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ON THE GRACEFULNESS OF THE DIGRAPHS $n \cdot \vec{C}_m$

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Abstract: A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of $D(V, E)$, while f' is called the induced edge's graceful labeling of D. In this paper we discuss the gracefulness of the digraph $n \cdot \vec{C}_m$ and prove that $n \cdot \vec{C}_m$ is a graceful digraph for $m = 19$ and even n.

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Key Words: digraph, directed cycles, graceful graph, graceful labeling

1. Introduction

A graph $G(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of G, while f' is called the induced edge's graceful labeling of G. A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)]$

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(mod $|E|+1$) for every directed edge (u,v) is a bijection, where $[v]$ (mod n) denotes the least positive residue of v modulo n. And, for any integers $a \le b$, let $[a,b]$ denote the set of all consecutive integers from a to b. Let $\vec{C}_m$ denote the directed cycle on m vertices. $n \cdot \vec{C}_m$ denotes the graph obtained from any n copies of $\vec{C}_m$ which have just one common vertex. Similarly, $n \cdot \vec{C}_m$ denote the digraphs obtained from any n copies of the directed cycle $\vec{C}_m$ which have just one common vertex.

As to the gracefulness of $n \cdot \vec{C}_m$ we know the following results: Ma proved in [8] that the gracefulness of $n \cdot \vec{C}_3$ implies that n is even, at same times he conjectured that the condition that n is even was also sufficient for $n \cdot \vec{C}_3$ to be graceful. In [5], the second author of this paper has showed this conjecture. Du and Cun proved in [6] that $n \cdot \vec{C}_{2k}$ is graceful for every integer $n \ge 1$ and $k \ge 1$, they also conjectured that the $n \cdot \vec{C}_m$ is graceful whenever m is odd and n is even. This conjecture has been proved when $m = 5, 7$ in [7], $m = 9, 11, 13$ in [8] and $m = 15, 17$ in [9], respectively.

In this paper, we will further discuss the gracefulness of the digraph $n \cdot \vec{C}_m$ and prove the digraph $n \cdot \vec{C}_m$ is graceful if $m = 19$ and n is even.

2 Main Results

Let $\vec{C}_m^1, \vec{C}_m^2, \cdots, \vec{C}_m^n$ denote the n directed cycles in $n \cdot \vec{C}_m$. The common vertex of $\vec{C}_m^i$'s is denoted by v_0, and, for every $i \in [1,n]$, other $m-1$ vertices of the $\vec{C}_m^i$ are denoted by v_j^i, $j = 1, \ldots, m-1$. For convenience, we put $v_0^1 = v_0^2 = \cdots = v_0^n = v_0$, and take subscripts j's modulo m. Obviously, $|E(n \cdot \vec{C}_m)| = mn$.

Suppose that $n \cdot \vec{C}_m$ is graceful and f and f' are its graceful labeling and the induced edge's graceful labeling, respectively.

Theorem 1. *For every even integer n, the digraph $n \cdot \vec{C}_{19}$ is graceful.*

Proof. Let $f(v_0) = 0$ and for other vertices, define:

$$f(v_j^i) = \begin{cases} (19 - \frac{j-1}{2})n + 1 - i, & j = 1, 3, 5;\ 1 \le i \le n, \\ 10n + 1 - i, & j = 7;\ 1 \le i \le n, \\ (19 - \frac{j-3}{2})n + 1 - i, & j = 9;\ 1 \le i \le n, \\ (j+3)n + 2 - i & j = 11;\ 1 \le i \le n, \\ 6n + 1 - i & j = 13;\ 1 \le i \le n, \\ \frac{j+1}{2}n + 1 - i, & j = 15, 17;\ 1 \le i \le n. \end{cases}$$

$$f(v_j^i) = \begin{cases} \frac{j-2}{2})n + i, & j = 2,4,6,8;\ 1 \le i \le n, \\ 4n + i, & j = 10;\ 1 \le i \le n, \\ 11n + 1 + i, & j = 12;\ 1 \le i \le n, \\ 4n + i, & j = 14;\ 1 \le i \le \frac{n}{2}, \\ 14n + \frac{n}{2} + i, & j = 14;\ \frac{n}{2} + 1 \le i \le n, \\ (28 - j)n + i, & j = 16,18;\ 1 \le i \le n. \end{cases}$$

