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SHIL'NIKOV HETEROCLINIC ORBITS IN A CHAOTIC SYSTEM

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Abstract: In this paper, a chaotic system is considered which exhibits a chaotic attractor with only two eqilibria for some parameters. The existence of heteroclinic orbits of Shil'nikov type in a chaotic system has been proved by using the undetermined coefficient method. As a result, the Shil'nikov criterion guarantees that the system has Smale horseshoes. Moreover, the geometric structures of attractor are determined by these heteroclinic orbits.

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1. Introduction

Chaos is very interesting nonlinear phenomenon and has been intensively studied in the last three decades. It should be noted that one of the commonly agreeable analytic for proving chaos in autonomous systems is based on the fundamental work of Shil'nikov and its subsequent embellishments and slight extension. This is known as the Shil'nikov method or Shil'nikov criterion today, and its role is in some sense equivalent to that of the Li-Yorke Lemma in the discrete setting. In this paper, a rigorous proof is given to show the existence of a chaotic attractor. Specifically, it will be shown that the chaotic system has one type of orbit that is the heteroclinic orbit of Shil'nikov type. Moreover, their precise algebraic expressions in a series expansion form will be derived

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with uniform convergence proved. By applying the Shil'nikov criterion, it is convinced that the chaotic system indeed is chaotic, with Smale horseshoes and the horseshoe type of chaos.

Consider the third-order autonomous system:

$$\frac{dx}{dt} = f(x), \quad t \in R,\ x \in R^3 \tag{1}$$

where the vector field $f(x) : R^3 \to R^3$, belong to class C^r $(r \geq 2)$.

A heteroclinic orbit, is similiarly defined except that there are two distinct saddle foci being connected by the orbit, one corresponding to the forward asymptotic time, and the other, to the reverse asymptotic time limit.

Denote by $\Sigma \in R^2$ a plane that cuts transversely across the recurrent system orbital flow, which occurs locally to heteroclinic orbits. Define a 2-D map P: $U \in \Sigma \to \Sigma$, called the Poincare map, where the neighborhood U designates those points that return to Σ at least once along the orbital flow of system. Then, P defines a 2-D discrete dynamical system:

$$x_{k+1} = P(x_k), \quad k = 0.1.2\ldots,$$

which characterizes the system (1). For the case of a heteroclinic orbit, the corresponding Poincare map is sometimes called a Shil'nikov map.

The heteroclinic Shil'nikov method, namely, the Shil'nikov criterion for the existence of chaos, is the following theorem.

Theorem 1.1. *Suppose that two distinct equilibrium points, denoted by x_e^1 and x_e^2, respectively, of system (1) are saddle foci, whose characteristic values γ_k and $\sigma_k \pm \omega_k$ $(k = 1, 2)$ satisfy the following Shil'nikov inequality:*

$$|\gamma_k| > |\sigma_k| > 0, \quad k = 1, 2$$

under constraint

$$\sigma_1\sigma_2 > 0, \text{ or } \gamma_1\gamma_2 > 0.$$

Suppose also that there exists a heteroclinic orbit joining x_e^1 and x_e^2. Then:

(I) the Shil'nikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;

(II) for any sufficiently small C^1-perturbation g of f, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in R^3 \tag{2}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of the Shil'nikov map defined near the heteroclinic orbit;

(III) both the original system (1) *and perturbed system* (2) *have horseshoe type of chaos.*

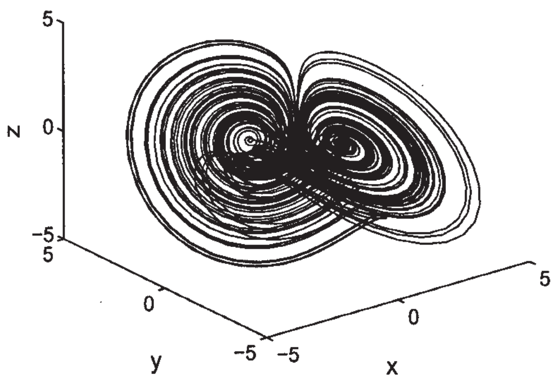

Figure 1: A two-scroll chaotic attractor for $R = 1$

2. A Simple Chaotic System

The chaotic system considered in this paper is following:

