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THE EVALUATION MAP FOR TORSION FREE SHEAVES ON A PROJECTIVE CURVE WITH ARITHMETIC GENUS ONE

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let C be an integral projective curve such that $p_a(C) = 1$, F a polystable vector bundle on C and F a torsion free sheaf on C. Here we study the general map $E \to F$ (injectivity or generic surjectivity or surjectivity) and the corresponding properties for the evaluation map $F \otimes H^0(C, \mathrm{Hom}(F, E)) \to E$.

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Let C be an integral complex projective curve such that $p_a(C) \geq 1$. For all torsion free sheaves A, B on C, let $e_{A,B} : H^0(C, \mathrm{Hom}(A, B)) \otimes A \to B$ denote the evaluation map. Here we consider the map $e_{A,B}$ when $p_a(C) = 1$. Hence either C is a smooth elliptic curve or $\mathbf{P}^1$ is the normalization map and C has a unique singular point which is either an ordinary node or an ordinary cusp. As in the smooth case for all coprime integers m, d and all $L \in \mathrm{Pic}^d(C)$ there is a unique rank m stable vector bundle E on C such that $\det(E) \cong L$ (see [5], Theorem 5.1). Hence for all integers r, d such that $r \geq 1$ there is a rank r polystable vector bundle E on C such that the indecomposable factors of E are pairwise non-isomorphic. Notice that $\mathrm{Hom}(A, B) = 0$ if A and B are polystable

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and either $\mu(A) > \mu(B)$ of $\mu(A) = \mu(B)$ and no indecomposable factor of A is isomorphic to an indecomposable factor of B.

First we will study $e_{A,B}$ when A, B are locally free and polystable and prove the following result.

Theorem 1. *Fix integers a, k, d, r such that $k > 0$, $r > 0$ and $dk > ra$. Let C be an integral projective curve such that $p_a(C) = 1$. Let E, F polystable vector bundles on C such that $\mathrm{rank}(F) = k$, $\deg(F) = a$, $\mathrm{rank}(E) = r$ and $\deg(E) = d$. We have $h^1(C, \mathrm{Hom}(F, E)) = 0$ and $h^0(C, \mathrm{Hom}(F, E)) = kd - ra$. Set $G := \mathrm{Im}(e_{F,E})$.*

(a) If $\mathrm{rank}(F) \geq \mathrm{rank}(E)$, then $\mathrm{rank}(G) = \mathrm{rank}(E)$.

(b) If $\mathrm{rank}(F) \leq \mathrm{rank}(E)$, then f is injective.

(c) If $\mathrm{rank}(F) > \mathrm{rank}(E)$, then f is surjective.

(d) If $\mathrm{rank}(F) < \mathrm{rank}(F)$, then f in injective and E/G is locally free.

Then, again when $p_a(C) = 1$, we will consider the case in which F, E may be not locally free. Let C be an integral singular projective curve such that $p_a(C) = 1$. Hence C has a unique singular point and this point is either an ordinary node or an ordinary cusp. For the classification of torsion free modules on the local ring of an ordinary node or an ordinary cusp, see [3] or [2], p. 24, or [4], Proposition 2 at p. 164, or [5], Lemmas 1.1 and 1.2. Let F be a rank r torsion free sheaf on C. Let $\tau(F)$ be the minimal integer t such that there is a rank r locally free subsheaf E of E such that $\mathrm{length}(F/E) = t$. We have $0 \leq \tau(F) \leq r$, and F is locally free if and only if $\tau(F) = 0$. Notice that the sheaf F/G is supported by P for any locally free rank r subsheaf G of F such that $\mathrm{length}(F/G) = \tau(F)$. $\tau(F)$ is the minimal integer t such that F is a subsheaf of a rank r locally free sheaf M such that $\mathrm{length}(M/F) = t$. For all pairs of integers (r, t) such that $r > 0$ and $0 \leq t \leq r$, there is a rank r torsion free sheaf F such that $\tau(F) = t$. For any torsion free sheaf F on C the fiber $F|\{P\}$ is a vector space of dimension $\mathrm{rank}(F) + \tau(F)$. Hence if G is locally free and there is a surjection $G \to F$, then $\mathrm{rank}(G) \geq \mathrm{rank}(F) + \tau(F)$. These observations shows that Theorem 1 does not hold (without some weakening of its stament) when either E or F is not locally free. To extend a weakened version of Theorem 1 to this more general case we first need to extend [1] to the case in which either E or F is not locally free. Let F be a rank r torsion free sheaf on C. We will say that F is *inner polystable* (resp. *outer polystable*) if there is a rank r polystable vector bundle E and an inclusion $j : E \to F$ (resp. $j : F \to E$) such that $\mathrm{length}(F/j(E)) = \tau(F)$ (resp. $\mathrm{length}(E/j(F)) = \tau(F)$).