Firstly, we show that f is an injective mapping from $V(n \cdot \vec{C}_{19})$ into $[0, 19n]$. For $j \in [0, 18]$, put$S_j = \{f(v_j^i) | 1 \le i \le n\}$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\} = \{0\}, \\
S_1 &= \{f(v_1^i) | 1 \le i \le n\} = \{19n + 1 - i | 1 \le i \le n\} \\
&= \{19n, 19n - 1, \cdots, 18n + 1\}, \\
S_2 &= \{f(v_2^i) | 1 \le i \le n\} = \{i | 1 \le i \le n\} = \{1, 2, \cdots, n\}, \\
S_3 &= \{f(v_3^i) | 1 \le i \le n\} = \{18n + 1 - i | 1 \le i \le n\} \\
&= \{18n, 18n - 1, \cdots, 17n + 1\}, \\
S_4 &= \{f(v_4^i) | 1 \le i \le n\} = \{n + i | 1 \le i \le n\} \\
&= \{n + 1, n + 2, \cdots, 2n\}, \\
S_5 &= \{f(v_5^i) | 1 \le i \le n\} = \{17n + 1 - i | 1 \le i \le n\} \\
&= \{17n, 17n - 1, \cdots, 16n + 1\}, \\
S_6 &= \{f(v_6^i) | 1 \le i \le n\} = \{2n + i | 1 \le i \le n\} \\
&= \{2n + 1, 2n + 2, \cdots, 3n\}, \\
S_7 &= \{f(v_7^i) | 1 \le i \le n\} = \{10n + 1 - i | 1 \le i \le n\} \\
&= \{10n, 10n - 1, \cdots, 9n + 1\} \\
S_8 &= \{f(v_8^i) | 1 \le i \le n\} = \{3n + i | 1 \le i \le n\} \\
&= \{3n + 1, 3n + 2, \cdots, 4n\}, \\
S_9 &= \{f(v_9^i) | 1 \le i \le n\} = \{16n + 1 - i | 1 \le i \le n\} \\
&= \{16n, 16n - 1, \cdots, 15n + 1\}, \\
S_{10} &= \{f(v_{10}^i) | 1 \le i \le n\} = \{6n + i | 1 \le i \le n\} \\
&= \{6n + 1, 6n + 2, \cdots, 7n\}, \\
S_{11} &= \{f(v_{11}^i) | 1 \le i \le n\} = \{14n + 2 - i | 1 \le i \le n\} \\
&= \{14n + 1, 14n, \cdots, 13n + 2\}, \\
S_{12} &= \{f(v_{12}^i) | 1 \le i \le n\} = \{11n + 1 + i | 1 \le i \le \frac{n}{2}\}
\end{aligned}$$

$$\cup \{4n+i|\frac{n}{2}+1\le i\le n\} = \{11n+2, 11n+3, \cdots, 11n+\frac{n}{2}+1\}$$
$$\cup \{4n+\frac{n}{2}1, 4n+\frac{n}{2}+2, \cdots, 5n\},$$
$$S_{13} = \{f(v_{13}^i)|1\le i\le n\} = \{6n+1-i|1\le i\le n\}$$
$$= \{6n, 6n-1, \cdots, 5n++1\},$$
$$S_{14} = \{f(v_{14}^i)|1\le i\le n\} = \{4n+i|1\le i\le \frac{n}{2}\}\cup\{14n+i|\frac{n}{2}+1\le i\le n\}$$
$$= \{4n+1, 4n+2, \cdots, 4n+\frac{n}{2}\}\cup\{14n+\frac{n}{2}+1, 14n+\frac{n}{2}+2, \cdots, 15n\},$$
$$S_{15} = \{f(v_{15}^i)|1\le i\le n\} = \{8n+1-i|1\le i\le n\}$$
$$= \{8n, 8n-1, \cdots, 7n++1\},$$
$$S_{16} = \{f(v_{16}^i)|1\le i\le n\} = \{12n+i|1\le i\le n\}$$
$$= \{12n+1, 12n+2, \cdots, 13n\},$$
$$S_{17} = \{f(v_{17}^i)|1\le i\le n\} = \{9n+i|1\le i\le n\}$$
$$= \{9n+1, 9n+2, \cdots, 10n\},$$
$$S_{18} = \{f(v_{18}^i)|1\le i\le n\} = \{10n+i|1\le i\le n\}$$
$$= \{10n+1, 10n+2, \cdots, 11n\}.$$

Then we see that $S_i \cap S_j = \emptyset$ for $i, j \in [0, 18]$ and $i \ne j$, which yields that f is an injection from $V(n \cdot \vec{C_{19}})$ into $[0, 18n]$.

