, if $i = 0$, the relation stands by assumption. The induction step is a direct consequence of the two identities: $\hat{A}_{i+1} = \vec{A}_i$ and $\hat{A}'_{i+1} = \vec{A}'_i$, since, $\dim \vec{A}_i = \dim \vec{A}'_i$ yields $\dim \hat{A}_{i+1} = \dim \hat{A}'_{i+1}$.

We can define for every i, $1 \le i \le r$ the set:

$$\Gamma_i = \{\hat{g}_i; \hat{A}_i \to \hat{A}'_i \text{ linear bijection } (1)|\varphi'_i \circ \hat{g}_i = \varphi_i \text{ and } (2) h'_i \circ \vec{g}_i = \hat{g}_i \circ h_i\}.$$

The Lemma 1 can then be extended as follows.

Lemma 2. *$\varphi_{i,i+1}\colon \Gamma_i \to \Gamma_{i+1}$, $\hat{g}_i \mapsto \vec{g}_i = \hat{g}_i|_{\vec{A}_i}$ is a bijection from Γ_i to Γ_{i+1}.*

Proof. For $i = 0$, the result is Lemma 1. We conclude the proof by induction after noting that if $x \in A_{i+1}$ then $h(x) \in A_i$ and $h^2(x) \in A_{i-1}$, therefore x and $\vec{A}_{i+1}$ span $\hat{A}_{i+1} = \vec{A}_i$; also, x, $h(x)$, and $\vec{A}_i$ span $\hat{A}_i = \vec{A}_{i-1}$. □

Corollary 1. *$\varphi_{0,\infty}\colon \Gamma_0 \overset{\varphi_{0,1}}{\to} \Gamma_1 \overset{\varphi_{1,2}}{\to} \Gamma_2 \ldots \to \Gamma_{r-1} \overset{\varphi_{r-1,r}}{\to} \Gamma_r = \Gamma_\infty$ defined by $\varphi_{0,\infty} = \varphi_{r-1,r} \circ \ldots \circ \varphi_{1,2} \circ \varphi_{0,1}$ and $\varphi_{0,\infty}(\hat{g}_0) = \hat{g}_0|_{\vec{A}_\infty}$ is a bijection from Γ_0 to Γ_∞, where $\Gamma_r = \Gamma_\infty = \hat{g}_\infty; \hat{A}_\infty \to \hat{A}'_\infty$ linear bijection $|\varphi'_\infty \circ \hat{g}_\infty = \varphi_\infty$ and $h'_\infty \circ \vec{g}_\infty = \hat{g}_\infty \circ h_\infty$, and $\varphi_\infty = \varphi_r$ and $\varphi'_\infty = \varphi'_r$.*

5. Isotropy Group of $h \in \mathrm{Hom}\,(\vec{A}, \hat{A})$

Let $\varphi = \varphi_0\colon \hat{A}_0 \to \mathbb{R}$ be a linear form such that $A_0 = A = \varphi^{-1}(1)$ and $\vec{A}_0 = \vec{A} = \operatorname{Ker}\varphi$. We know that the affine group $GA(A_0)$ can be identified with the group $G = \{\hat{g} \in GL(\hat{A}_0); \varphi \circ \hat{g} = \varphi\}$.

Let $\vec{A}_\infty$ be the largest subspace of $\vec{A}$ such that $h_\infty(\vec{A}_\infty) \subset \vec{A}_\infty$, $\vec{A}_\infty = \varphi_\infty^{-1}(1)$, $\vec{A}_\infty = \operatorname{Ker}\varphi_\infty$, and $h_\infty = h|_{\vec{A}_\infty}$. In this case also, the group of affine transformations of $A_\infty = A_r$, i.e., $GA(A_\infty) = GA(A_r)$, can be identified with the Lie group $G_\infty = \{\hat{g}_\infty \in GL(\hat{A}_\infty); \varphi_\infty \circ \hat{g}_\infty = \varphi_\infty\}$, while the isotropy group of h is defined by $G_h = \{\hat{g} \in GL(\hat{A}_0); \varphi \circ \hat{g} = \varphi$ and $\hat{g} \circ h = h \circ \vec{g}\}$. It is seen that $G_h = \Gamma_0$ (here Γ_0 is obtained through substituting $A'_0 = A_0$, $\varphi' = \varphi$, $h' = h$ and $r' = r$ in Lemmas 1 and 2).