$$\begin{aligned} \dot{x} &= -y - x, \\ \dot{y} &= -xz, \\ \dot{z} &= xy + R\,, \end{aligned} \tag{3}$$

where R is parameter. With the typical parameter value of $R = 1$, the corresponding numerical attractor can be found and is depicted in Figure 1. Note that system (3) has two equilibria: $O_1(\sqrt{R}, -\sqrt{R}, 0)$ and $O_2(-\sqrt{R}, \sqrt{R}, 0)$. From Figure 1 one can see that in this attractor, outside trajectories are typically attracted into the vicinity of its steady state, and they are alternatively swirling between the two equilibria O_1 and O_2. Intuitively, there should be a heteroclinic orbit joining O_1 and O_2. Now we will confirm this heteroclinic orbit in the next, where one heteroclinic orbit is found with a precise algebraic expression.

3. The Heteroclinic Orbits

First, note that the linearized system of (3) corresponding to the equilibrium point O_1 has the following form:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{4}$$

where

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -\sqrt{R} \\ -\sqrt{R} & \sqrt{R} & 0 \end{pmatrix}. \tag{5}$$

The characteristic polynomical of the Jacobian A is

$$\lambda^3 + \lambda^2 + R\lambda + 2R = 0\,. \tag{6}$$

Letting $\lambda = \mu - \frac{1}{3}$ yields, $\mu^3 + p\mu + q = 0$, where

$$p = R - \frac{1}{3}, \quad q = \frac{5}{3}R - \frac{2}{27}. \tag{7}$$

Denote $\Delta = (\frac{q}{2})^2 + (\frac{p}{3})^3$, when $\Delta > 0$, the algebraic equation $\mu^3 + p\mu + q = 0$ has a unique negative real root and a conjugate pair of complex roots,with

$$\begin{aligned} \alpha_1 &= \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}; \\ \beta_1 &= -\frac{1}{2}(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}); \\ \gamma_1 &= \frac{\sqrt{3}}{2}(\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}); \end{aligned} \tag{8}$$

Therefore, when $\Delta > 0$, three roots of equation (6) are:

$$\lambda_1 = -\frac{1}{3} + \alpha_1; \quad \lambda_2 = -\frac{1}{3} + \beta_1 + \gamma_1 i; \quad \lambda_3 = -\frac{1}{3} + \beta_1 - \gamma_1 i. \tag{9}$$

Here we take the parameters $R = 1$, for example, we can get the approximately result $\lambda_1 = -1.3532$, $\lambda_2 = 0.1766 + 1.2028i$, $\lambda_3 = 0.1766 - 1.2028i$. We can know that the roots of (6) satisfy Theorem 1.1.

Second, it follows from (3) that:

$$y = -(\dot{x} + x), \quad z = \frac{\ddot{x} + \dot{x}}{x}, \tag{10}$$

$$\frac{d}{dt}(\frac{\ddot{x} + \dot{x}}{x}) + x(\dot{x} + x) - R = 0\,. \tag{11}$$

If $x(t)$ is found, then $y(t)$, $z(t)$ will also be determined. Therefore, finding the heteroclinic orbit of (3) is now changed to seeking a function $\varphi(t)$ such that

$x(t) = \varphi(t)$ satisfying (11) and

$$\varphi(t) \longrightarrow \sqrt{R},\ t \longrightarrow +\infty\,, \qquad \varphi(t) \longrightarrow -\sqrt{R},\ t \longrightarrow -\infty\,, \tag{12}$$

or

$$\varphi(t) \longrightarrow \sqrt{R},\ t \longrightarrow -\infty\,, \qquad \varphi(t) \longrightarrow -\sqrt{R},\ t \longrightarrow +\infty\,. \tag{13}$$

Without loss of generality, one may stipulate a definite as following: from O_1 to O_2 corresponds to $t \longrightarrow +\infty$, while from O_2 to O_1 corresponds to $t \longrightarrow -\infty$.

Next, suppose that, for $t > 0$,

$$\varphi(t) = -\sqrt{R} + \sum_{k=1}^{\infty} a_k e^{k\alpha t}\,, \tag{14}$$

where $\alpha < 0$ is an undetermined constant, and a_k $(k \geq 1)$ are also undetermined coefficients.