Secondly, we show the induced edges labeling f' is a bijection from $E(n\cdot\vec{C}_{19})$ onto $[1, 19n]$.

For $j \in [0, 18]$, set $B_{j,1} = \{[f(v_j^{i+1}) - f(v_j^i)] \pmod{19n+1}|1 \le i \le \frac{n}{2}\}$, $B_{j,2} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{19n+1}|\frac{n}{2}+1 \le i \le n\}$ or $B_{j,e} = \{[f(v_j^{i+1}) - f(v_j^i)] \pmod{19n+1}|i = 2, 4, \cdots, n\}$, $B_{j,o} = \{[f(v_{j+1}^i) - f(v_j^i)] \pmod{19n+1}|i = 1, 3, \cdots, n-1\}$ and $B_j = B_{j,1} \cup B_{j,2}$or $B_j = B_{j,e} \cup B_{j,o}$ and $B = \bigcup_{j=0}^{18} B_j$. Then, in order to prove that f' is a bijection it suffices to show $B = [1, 19n]$ or equivalently $[1, 19n] \subseteq B$.

First, we show the $\{1, 3, \cdots, 19n-1\} \subseteq B$.

(1) For $j = 12$ we have that

$$B_{12,2} = \{6n+1-i-(4n+i) = 2n-2i+1|\tfrac{n}{2}+1 \le i \le n\}$$
$$= \{1, 3, \cdots, n-1\},$$

(2) For $j = 17, 14, 15$ we have

$$B_{17} \cup B_{14,1} \cup B_{15}$$

$$= \{10n+i-(9n+1-i) = n-1+2i|1 \le i \le n\} \cup \{8n+1-i-(4n+i)$$
$$= 4n-2i+1|1 \le i \le \frac{n}{2}\} \cup \{12n+i-(8n+1-i) = 4n+2i-1|1 \le i \le n\}$$
$$= \{n+1, n+3, \cdots, 3n-1\} \cup \{3n+1, 3n+3, \cdots, 4n-1\}$$
$$\cup \{4n+1, 4n+3, \cdots, 6n-1\} = \{n+1n+3, \cdots, 6n-1\},$$

which and (1) imply $\{1, 3, \cdots, 6n-1\} \subseteq B$.

(3) For $j = 6, 18, 13$ we have

$$B_7 \cup B_{18,e} \cup B_{13,2} = \{10n+1-i-(2n+i) = 8n+1-2i|\ 1 \le i \le n\}$$
$$\cup \{0-(10n+i) = 9n+1-i|i = 2, 4, \cdots, n\}$$
$$\cup \{14n+i-(6n+1-i) = 8n+2i-1|\frac{n}{2}+1 \le i \le n\},$$
$$= \{6n+1, 6n+3, \cdots, 8n-1\} \cup \{8n+1, 8n+3, \cdots, 9n-1\}$$
$$\cup \{9n+1, 9n+3, \cdots, 10n-1\}.$$

which and (2) imply $\{1, 3, \cdots, 9n-1\} \subseteq B$.

(4) For $j = 11, 8, 12$ we have

$$B_{11,2} \cup B_8 \cup B_{12,1} = \{4n+i-(14n+2-i) = 9n+2i-1|\frac{n}{2}+1 \le i \le n\}$$
$$\cup \{16n+1-i-(3n+i) = 13n+1-2i|\ 1 \le i \le n\}$$
$$\cup \{6n+1-i-(11n+1+i) = 14n-2i+1|1 \le i \le \frac{n}{2}\},$$
$$= \{10n+1, 10n+3, \cdots, 11n-1\} \cup \{11n+1, 11n+3, \cdots, 13n-1\}$$
$$\cup \{13n+1, 13n+3, \cdots, 14n-1\},$$

which and (3) imply $\{1, 3, \cdots, 14n-1\} \subseteq B$.