If $\hat{g} \in G_h$, then, the identities: (1) $\varphi \circ \hat{g} = \varphi$ and (2) $\hat{g} \circ h = h \circ \vec{g}$ (restricted to $\vec{A}_0$ and $\vec{A}_\infty$) produce in conjunction with the condition $h_\infty(A_\infty) \subset A_\infty$, two other identities: (1_∞) $\varphi_\infty \circ \hat{g}_\infty = \varphi_\infty$ and (2_∞) $\vec{g}_\infty \circ h_\infty = h_\infty \circ \vec{g}_\infty$, respectively, where $\hat{g}_\infty = \hat{g}|_{A_\infty}$ and $\vec{g}_\infty = \hat{g}|_{\vec{A}_\infty}$. Hence, $\hat{g}_\infty$ is a linear transformation over A_∞ satisfying the relations (1_∞) and (2_∞), so, $g_\infty \in \{\hat{g}'_\infty \in G_\infty; \vec{g}'_\infty \circ h_\infty = h_\infty \circ \vec{g}'_\infty\} = \Gamma_r = \Gamma_\infty$. Since $\varphi_{0,\infty}$ is a bijection, it is also an isomorphism from G_h (isotropy group of h) to G_∞. We can then state the following theorem.

Theorem 3. *The isotropy group of h in $GA(A)$ is isomorphic to the group of affine transformations of A_∞, whose linear part commutes with h_∞.*

Corollary 2. *We have:*

— The dimension of the stabilizer G_h of h is greater or equal to two times the dimension of $\vec{A}_\infty$.

— The orbit O_h of h is open if and only if $\vec{A}_\infty = 0$.

Proof. We begin by noting that $G_h = G_\infty = \{g \in GA(A_\infty); \vec{g} \circ h_\infty =$

$h_\infty \circ \vec{g}\}$. Let $C(h_\infty)$ be the commuting part of h_∞ in the algebra End $(\vec{A}_\infty)$. It is a known result that $\dim C(h_\infty) \geq \dim \vec{A}_\infty$, also since the Lie algebra of the Lie group G_∞ is isomorphic to $\vec{A}_\infty \times C(h_\infty)$, then, $\dim G_h = \dim \vec{A}_\infty + \dim C(h_\infty) \geq 2 \dim \vec{A}_\infty$. Finally, the orbit O_h of h is open if and only if the isotropy group G_h is "trivial", i.e., if and only if $\vec{A}_\infty = 0$. □

6. Study of the Orbits of the Coadjoint Representation of the Affine Group

Let h and h' be two elements of Hom $(\vec{A}, \hat{A})$, (A_i) and (A'_i) the vector space sequences associated with the triplets $(\hat{A}, \varphi, h)$ and $(\hat{A}, \varphi, h')$, respectively. Again, $\vec{A}_\infty$ and $\vec{A}'_\infty$ denote the largest vector subspaces of $\vec{A}$ stable by h and h', respectively, and $h_\infty = h|_{\vec{A}_\infty}$ and $h'_\infty = h'|_{\vec{A}'_\infty}$ are the restrictions of h and h' to $\vec{A}_\infty$ and $\vec{A}'_\infty$ respectively.

