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BRILL-NOETHER THEORY FOR STABLE VECTOR BUNDLES WITH FIXED DETERMINANT ON A SMOOTH CURVE

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Abstract: Here we collect a few existence results on the Brill-Noether theory for stable vector bundles with fixed determinant on a smooth curve.

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Let X be a smooth and connected projective curve of genus $g \geq 3$. For all integers n, d and any $L \in \mathrm{Pic}^d(X)$ let $U(n,d)$ (resp. $U(n,L)$) the variety of all stable vector bundles with rank n and degree d (resp. rank n and determinant isomorphic to L). For all integers n, d, k and all $L \in \mathrm{Pic}^d(X)$ set $W^{k-1}_{n,d} := \{E \in U(n,d) : h^0(X,E) \geq k\}$ and $W^{k-1}_{n,L} := \{E \in U(n,L) : h^0(X,E) \geq k\}$. Set $\rho(n,d,g,k) := n^2(g-1)+1k(k-d+n(g-1))$ and $\rho'(n,d,g,k) := \rho(n,d,g,k)-g$. The integer $\rho(n,d,g,k)$ (resp. $\rho'(n,d,g,k)$) is the "expected" dimension of every irreducible component of $W^{k-1}_{n,d}$ (resp. $W^{k-1}_{n,L}$) and a lower bound for any such irreducible component.

Here we look at the existence results proved in [1] and [2] and adapt them to the set-up in which the determinant is fixed

Lemma 1. *Let T be an integral variety parametrizing finite-to-one unstable vector bundles with rank n and fixed determinant on X. Then* $\dim(T) \leq$

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$n^2(g-1)-g$.

Proof. The result is obvious if $n=1$, because every line bundle is stable and hence $T=\emptyset$. Assume $n\geq 2$ and that the result is true for all ranks $n'\leq n-1$. Fix a general $t\in T$ and let E be the associated vector bundle. Set $L:=\det(E)$. Since E is not stable, there is a filtration $0:=E_0\subset E_1\subset\cdots\subset E_r:=E$ by subbundles such that $r\geq 2$, each E_i/E_{i-1} is stable and $\mu(E_i/E_{i-1})\leq\mu(E_{i-1}/E_{i-2})$ for all $i\geq 2$. Set $m:=\mathrm{rank}(E_{r-1})$. In an open subset of T the integer r and the ranks and degree of all vector bundles E_i/E_{i-1}, $1\leq i\leq r$, are constant. Varying t T, E_{r-1} varies in a family of dimension at most $m^2(g-1)+1$ (and even at most $m^2(g-1)$ by [1], Lemma 4.1, if $r\geq 3$). For a fixed E_{r-1} we need to compute the dimension of all the possible vector bundles E/E_{r-1}, E varying in T, and, for fixed E_{r-1} and E/E_{r-1}, all bundles which are extension of E/E_{r-1} by E_{r-1}. The latter computation is done at the end of the proof of [1], Lemma 4.1. The former computation follows from the fact that $\dim(U((n-m),M))=(n-m)^2(g-1)+1-g$, where $M:=L\otimes\det(E_{r-1})^*$. □

Theorem 1. *Fix integers n,d,k such that $n>d>0$ and $n\leq d+(n-k)g$, and $L\in Pic^d(X)$. Then $W^{k-1}_{n,L}$ is non-empty, irreducible and with the expected dimension $\rho'(n,d,g,k)$. Every $E\in W^{k-1}_{n,L}$ fits in an exact sequence*

$$0\to\mathcal{O}_X^{\oplus k}\to E\to F\to 0 \tag{1}$$

with $\det(F)\cong L$ and if E is general in $W^{k-1}_{n,L}$, then F is a general element of $U(n-k,L)$.

Proof. To get the non-emptyness we follow step by step the proofs in [1], Section 5 and Section 6. In our set-up the vector bundle H in [1], diagram (5), is forced to have the fixed determinant $L\otimes\det(G_1)^*$. Each time [1] quotes [1], Lemma 4.1, we must use either Lemma 1 or [1], Lemma 4.1, or both. Since $\det(H)$ is fixed in [1], equation (7), we must use the integer $\dim(\mathcal{M}(i,d))-g$ instead of integer $\dim(\mathcal{M}(i,d))$. The proof of the irreducibility of $W^{k-1}_{n,L}$ is the same as the irreducibility of $W^{k-1}_{n,d}$ proved in the first part of [1], Section 4, just quoting Lemma 1 instead of [1], Lemma 4.1. The last assertion follows from [1], Remark 3.2. □

Remark 1. Fix integers $l\geq 1$ and $n\geq gl+1$ and $L\in\mathrm{Pic}^{n+gl}(X)$. The proof of [2], Theorem B-1 of Section 2, gives $W^{n+l-1}_{n,L}\cong U(n,L)$.

Remark 2. Fix integers n, d such that $0 < n < d < 2n$. Let E be any stable vector bundles on X such that $\mathrm{rank}(E) = n$, $\deg(E) = d$ and $h^0(X, E) \geq n$. By [2], bottom of p. 28, E fits in an exact sequence

$$0 \to \mathcal{O}_X^{\oplus n} \to E \to \Theta \to 0 \qquad (2)$$

in which Θ is a degree d skyscraper sheaf. Hence $\det(E) \cong \mathcal{O}_X(D)$ with D an effective degree d divisor. The converse is true for general D. Thus we see that $W^n_{n,L} \neq \emptyset$ if and only if L belongs to an integral subvariety of $\mathrm{Pic}^d(X)$ with dimension $\min\{d, g\}$.

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