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STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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Abstract: In this paper, the authors investigate the generalized Hyers-Ulam-Rassias stability and Ulam-Gavruta-Rassias stability of a new quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z) = f(x+y)+f(x+z)+f(y-z)+f(x)+f(y)+f(z).$$

The Hyers-Ulam stability of this equation on bounded domain is also discussed.

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1. Introduction

The stability problem for functional equations was originally stated by S.M. Ulam [25] in 1940. In 1941, this problem was solved by D.H. Hyers [8] in the case of Banach spaces. There he introduced the Hyers-Ulam stability. In 1978, Th.M. Rassias [22] extended the Hyers-Ulam stability by considering variables. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. Hyers-Ulam stability has been generalized to the function case by D. Gavruta [7]. Since then the stability problems for various functional equations have been extensively investigated by a number of mathematicians [3], [4], [12], [13], [23]. For more detailed definitions for the terminologies, one

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can refer to [1], [16].

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence the above functional equation is called the quadratic functional equation or the Euler-Lagrange functional equation and every solution of the quadratic equation (1.1) is called a quadratic function. The functional equation (1.1) is a familiar equation and this equation was dealt by many authors F. Skof [24], P.W. Cholewa [5], S. Czerwik [6] and J.M. Rassias [18].

S.M. Jung and P.K. Sahoo [15] investigated the Hyers-Ulam stability of the quadratic functional equation of Pexider type

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y).$$

The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z)$$

was discussed by J.H. Bae and K.W. Jun [2]. J.M. Rassias [21] established Hyers-Ulam-Rassias stability of a quadratic functional equation on several variables.

In this paper, the authors introduced the following functional equation

$$\begin{aligned} f(x+y+z) &+ f(x+y-z) + f(x-y+z) \\ &= f(x+y) + f(x+z) + f(y-z) + f(x) + f(y) + f(z)\,, \end{aligned} \tag{1.2}$$

which is some what different from the above mentioned forms and discuss its various stability problems. The general solution and the generalized Hyers-Ulam-Rassias stability for quadratic functional equation

$$f(2x+y) + f(2x-y) = 6f(x) + f(x+y) + f(x-y) \tag{1.3}$$

was obtained by I.S. Chang and H.M. Kim [3]

In Section 2, we establish the general solution for equation (1.2) and (1.3), which are equivalent to equation (1.1). In Section 3, we investigate the Hyers-Ulam-Rassias stability of equation (1.2). In Section 4, we discuss the Ulam-Gavruta-Rassias stability of (1.2). The Hyers-Ulam stability of equation (1.2) on bounded domain is discussed in Section 5.

2. Solution of Equation (1.2)

We denote E_1 and E_2 be real vector spaces and $\mathbb{N}$ be the set of all natural numbers.

Theorem 2.1. *A function $f : E_1 \to E_2$ satisfies the functional equation (1.1) if and only if $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if $f : E_1 \to E_2$ satisfies the functional equation (1.3).*

Proof. Let $f : E_1 \to E_2$ satisfies (1.1). Letting $x = y = 0$ in (1.1), we get $f(0) = 0$. Replacing $x = 0$ in (1.1), we obtain $f(y) = f(-y)$. Therefore f is even. Replacing $y = x$ and $y = 2x$ in (1.1), we get $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$. In general for any positive integer n, $f(nx) = n^2 f(x)$. Replacing (x, y) by $(x + y, z)$ in (1.1), we obtain

$$f(x + y + z) + f(x + y - z) = 2f(x + y) + 2f(z). \tag{2.1}$$

Again, replacing x by $x + z$ in (1.1), we obtain

$$f(x + y + z) + f(x - y + z) = 2f(x + z) + 2f(y). \tag{2.2}$$

Further replacing y by $y - z$ in (1.1), we get

$$f(x + y - z) + f(x - y + z) = 2f(x) + 2f(y - z). \tag{2.3}$$

Adding (2.1),(2.2) and (2.3), we arrive to (1.2)

Let $f : E_1 \to E_2$ satisfies (1.2). Letting $x = y = z = 0$ in (1.2), we get $f(0) = 0$. Replacing $x = z = 0$ in (1.2), we obtain $f(y) = f(-y)$. Therefore f is even. Replacing $y = x, z = 0$ and $y = z = x$ in (1.2), we get $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$. In general for any positive integer n, $f(nx) = n^2 f(x)$. Letting $y = x$ in (1.2), we arrive to (1.3).