We substitute (14) into (11) and then comparing the coefficients of $e^{k\alpha t}$ $(k \geq 1)$ of the same power terms:

$$1 = \varphi(t)\frac{1}{\varphi(t)} = (-\sqrt{R} + \sum_{k=1}^{\infty} a_k e^{k\alpha t})(-\frac{1}{\sqrt{R}} + B_k e^{k\alpha t})\,, \tag{15}$$

where $B_1 = -\frac{1}{R}a_1$,

$$B_k = \frac{1}{\sqrt{R}}(-\frac{1}{\sqrt{R}}a_k + \sum_{i+j=k} a_i B_j), \quad i \geq 1,\ j \geq 1,\ k > 1\,. \tag{16}$$

So we can have the following results: for $n = 1$:

$$a_1[\frac{\alpha^2}{\sqrt{R}}(\alpha+1) + (\alpha+2)\sqrt{R}] = 0; \tag{17}$$

for $n = 2$:

$$a_2 = \frac{1}{\omega(2\alpha)}\frac{a_1^2}{\sqrt{R}}[(\alpha+1)R - 2\alpha^2(\alpha+1)]\,, \tag{18}$$

where $\omega(2\alpha) = (2\alpha)^3 + (2\alpha)^2 + R(2\alpha) + 2R = 0$; for $n \geq 3$:

$$a_n = \frac{1}{\omega(n\alpha)}[-n\alpha^2 \sum_{i+j=n} b_i c_j + \sqrt{R} \sum_{i+j=n} (j\alpha+1)a_i a_j]\,, \tag{19}$$

where

$$\omega(n\alpha) = (n\alpha)^3 + (n\alpha)^2 + R(n\alpha) + 2R = 0\,, \tag{20}$$

$$b_i = i(i\alpha+1)a_i\,, \tag{21}$$

$$c_j = \sum_{(l_1,\ldots,l_j)\in S_j} \prod_{k=1}^{j} (\frac{a_k}{\sqrt{R}})^{l_k} \frac{(\sum_{k=1}^{j})!}{\prod_{k=1}^{j}(l_k!)}\,, \tag{22}$$

with

$$S_j = \{(l_1, ..., l_j) | l_1 \geq 0, ...l_j \geq 0, \sum_{i=1}^{j} i * l_i = j\} . \tag{23}$$

Here we assume $a_1 \neq 0$, otherwise, one can inductively obtain $a_k = 0$ for $k > 1$. In this case, we might be surprising to see that (20) is nothing but the characteristic polynomial of the Jacobian of the linearized of the chaotic system (3) evaluated at the equilibrium point O_1. Compare the (6) and (20), we can know that (20) has the unique root, so $\omega(n\alpha) \neq 0$, $k > 1$. Consequently, a_k $(k \geq 2)$ is completely determined by R, α, and a_1, which has the following form:

$$a_k = \varphi_k a_1^k, \quad k > 1 , \tag{24}$$

where $\varphi_k(k > 1)$ are some known functions depending on α, R.

Note that (11) has symmetry: if $x(t)$ is a solution, then $-x(-t)$ is also a solution. So,

$$\varphi(t) = \sqrt{R} - \sum_{k=1}^{\infty} a_k e^{-k\alpha t}, \quad t < 0 , \tag{25}$$

is also a solution of (11). From the continuity of the solution, We have

$$\sum_{k=1}^{\infty} = \sqrt{R} \tag{26}$$

which will determine the value of a_1. Thus, if R satisfy some conditions (for example $R = 1$), system (3) has a heteroclinic orbit, which connects the equilibria O_1 and O_2, and takes the following form:

$$\varphi(t) = \begin{cases} -\sqrt{R} + \sum_{k=1}^{\infty} a_k e^{k\alpha t}, & t > 0; \\ 0, & t = 0; \\ \sqrt{R} - \sum_{k=1}^{\infty} a_k e^{-k\alpha t}, & t < 0. \end{cases} \tag{27}$$