If we assume that h and h' belong to the same orbit of $GA(A)$, then, there exists $g \in GA(A)$ such that $h' \circ \vec{g} = \hat{g} \circ h$ and $\varphi \circ \hat{g} = \varphi$, therefore: $\hat{g} \in \Gamma = \{\hat{g}_0 \colon \hat{A}_0 \tilde{\rightarrow} \hat{A}_0$ linear isomorphism $\varphi_0 \circ \hat{g}_0 = \varphi_0 (1)$ and $h'_0 \circ \vec{g}_0 = \hat{g}_0 \circ h_0 (2)\}$ (for Γ_0, we set $A_0 = A$, $\hat{A}_0 = \hat{A}$, $\varphi_0 = \varphi$, $h_0 = h$, and $h'_0 = h'$). We know then, by application of Lemma 1, that $\vec{g} = \hat{g}|_{A_\infty} \in \Gamma_1$ where, $\Gamma_1 = \{\hat{g}_1 \colon \hat{A}_1 \tilde{\rightarrow} \hat{A}'_1$ isomorphism $\varphi'_1 \circ \hat{g}_1 = \varphi_1 (1)$ and $h'_1 \circ \vec{g}_1 = \hat{g}_1 \circ h_1 (2)\}$ (for Γ_1, $\hat{A}_1 = \vec{A}_0 = \hat{A}'_1$, $\varphi_1 = \varphi_0 \circ h_0$, $\varphi'_1 = \varphi_0 \circ h'_0$, $h_1 = h_0|_{\vec{A}_1}$, $h'_1 = h'_0|_{\vec{A}'_1}$, $\vec{A}_1 = \operatorname{Ker} \varphi_1 = h_0^{-1}(A_0)$, and $\vec{A}'_1 = \operatorname{Ker} \varphi'_1 = {h'}_0^{-1}(\vec{A}_0)$).

Consequently, $\dim \vec{A}_1 = \dim \vec{A}'_1$, and by direct induction, $\dim \vec{A}_\infty = \dim \vec{A}'_\infty$, the two vector subspace sequences associated with the triplets $(\hat{A}, \varphi, h)$ and $(\hat{A}, \varphi, h')$ become constant beyond the same rank r, if and only if h and h' belong to the same orbit.

By an appropriate restriction of (1) $\varphi \circ \hat{g}$ and (2) $h' \circ \vec{g} = \hat{g} \circ h$, where $g \in GA(A_0)$, an element $\hat{g}_\infty = \hat{g}|_{\hat{A}_\infty} \in \Gamma_r = \Gamma_\infty$ can be obtained such that $\vec{g}_\infty = \hat{g}|_{\vec{A}_\infty}$ becomes a linear isomorphism from $\vec{A}_\infty$ to $\vec{A}'_\infty$ and $h' \circ \vec{g}_\infty = \vec{g}_\infty \circ h_\infty$ (or equivalently, $h'_\infty = \vec{g}_\infty \circ h_\infty \circ \vec{g}_\infty^{-1}$).

Inversely, let us assume that $\dim \vec{A}_\infty = \dim \vec{A}'_\infty$ and that there exists a linear isomorphism $\vec{g}\infty$ from $\vec{A}_\infty$ to $\vec{A}'_\infty$ such that $h' \circ \vec{g}_\infty = \vec{g}_\infty \circ h_\infty$ ($h_\infty = h|_{\vec{A}_\infty}$ and $h'_\infty = h'|_{\vec{A}'_\infty}$).

Let r be the smallest integer in $\vec{A}_r = \vec{A}_\infty$ and $\vec{A}'_r = \vec{A}'_\infty$. Let $\vec{g}_r = \vec{g}_\infty$, then, there exists a unique linear isomorphism $\hat{g}_r$ from $\hat{A}_r$ to $\hat{A}'_r$ such that $\hat{g}_r|_{\vec{A}_r} = \vec{g}_r = \vec{g}_\infty$ and $h'_r \circ \vec{g}_r = \vec{g}_r \circ h_r$, where $h_r = h_\infty$ and $h'_r = h'_\infty$; thus, $\hat{g}_r \in \Gamma_r$ (the identity $\varphi'_r \circ \hat{g}_r = \varphi_r$ is satisfied by the definition of the two sequences $(\vec{A}_i)$ and $(\vec{A}'_i)$). We conclude using the bijection between Γ_0 and $\Gamma_r = \Gamma_\infty$, mentioned in Corollary 3, and asserting the existence of a linear isomorphism $\hat{g}_0 \colon \hat{A}_0 \xrightarrow{\sim} \hat{A}_0$ satisfying the two identities: (1) $\varphi_0 \circ \hat{g}_0 = \varphi_0$ and (2) $h'_0 \circ \vec{g}_0 = \hat{g}_0 \circ h_0$.