In order to prove the functional equation (1.3) implies (1.1), one can refer to the proof of Theorem 2.1 as it is given in [3]. Hence the functional equation (1.1), (1.2) and (1.3) are equivalent. The functional equation (1.1) has the quadratic equation and hence the functional equation (1.2) and (1.3) also have the same solution. □

3. Hyers-ULAM-Rassias Stability of (1.2)

Let E_1 be a normed space and let E_2 be a Banach space. Define a function $\varphi : E_1 \times E_1 \to \mathbb{R}^+$ such that

$$\sum_{k=0}^{\infty} \frac{\varphi(3^k x, 3^k y, 3^k z)}{9^k} \text{ converges and } \lim_{i\to\infty} \frac{\varphi(3^i x, 3^i y, 3^i z)}{9^i} = 0 \tag{3.1}$$

for all $x, y \in E_1$. Now suppose, $Df(x,y,z) = f(x+y+z) + f(x+y-z) + f(x-y+z) - f(x+y) - f(x+z) - f(y-z) - f(x) - f(y) - f(z)$. We state the following theorem.

Theorem 3.1. *Suppose $f : E_1 \to E_2$ satisfies*

$$\| Df(x,y,z) \| \leq \varphi(x,y,z) \tag{3.2}$$

for all $x, y, z \in E_1$, then there exits a unique quadratic function $F : E_1 \to E_2$ which satisfies the equation (1.2) and the inequality

$$\| f(x) - F(x) \| \leq \frac{1}{9} \sum_{k=0}^{\infty} \frac{\varphi(3^k x, 3^k y, 3^k z)}{9^k} \tag{3.3}$$

for all $x \in E_1$. The function $F(x)$ is given by

$$F(x) = \lim_{i\to\infty} \frac{f(3^i x)}{9^i} \tag{3.4}$$

for all $x \in E_1$.

Proof. Setting $y = z = x$ in (3.2), we obtain

$$\left\| \frac{f(3x)}{9} - f(x) \right\| \leq \frac{1}{9} \varphi(x,x,x) \tag{3.5}$$

for all $x \in E_1$. Replacing x by $3x$ in (3.5), dividing by 9, and summing the resultant inequality with (3.5), we obtain,

$$\left\| \frac{f(3^2 x)}{9^2} - f(x) \right\| \leq \frac{1}{9} \left[\varphi(x,x,x) + \frac{\varphi(3x,3x,3x)}{9} \right] \tag{3.6}$$

for all $x \in E_1$. Using induction on a positive integer i we obtain that

$$\left\| \frac{f(3^i x)}{9^i} - f(x) \right\| \leq \frac{1}{9} \sum_{k=0}^{i-1} \frac{\varphi(3^k x, 3^k x, 3^k x)}{9^k} \leq \frac{1}{9} \sum_{k=0}^{\infty} \frac{\varphi(3^k x, 3^k x, 3^k x)}{9^k} \tag{3.7}$$

for all $x \in E_1$. To prove that $\{f(3^i x)/9^i\}$ is Cauchy sequence. Replacing x by $3^j x$ and dividing by 9^j in (3.7) for any $i, j > 0$, we obtain

$$\left\| \frac{f(3^i 3^j x)}{9^{i+j}} - \frac{f(3^j x)}{9^j} \right\| \leq \frac{1}{9} \sum_{k=0}^{i-1} \frac{\varphi(3^{k+j}x, 3^{k+j}x, 3^{k+j}x)}{9^{k+j}}$$
$$\leq \frac{1}{9} \sum_{k=0}^{\infty} \frac{\varphi(3^{k+j}x, 3^{k+j}x, 3^{k+j}x)}{9^{k+j}}. \tag{3.8}$$