Theorem 2. *Fix integers r, k, a, d, τ, η such that $r > 0$, $k > 0$, $0 \leq \tau \leq r$, $0 \leq \eta k$ and $(a + \eta)/k < (d - \tau)/r$. Let C be an integral singular curve with*

$p_a(C) = 1$ and let P its singular point. Let F be an outer polystable torsion free sheaf on C such that $\mathrm{rank}(F) = k$, $\deg(F) = k$ *and* $\tau(F) = \eta$. *Let E be an inner polystable torsion free sheaf on C such that* $\mathrm{rank}(E) = r$, $\deg(E) = r$ *and $\tau(E) = \tau$. Fix a general $f \in H^0(C, \mathrm{Hom}(F, E))$.*

(a) If $k \le r$, then f is injective.

(b) If $k > r$, then $\mathrm{Coker}(f)$ *is either $\{0\}$ or its supported by P.*

(c) Assume F locally free and that $k > r + \tau$ and $a/k < \lfloor (d - \tau)/(r + \tau) \rfloor$. Then there is a sheaf A such that $\deg(A) = d$, $\mathrm{rank}(A) = r$, $\tau(A) = \tau$ *and for which a general $h \in H^0(C, \mathrm{Hom}(F, A))$ is surjective.*

Theorem 3. *Fix integers r, k, a, d, τ such that $r > 0$, $k > 0$, $0 \le \tau \le r$, and $a/k < d/r$. Let C be an integral singular curve with $p_a(C) = 1$ and let P its singular point. Let F be a polystable vector bundle on C such that* $\mathrm{rank}(F) = k$, $\deg(F) = k$ *and $\tau(F) = \eta$. Let E be an inner polystable torsion free sheaf on C such that* $\mathrm{rank}(E) = r$, $\deg(E) = r$ *and $\tau(E) = \tau$. Assume that E is semistable, inner polystable and outer polystable. We have $h^1(C, \mathrm{Hom}(F, E)) = 0$ and $h^0(C, \mathrm{Hom}(F, E)) = kd - ra$.*

(a) If $k(k(d + \tau) - ra) \le r$, then $e_{F,E}$ is injective.

(b) If $k(k(d - \tau) - ra) > r$, then either $e_{F,E}$ is surjective or $\mathrm{Coker}(e_{F,E})$ *is supported by P.*

(c) Assume F locally free, $a/k < \lfloor (d-\tau)/(r+\tau) \rfloor$, and $k(k(d-\tau)-(r+\tau)a) > r + \tau$. Then there is a sheaf A such that $\deg(A) = d$, $\mathrm{rank}(A) = r$, $\tau(A) = \tau$ *and for which the evaluation map $e_{F,A}$ is surjective.*

Remark 1. Let C be an integral projective curve such that $p_a(C) = 1$. Let A be a vector bundle on C and G, F torsion free sheaves on C. Both $\mathrm{Hom}(G, A)$ and $\mathrm{Hom}(A, F)$ are torsion free. Obviously, $\deg(\mathrm{Hom}(A, F)) = \mathrm{rank}(A) \cdot \mathrm{rank}(F)(\mu(F) - \mu(A))$. Since C is Gorenstein, every torsion free sheaf on C is reflexive. Hence we also get $\deg(\mathrm{Hom}(G, A)) = \mathrm{rank}(A) \cdot \mathrm{rank}(G)(\mu(A) - \mu(G))$.

Proof of Theorem 1. Duality fo Cohen-Macaulay schemes gives $h^1(C, \mathrm{Hom}(F, E)) = h^0(C, \mathrm{Hom}(E, F))$. Since E, F are polystable and $\mu(E) > \mu(F)$, we have $h^0(C, \mathrm{Hom}(E, F)) = 0$. Hence $h^0(C, \mathrm{Hom}(F, E)) = 0$. Thus $h^0(C, \mathrm{Hom}(F, E)) = kd - ra$ (Riemann-Roch). Set $G := \mathrm{Im}(e_{F,E}$. Notice that $\mathrm{rank}(G) \le \min\{r, k(kd - ra)\}$. Hence to prove both parts (a) and (b) we may assume $G \ne E$. Set $\rho := \mathrm{rank}(G)$ and $y := \deg(G)$. We have $h^1(C, \mathrm{Hom}(F, G)) = 0$ (and hence $h^0(C, \mathrm{Hom}(F, G)) = a\rho - yk$) unless $\mu_-(G) = \mu(F)$ and the minimal slope subquotient of G has as a factor a direct factor of F. The latter case implies that G has a direct factor D isomorphic to a direct factor of F and in the latter case we have $h^0(C, \mathrm{Hom}(F, G)) = a\rho - yk + h^0(C, \mathrm{Hom}(D, D))$.