A naturally arising question is whether or not a_1 exist. Fortunately, numerical simulation shows that such a_1 indeed exist. For this reason, one may assume that they exist and then goes ahead to find them. Consider the following algebraic equation:

$$F(a_1) = \varphi_n a_1^n + \varphi_{n-1} a_1^{n-1} + ... + \varphi_2 a_1^2 + a_1 - \sqrt{R} = 0 , \tag{28}$$

for $n > 1$. Obviously, $F(0) = -\sqrt{R} < 0$. In addition, one can prove that $\omega(n\alpha) > 0$ and $\varphi_n > 0$ when n is sufficiently large. Therefore, $F(a_1) > 0$ for a_1 and n sufficiently large. These imply that equation (27) has at least one real root with respect to a_1.

4. Uniform Convergence

In the next, the uniform convergence of the series expansion (14) of the heteroclinic orbits is proven. For simplicity, only the case where system (3) has the special parameter set that generate the existence of a one-scroll attractor is discussed. For some other parameter sets, if the heteroclinic orbit exists, the proof is similar.

When $R = 1$, the value of a_1 and α can be determined by (17) $\rightarrow$ (22) and (25). So, a_k $k \geq 2$ can also be determined, with $\sum_{k=1}^{\infty} = 1$. Thus, a_k is bounded; that is, there exists a $C > 0$ such that

$$|a_k| \leq C, \quad k = 1, 2, \dots, \tag{29}$$

consequently,

$$\sum_{k=1}^{\infty} |a_k e^{k\alpha t}| \leq C \sum_{k=1}^{\infty} e^{k\alpha t} \tag{30}$$

is convergent on $(0, +\infty)$. So, $-\sqrt{R} + \sum_{k=1}^{\infty} a_k e^{k\alpha t}$ is also convergent on $(0, +\infty)$. Similarly, it can be proved that $\sqrt{R} - \sum_{k=1}^{\infty} a_k e^{-k\alpha t}$ is also convergent on $(-\infty, 0)$. Hence, there exists an orbit $\gamma(t)$ of system (3) such that $\lim_{t\rightarrow+\infty} \gamma(t) = O_2$ and $\lim_{t\rightarrow-\infty} \gamma(t) = O_1$.

5. Conclusion

By applying the undetermined coefficient method, the orbit in the considered chaotic system, heteroclinic orbit, have been identified and reported in this paper, with explicit and convergent algebraic expressions derived. It has been shown that with the typical parameter value of $R = 1$, the chaotic system has one heteroclinic orbit of Shil'nikov type, implying by the Shil'nikov criterion that that the chaotic system has Smale horseshoes and has horseshoe chaos.

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References

[1] L.P. Cělikovsky, A contribution of the problem of the structure of an extended neighborhood of rough equlibrium state of saddle-focus type, *Math. U.S.S.R.-Sbornik*, **10**, 91-102, Translated by F.A. Cezus.

[2] S. Cělikovsky, G. Chen, On a generalized lorenz canonial form of chaotic systems, *Int. J. Bifurcation and Chaos*, **12** (2002), 1789-1812.

[3] J. Palis, F. Takens, Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcation, *Cambridge University Press*, **35** (1993).

[4] V.D. Schrier, Maas, The diffusionless Lorenz equations Cělikovsky bifurcations and reduction to an explicit map, *Phys. D*, **1**, No. 2 (2000), 19-36.

[5] F.Y. Sun, Global chaos synchronization between two new different chaotic systems via active control, *Chin. Phys. Lett.*, **23** (2006), 32-34.

[6] F.Y. Sun, Y. Zhao, T.S. Zhou, Identify fully-uncertain parameters and design controllers based on synchronization, *Chaos, Solitons and Fractals*, To Appear.

[7] S. Wiggins, *Global Bifurcation and Chaos*, Springer-Velag, New York (1998).

[8] T.S. Zhou, G. Chen, S. Cělikovsky Šilnikov, Chaos in the generalized Lorenz canonical form of dynamical systems, *Nonlinear Dynamics*, **39** (2005), 319-34.

[9] T.S. Zhou, G. Chen, Y. Tang, The complex dynamical behaviors of the chaotic Chen's system, *Int. J. Bifurcation and Chaos*, **13** (2003), 2561-2574.