Since the right-hand side of the inequality (3.8) tends to 0 as j tends to infinity, the sequence $\{f(3^i x)/9^i\}$ is a Cauchy sequence. Define $F(x) = \lim_{i\to\infty} \frac{f(3^i x)}{9^i}$ for all $x \in E_1$. By letting $i \to \infty$ in (3.7), we arrive our result (3.3). Now we need to show that $F(x)$ satisfies (1.2). Replacing $(x,\ y,\ z)$ by $(3^i x, 3^i y, 3^j z)$ in (3.2) and dividing by 9^i we get,

$$\frac{1}{9^i} \left\| Df(3^i x, 3^i y, 3^j z) \right\| \leq \frac{\varphi(3^i x, 3^i y, 3^i z)}{9^i}.$$

Taking the limit as $i \to \infty$ in the last inequality and using (3.1) and (3.4), we see that $F(x)$ satisfies (1.2). To prove the uniqueness of $F(x)$. Let $F' : E_1 \to E_2$ be another quadratic function satisfies (1.2) and (3.4) then we have, $F(3^i x) = 9^i F(x)$ and $F'(3^i x) = 9^i F'(x)$ for all $x \in E_1$ and $i \in \mathbb{N}$. Now

$$\begin{aligned}
\left\| F(x) - F'(x) \right\| &= \frac{1}{9^i} \left\| F(3^i x) - F'(3^i x) \right\| \\
&\leq \frac{1}{9^i} \left(\left\| F(3^i x) - f(3^i x) \right\| + \left\| f(3^i x) - F'(3^i x) \right\| \right) \\
&\leq \frac{1}{9^i} \frac{2}{9} \sum_{k=0}^{i-1} \frac{\varphi(3^{k+i}x, 3^{k+i}x, 3^{k+i}x)}{9^k} \\
&\leq \frac{2}{9} \sum_{k=0}^{\infty} \frac{\varphi(3^{k+i}x, 3^{k+i}x, 3^{k+i}x)}{9^{k+i}} \quad \to 0 \quad \text{as} \quad i \to \infty,
\end{aligned}$$

for all $x \in E_1$. Therefore $F(x)$ is unique. This completes the proof of the theorem. □

The following corollaries are immediate consequences of main Theorem 3.1 concerning the stability of the equation (1.2).

Corollary 3.2. *Let E_1 be a real normed space and E_2 be a Banach space. Let ϵ, p be real numbers with $\epsilon \geq 0, p < 2$, or $p > 2$ and $f : E_1 \to E_2$ which satisfies*

$$\|Df(x, y, z)\| \leq \epsilon \left(\|x\|^p + \|y\|^p + \|z\|^p \right) \tag{3.9}$$

for all $x, y, z \in E_1$. Then there exists a unique quadratic function $F : E_1 \to E_2$ which satisfies (1.2) and the inequality

$$\|f(x) - F(x)\| \le \frac{3\epsilon}{\|9 - 3^p\|} \|x\|^p \tag{3.10}$$

for all $x \in E_1$. The function $F(x)$ is defined in (3.4). Moreover if f is measurable, $f(tx)$ is continuous for each fixed $t \in \mathbb{R}$ and $x \in E_1$, then $f(tx) = t^2 f(x)$.

Corollary 3.3. *Let E_1 be a real normed space and E_2 be a Banach space. Let ϵ be real number. If $f : E_1 \to E_2$ satisfies*

$$\|Df(x, y, z)\| \le \epsilon \tag{3.11}$$

for all $x, y, z \in E_1$, then there exists a unique quadratic function $F : E_1 \to E_2$ which satisfies (1.2) and the inequality

$$\|f(x) - F(x)\| \le \frac{3\epsilon}{8} \tag{3.12}$$

for all $x \in E_1$. The function $F(x)$ is defined in (3.4). Moreover if f is measurable, $f(tx)$ is continuous for each fixed $t \in \mathbb{R}$ and $x \in E_1$, then $f(tx) = t^2 f(x)$.