Write $F \otimes H^0((C, \mathrm{Hom}(F, G)) = D \oplus F'$ and $G = D \oplus G'$ in such a way that in this decomposition $e_{F,E}$ has diagonal form. In order to prove part (a) it is sufficient to show that $G' = \{0\}$. In order to prove part (b) it is sufficient to obtain a contradiction. Assume $G' \neq \{0\}$. F' is polystable. By [5], Theorem 5.1, there is a polystable vector bundle A on C such that $\deg(A) = \deg(G')$ and $\mathrm{rank}(A) = \mathrm{rank}(G')$. By [1] there is a surjections $F' \to A$ and an inclusion $A \to E$. Composing with the identity map $D \to D$ we get a morphism $F \otimes H^0((C, \mathrm{Hom}(F, G)) \to E$ whose image is isomorphic to $D \oplus A$. Since every map $F \to E$ is contained in G, we get $G' \cong A$. Since there are many pairwise non-isomorphic polystable vector bundles with the same properties as A (e.g. with different determinant), we obtained a contradiction. □

Lemma 1. *Let C be a singular integral projective curve such that $p_a(C) = 1$. Fix integers $r \geq \tau > 0$ and u. Set $v := \lfloor u - \tau/r \lfloor$. Let G be a polystable vector bundle on X such that $\mathrm{rank}(G) = r + \tau$ and $\deg(G) = (r + \tau)v$. Then there exist a rank r torsion free sheaf A on C such that $\deg(A) = u$ and $\tau(A) = \tau$ and a surjection $h : E \to A$.*

Proof. Set $\{P\} := \mathrm{Sing}(C)$. Fix $L_i \in \mathrm{Pic}^v(C)$, $1 \leq i \leq \tau$, a take a general $h : \oplus_{i=1}^{\tau} \mathcal{I}_P \otimes M_i \to G$. It it easy to check that $A := \mathrm{Coker}(h)$ is locally free and with the prescribed invariants. □

Proof of Theorem 2. By assumption there are polystable vector bundles A, B respectively of rank r and k and inclusions $i : F \to A$, $j : B \to E$ such that $\mathrm{length}(A/i(F)) = \eta$ and $\mathrm{length}(E/j(B)) = \tau$. Fix a general $h \in H^0(C, \mathrm{Hom}(A, B))$. Since $\mu(A) < \mu(B)$ [1] gives that h is injective if $k \leq r$, while h is surjective if $k > r$. Taking $j \circ h \circ i$ and using the openness of injectivity we get (a). We also easily get (b). Now we will prove part (c). Take G as in Lemma 1. By [1] there is a surjection $F \to G$. Then use a surjection $G \to A$ as in Lemma 1 and the openness of surjectivity. □

Proof of Theorem 3. Since C is Gorenstein, every coherent torsion free sheaf on C is reflexive. Since F is locally free and E is reflexive, duality for the Cohen-Macaulay scheme C gives $h^1(C, \mathrm{Hom}(F, E)) = h^0(C, \mathrm{Hom}(E, F))$. Since E, F are semistable and $\mu(E) > \mu(F)$, we have $h^0(C, \mathrm{Hom}(E, F)) = 0$. Hence $h^0(C, \mathrm{Hom}(F, E)) = 0$. Thus $h^0(C, \mathrm{Hom}(F, E)) = kd - ra$ (Riemann-Roch and the local freeness of F). Take rank r polystable vector bundles on C equipped by inclusions $i : G \to E$ and $j : E \to M$ such that $\mathrm{length}(E/j(G)) = \mathrm{length}(M/j(E)) = \tau$. Under the assumptions of part (a) (resp. (b)) the map $e_{F,M}$ is injective (resp. the map $e_{F,G}$ is surjective) by Theorem 1. We immediately get parts (a) and (b). For part (c) use Lemma 1 and Theorem 1 applied to the pair (F, G). □

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