The identity (1) and $\hat{g}_0 \in GL(\hat{A}_0)$ show that $g_0 = \hat{g}_0|_{A_0}$ is an affine transformation of $A = A_0$. The identity (2), on the other hand, shows that h and h' belong to the same orbit.

We can then state the following theorem.

Theorem 4. *Let h and h' be in* Hom $(\vec{A}, \hat{A})$. *Let $\vec{A}_\infty$ and $\vec{A}'_\infty$ be the largest vector subspaces of $\vec{A}$ stable h and h', respectively. Let h_∞ and h'_∞ their restrictions to A_∞ and A'_∞, respectively. Then, h and h' belong to the same orbit of $GA(A)$ if and only if $\vec{A}_\infty$ and $\vec{A}'_\infty$ have the same dimension and there exists a linear isomorphism $\vec{g}_\infty$ from $\vec{A}_\infty$ to $\vec{A}'_\infty$ satisfying $h'_\infty = \vec{g}_\infty \circ h_\infty \circ \vec{g}_\infty^{-1}$.*

7. Final Observations

Concerning the study of the isotropy groups of the coadjoint action of the affine group $GA(A)$. Let us consider an element h in Hom $(\vec{A}, \hat{A})$: the dual of the Lie algebra associated with $GA(A)$ ($h \colon \vec{A} \to \hat{A}$ is a linear transformation). If $\varphi \colon \hat{A} \to \mathbb{R}$ is a linear form over $\hat{A}$ with $A = \varphi^{-1}(1)$ and $\vec{A} = \operatorname{Ker}\varphi$, then, as was done previously, we can construct a sequence of vector subspaces $(\vec{A}_i)_{0 \le i \le r}$ of $\vec{A}$ from the triplet $(\hat{A}, \varphi, h)$. This sequence is constant beyond r (the smallest integer such that $\vec{A}_r = \vec{A}_\infty =$ largest vector subspace of $\vec{A} = \vec{A}_0$ that is stable by $h = h_0$). Then, we define a sequence of affine spaces $(A_i)_{0 \le i \le r}$ by the recurrence: $A_0 = A$, $A_i = h_{i-1}^{-1}$ for $i \ge 1$, where, $h_i = h_{i-1}|_{\vec{A}_i}$, and, $\hat{A}_i = \vec{A}_{i-1}$ (for $i = 0$, we set $\vec{A}_{i-1} = \hat{A}_0$).

Let x be an arbitrarily fixed element in $A_r = A_\infty$; it is quite easy to see that: $h(x) \in A_{r-1}$, $h^2(x) \in A_{r-2}$, ..., $h^r(x) \in A_0$, and that: $\vec{A}_r = \vec{A}_\infty$, and, x span $\vec{A}_{r-1} = \hat{A}_r$, $\vec{A}_\infty$, x, and, $h(x)$ span $\vec{A}_{r-2}$, ..., and finally, $\vec{A}_\infty$, x, $h(x)$, ..., and, $h_r(x)$ span the vector space $\hat{A}_0$ (recall that if $\dim_{\mathbb{R}} A_0 = n$, then, $\dim_{\mathbb{R}} \hat{A}_0 = n + 1$).

Let G be the isotropy group of h, element of $GA(A0)$, and, $G_\infty = \{g \in GA(A_\infty); \vec{g} \circ h_\infty = h_\infty \circ \vec{g}\}$ ($h_\infty = h_r = h|_{\vec{A}_\infty}$). It is not difficult to show that the transformation $\psi \colon G_h \xrightarrow{\sim} G_\infty$, $g \mapsto \psi(g) = g|_{A_\infty}$ is an isomorphism of groups.

Indeed, given an element $g_r \in G_\infty$ we can construct an "enhancement" $\hat{g}_0$ of $\hat{g}_r$ such that $\hat{g}_0|_{A_0} = g_0 \in G_h$, by setting for every $j \in \{0, 1, \ldots, r\}$,

$\hat{g}_0(h_0^j(x) = h_0^j((g_r(x))$ and $\hat{g}_0|_{\vec{A}_r} = \vec{g}_r$.

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