4. Ulam-Gavruta-Rassias Stability of Equation (1.2)

In 1982, J.M. Rassias [17] for the first time introduced a generalization of Hyers Stability Theorem which allows the Cauchy difference to be unbounded. We state that the result as follows.

Theorem 4.1. (see [17]) *Let $f : E \to E'$ ba a mapping from a normed vector space E in to a Banach space E' satisfies subject to the inequality*

$$\| f(x + y) - f(x) - f(y) \| \le \epsilon \|x\|^p \|y\|^p$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x, y \in E$ and $L : E \to E$ is the unique additive mapping which satisfies

$$\| f(x) - L(x) \| \le \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p}$$

for all $x \in E$. If $p > \frac{1}{2}$ then the inequality holds for all $x, y \in E$ and the limit

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x, y \in E$, and $A : E \to E'$ is the unique additive mapping which satisfies

$$\| f(x) - A(x) \| \leq \frac{\epsilon}{2^{2p} - 2} \|x\|^{2p}$$

for all $x \in E$. If in addition $f : E \to E'$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $f \in \mathbb{R}$ for each fixed $x \in E$, then L is $\mathbb{R}$-linear mapping.

Using Theorem 4.1 above, Ulam-Gavruta-Rassias stability for the functional equation (1.2) is as follows.

Theorem 4.2. *Let $f : E_1 \to E_2$ ba a mapping from a normed vector space E_1 in to a Banach space E_2 which satisfies subject to the inequality*

$$\| Df(x, y, z) \| \leq \epsilon \|x\|^p \|y\|^p \|z\|^p \tag{4.1}$$

for all $x, y, z \in E_1$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{2}{3}$. Let

$$Q(x) = \lim_{i\to\infty} \frac{f(3^i x)}{9^i}, \tag{4.2}$$

then a unique quadratic function $Q : E_1 \to E_2$ satisfy

$$\| f(x) - Q(x) \| \leq \frac{\epsilon}{3^2 - 3^{3p}} \|x\|^{3p} \tag{4.3}$$

for all $x \in E_1$.

Proof. Replacing $y = z = x$ in (4.1), we obtain

$$\left\| \frac{f(3x)}{9} - f(x) \right\| \leq \frac{\epsilon}{9} \|x\|^{3p} \tag{4.4}$$

for all $x \in E_1$. Replacing x by $3x$ dividing by 9 in (4.4) and summing the resultant inequality with (4.4), we arrive to

$$\left\| \frac{f(3^2 x)}{9^2} - f(x) \right\| \leq \frac{\epsilon}{9} \left[1 + \frac{3^{2p}}{9}\right] \|x\|^{3p} \tag{4.5}$$

for all $x \in E_1$. Using induction on a positive integer i we obtain that

$$\left\| \frac{f(3^i x)}{9^i} - f(x) \right\| \leq \frac{\epsilon}{9} \sum_{k=0}^{i-1} \frac{(3^{3p})^k}{9^k} \|x\|^{3p} \leq \frac{\epsilon}{9} \sum_{k=0}^{\infty} \frac{(3^{3p})^k}{9^k} \|x\|^{3p} \tag{4.6}$$

for all $x \in E_1$. To prove that $\{f(3^i x)/9^i\}$ is a Cauchy sequence, we replace x by $3^j x$ and divide by 9^j in (4.6), for $i, j > 0$ we obtain

$$\left\| \frac{f(3^i 3^j x)}{9^{i+j}} - \frac{f(3^j x)}{9^j} \right\| \leq \frac{1}{9^j} \left\| \frac{f(3^{i+j} x)}{9^i} - f(3^j x) \right\|$$
$$\leq \frac{1}{9^j} \frac{\epsilon}{9} \sum_{k=0}^{i-1} \frac{(3^{3p})^k}{9^i} \left\| 3^j x \right\|^{3p} \leq \frac{\epsilon}{9} \sum_{k=0}^{\infty} \frac{(3^{3p})^{k+j}}{9^{i+j}} \|x\|^{3p}. \quad (4.7)$$