(5) For $j = 4, 2, 0$ we have

$$B_4 \cup B_2 \cup B_{0,e} = \{17n+1-i-(n+i) = 16n-2i+1|1 \le i \le n\}$$
$$\cup \{18n+1-i-(i) = 18n+1-2i|\ 1 \le i \le n\}$$
$$\cup \{18n+1-i-(0) = 18n-i+1|i = 2, 4, \cdots, n\},$$
$$= \{14n+1, 14n+3, \cdots, 16n-1\} \cup \{16n+1, 16n+3, \cdots, 18n-1\}$$
$$\cup \{18n+1, 18n+3, \cdots, 19n-1\},$$

which and (4) imply $\{1, 3, \cdots, 19n-1\} \subseteq B$.

Next, we show $\{2, 4, \cdots, 19n\} \subseteq B$.

(6) For $j = 1, 3, 5$, we have

$$\begin{aligned} B_1 \cup B_3 \cup B_5 = \{ & i-(19n+1-i)=2i|\ 1\le i\le n\} \\ & \cup \{n+i-(18n+1-i)=2n+2i|\ 1\le i\le n\} \\ & \cup \{2n+i-(17n+1-i)=4n+2i|\ 1\le i\le n\}\{2,4,\cdots,6n\}, \end{aligned}$$

(7) For $j = 10, 18, 9$ we have

$$\begin{aligned} B_{10} \cup B_{18,o} \cup B_9 = & \{14n+2-i-(6n+i)=8n-2i+2|1\le i\le n\} \\ & \cup \{o-(10n+i)=9n+1-i|i=1,3,\cdots,n-1\} \\ & \cup \{6n+i-(16n+1-i)=9n+2i|1\le i\le n\}, \\ = & \{6n+2,6n+4,\cdots,8n\}\cup\{8n+2,8n+4,\cdots,9n\}\cup\{9n+2,9n+4,\cdots,11n\}. \end{aligned}$$

which and (6) imply $\{2, 4, \cdots, 9n\} \subseteq B$.

(8) For $j = 12, 10, 14$, we have

$$\begin{aligned} B_{12} \cup B_{10} \cup B_{14} = & \{6n+1-i-(12n+i)=11n+2-2i\} \\ & \cup \{7n+1-i-(11n+i)=13n+2-2i\} \\ & \cup\{5n+1-i-(7n+i)=15n+2-2i|\ 1\le i\le n)=\{9n+2,9n+4,\cdots,11n\}, \end{aligned}$$

which and (7) imply $\{2, 4, \cdots, 15n\} \subseteq B$.

(9) For $j = 14, 7, 16$ we have

$$\begin{aligned} B_{14,2} \cup B_7 \cup B_{16} = & (\{\{8n+1-i-(14n+i)=13n+2-2i|\frac{n}{2}+1\le i\le n\} \\ & \cup \{3n+i-(10n+1-i)=12n+2i|\ 1\le i\le n\} \\ & \cup \{9n+1-i-(12n+i)=16n+2-2i|1\le i\le n\} \\ = & \{11n+2,11n+4,\cdots,12n\}\cup\{12n+2,12n+4,\cdots,14n\} \\ & \cup \{14n+2,14n+4,\cdots,16n\}=\{11n+2,11n+4,\cdots,15n\}, \end{aligned}$$

which and (8) imply $\{2, 4, \cdots, 16n\} \subseteq B$.

(10) For $j = 11, 13, 0$ we have

$$\begin{aligned} B_{11,1} \cup B_{13,1} \cup B_{0,o} = & (\{\{11n+1+i-(14n+2-i)=16n+2i|1\le i\le \frac{n}{2}\} \\ & \cup \{4n+i-(6n+1-i)=17n+2i|\ 1\le i\le \frac{n}{2}\} \\ & \cup \{19n+1-i-(0)=19n+1-i|i=1,3,\cdots,n-1\} \\ = & \{16n+2,16n+4,\cdots,17n\}\cup\{17n+2,17n+4,\cdots,18n\} \end{aligned}$$

$$\cup \{18n+2, 18n+4, \cdots, 19n\} = \{16n+2, 16n+4, \cdots, 19n\},$$

which and (9) imply $\{2, 4, \cdots, 19n\} \subseteq B$.

We thus prove that $[1, 19n] \subseteq B$, which completes the proof. □

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