The right-hand side of the inequality (4.7) tends to 0 as j tends to infinity. Hence the sequence $\{f(3^i x)/9^i\}$ is a Cauchy sequence. Define

$$Q(x) = \lim_{i \to \infty} \frac{f(3^i x)}{9^i} \quad \forall x \in E_1. \quad (4.8)$$

Now letting $i \to \infty$ in (4.6), we arrive to (4.3). Now to show that Q satisfies (1.2), replace $(x,\ y, z)$ by $(3^i x,\ 3^i y, 3^i z)$ and divide by 9^i in (4.1), we see that Q satisfies (1.2). In order to prove the uniqueness of Q, we assume $Q' : E_1 \to E_2$ be another quadratic function satisfying (1.2) and (4.2) then we have, $Q(3^i x) = 9^i Q(x)$ and $Q'(3^i x) = 9^i Q'(x)$ for all $x \in E_1$ and $i \in \mathbb{N}$. But

$$\left\| Q(x) - Q'(x) \right\| \leq \frac{1}{9^i} \left(\left\| Q(3^i x) - f(3^i x) \right\| + \left\| f(3^i x) - Q'(3^i x) \right\| \right)$$
$$\leq \frac{1}{9^i} \left[\frac{2\epsilon}{9 - 3^{3p}} \right] \left\| 3^i x \right\|^{3p} \quad \to 0 \quad \text{as} \quad i \to \infty. \quad (4.9)$$

for all $x \in E_1$. Therefore Q is unique. □

Theorem 4.3. *Let $f : E_1 \to E_2$ ba a mapping from a normed vector space E_1 in to a Banach space E_2 satisfies subject to the inequality*

$$\| Df(x, y, z) \| \leq \epsilon \|x\|^p \|y\|^p \|z\|^p \quad (4.10)$$

for all $x, y, z \in E_1$, where ϵ and p are constants with $\epsilon > 0$ and $p > \frac{2}{3}$. Let

$$Q(x) = \lim_{i \to \infty} 9^i f\left(\frac{x}{3^i}\right), \quad (4.11)$$

a unique quadratic function $Q : E_1 \to E_2$ satisfying

$$\| f(x) - Q(x) \| \leq \frac{\epsilon}{3^{3p} - 3^2} \|x\|^{3p} \quad (4.12)$$

exists for all $x \in E$.

Proof. Replacing x by $\frac{x}{3}$ in (4.4), we obtain

$$\left\|f(x) - 9f\left(\frac{x}{3}\right)\right\| \leq \frac{\epsilon}{3^{3p}} \|x\|^{3p} \tag{4.13}$$

for all $x \in E_1$. Again Replacing x by $3x$ and multiply by 9 in (4.12) and summing the resultant inequality with (4.12), we arrive

$$\left\|f(x) - 9^2 f\left(\frac{x}{3^2}\right)\right\| \leq \frac{\epsilon}{3^{3p}} \left[1 + \frac{9^k}{3^{3p}}\right] \|x\|^{3p} \tag{4.14}$$

for all $x \in E_1$. Using induction on a positive integer i we obtain that

$$\left\|f(x) - 9^i f\left(\frac{x}{3^i}\right)\right\| \leq \frac{\epsilon}{3^{3p}} \sum_{k=0}^{i-1} \frac{9^k}{(3^{3p})^k} \|x\|^{3p} \leq \frac{\epsilon}{3^{3p}} \sum_{k=0}^{\infty} \frac{9^k}{(3^{3p})^k} \|x\|^{3p} \tag{4.15}$$

for all $x \in E_1$. In order to prove $\left\{9^i f\left(\frac{x}{3^i}\right)\right\}$ is Cauchy sequence, replace x by $\frac{x}{3^j}$ and multiply by 9^j in (4.15), for $i, j > 0$, we have

$$\begin{aligned} \left\|9^j f\left(\frac{x}{3^j}\right) - 9^{i+j} f\left(\frac{x}{3^{i+j}}\right)\right\| &\leq 9^j \left\|f\left(\frac{x}{3^j}\right) - 9^i f\left(\frac{x}{3^{i+j}}\right)\right\| \\ &\leq 9^j \frac{\epsilon}{3^{3p}} \sum_{k=0}^{i-1} \frac{9^k}{(3^{3p})^k} \left\|\frac{x}{3^j}\right\|^{3p} \leq \frac{\epsilon}{3^{3p}} \sum_{k=0}^{\infty} \frac{9^{k+j}}{(3^{3p})^{k+j}} \|x\|^{3p}. \end{aligned} \tag{4.16}$$

Since the right-hand side of the inequality (4.16) tends to 0 as j tends to infinity. The sequence $\left\{9^i f\left(\frac{x}{3^i}\right)\right\}$ is a Cauchy sequence and define

$$Q(x) = \lim_{i\to\infty} 9^i f\left(\frac{x}{3^i}\right) \quad \forall x \in E_1. \tag{4.17}$$

To prove Q is unique and satisfies the equation (1.2), we follow the same argument used in Theorem 4.2 and the result can be easily obtained. □

5. Hyers-Ulam Stability of (1.2) on Bounded Domain

In this section, we use the following notation. Let n be a positive integer, $r > 0$ be a constant, E be a Banach space and $I^n = [-r, r]^n$.

The stability of the quadratic functional equation (1.1) on a bounded real interval was presented by S.M. Jung [14] in the following theorem.

Theorem 5.1. (see [14]) *Suppose $f : I^n \to E$ satisfies*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \leq \epsilon \tag{5.1}$$

for some $\epsilon > 0$ and for all $x, y \in I^n$ with $x+y, x-y \in I^n$, then there exits a quadratic function $Q : \mathbb{R}^n \to E$ such that

$$\| f(x) - Q(x) \| \leq (2912n^2 + 1872n + 334)\epsilon \tag{5.2}$$

for all $x \in I^n$.

Using the above theorem and the Hyers-Ulam stability of the quadratic functional equation (1.2) on a bounded domain, we obtained the following theorem.

Theorem 5.2. *Suppose $f : I^n \to E$ satisfies the inequality (3.11) for all $x, y, z \in I^n$ with $x+y+z$, $x+y-z$, $x-y+z$, $x+y$, $x+z$, $y-z \in I^n$, then there exits a quadratic function $Q : \mathbb{R}^n \to E$ such that*

$$\| f(x) - Q(x) \| \leq \frac{4}{3}(2912n^2 + 1872n + 334)\epsilon \tag{5.3}$$

for all $x \in I^n$.

Proof. Setting $x = y = z = 0$ in (3.11), we get

$$\| f(0) \| \leq \frac{\epsilon}{3}. \tag{5.4}$$

Letting $z = 0$ in (3.11), we obtain

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) - f(0) \| \leq \epsilon. \tag{5.5}$$

Now, using (5.4) and (5.5), we arrive to

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\qquad \leq \|f(x+y) + f(x-y) - 2f(x) - 2f(y) - f(0)\| + \|f(0)\| \leq \frac{4}{3}\epsilon \end{aligned}$$

for $x, y \in I^n$ and $x+y, x-y \in I^n$. Using Theorem 5.1, there exists a quadratic function $Q : \mathbb{R}^n \to E$ satisfying

$$\| f(x) - Q(x) \| \leq \frac{4}{3}(2912n^2 + 1872n + 334)\epsilon$$

for all $x \in I^n$. □

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