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OPTIMAL BIRTH CONTROL FOR AN AGE-DEPENDENT N-DIMENSIONAL FOOD CHAIN MODEL II. FREE HORIZON PROBLEMS

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Abstract: We study optimal birth policies for an age-dependent n-dimensional food chain model, which is controlled by fertility. New results on problems with free final time and integral phase constraints are presented, the approximate controllability of system is discussed.

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1. Introduction

We continue the study initiated in [2]. This paper presents further new results on several optimal birth control problems. We first investigate the problem with fixed final state and free final time, of which the time-optimal problem is a special case. Then we examine problems with integral phase constraints. Finally we study the approximate controllability of controlled system. It is supposed that the reader is familiar with the terminology and notation in [2].

2. Problems with Fixed Terminal and Free Horizon

Consider the optimal control problem

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Minimize

$$J(p,\beta)=\int_0^{t_1}\int_0^{a_+} L(p_1(a,t),\cdots,p_n(a,t),\beta_1(t),\cdots,\beta_n(t))\,\mathrm{d}a\mathrm{d}t, \quad (1)$$

where $(p(a,t),\beta(t))$, $p(a,t) = (p_1(a,t),\cdots,p_n(a,t))$, $\beta(t) = (\beta_1(t),\cdots,\beta_n(t))$, is subject to

$$\begin{cases}\frac{\partial p_1}{\partial t}+\frac{\partial p_1}{\partial a}=-\mu_1(a,t)p_1-\lambda_1(a,t)P_2(t)p_1,\\ \frac{\partial p_i}{\partial t}+\frac{\partial p_i}{\partial a}=-\mu_i(a,t)p_i+\lambda_{2i-2}(a,t)P_{i-1}(t)p_i-\lambda_{2i-1}(a,t)P_{i+1}(t)p_i,\\ \qquad i=2,3,\cdots,n-1,\\ \frac{\partial p_n}{\partial t}+\frac{\partial p_n}{\partial a}=-\mu_n(a,t)p_n+\lambda_{2n-2}(a,t)P_{n-1}(t)p_n,\\ p_i(0,t)=\beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\,\mathrm{d}a,\ i=1,2,\cdots,n,\\ p_i(a,0)=p_{i0}(a),\ i=1,2,\cdots,n,\\ P_i(t)=\int_0^{a_+}p_i(a,t)\,\mathrm{d}a,\ i=1,2,\cdots,n,\quad (a,t)\in Q,\end{cases} \quad (2)$$

and

$$p_i(a,t_1)=p_i^0(a), i=1,2,\cdots,n. \quad (3)$$

Here $t_1>0$ is not fixed, p_i^0 is prescribed nonnegative function.

For each $t_1>0$, choose a measurable function $v\geq 0$, define the time transformation

$$t(\tau)=\int_0^{\tau} v(s)\,\mathrm{d}s,\ t(1)=t_1 \quad (4)$$

and

$$p_i(a,\tau)=p_i(a,t(\tau)),\ \beta_i(\tau)=\begin{cases}\beta_i(t(\tau)), & \tau\in S_1,\\ \text{arbitrary}, & \tau\in S_2,\end{cases}\quad i=1,2,\cdots,n, \quad (5)$$

where

$$S_1=\{\tau\in[0,1]:\ t(\tau)>0\},\quad S_2=\{\tau\in[0,1]:\ t(\tau)=0\}. \quad (6)$$

If we define similarly $\mu_i(a,\tau),\lambda_i(a,\tau),m_i(a,\tau)$, then $(p(a,\tau),\beta(\tau))$ satisfies

$$\begin{cases}\frac{\partial p_1}{\partial \tau}+v(\tau)\frac{\partial p_1}{\partial a}=-[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2(\tau)]p_1v(\tau),\\ \frac{\partial p_k}{\partial \tau}+v(\tau)\frac{\partial p_k}{\partial a}=-[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}(\tau)\\ \qquad -\lambda_{2k-2}(a,\tau)P_{k-1}(\tau)]p_kv(\tau),\\ \qquad k=2,3,\cdots,n-1,\\ \frac{\partial p_n}{\partial \tau}+v(\tau)\frac{\partial p_n}{\partial a}=-[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}(\tau)]p_nv(\tau),\\ v(\tau)p_i(0,\tau)=v(\tau)\beta_i(\tau)\int_{a_1}^{a_2}m_i(a,\tau)p_i(a,\tau)\,\mathrm{d}a,\\ p_i(a,0)=p_{i0}(a),\\ P_i(\tau)=\int_0^{a_+}p_i(a,\tau)\,\mathrm{d}a,\ i=1,2,\cdots,n,\ (a,\tau)\in[0,a_+]\times[0,1].\end{cases} \quad (7)$$

and

$$p_i(a,1)=p_i^0(a),\quad i=1,2,\cdots,n. \quad (8)$$

Consequently, if $(p^*(a,t),\beta^*(t),t_1^*)$ is a solution of problem (1)-(3), and $v^*(\tau)$

is a measurable function corresponding to t_1^*, then $(p^*(a,\tau),\beta^*(\tau),v^*(\tau))$ must be a solution to the following problem:

$$\text{Minimize } J(p,\beta) = \int_0^1\int_0^{a_+} v(\tau)L(p_1(a,\tau),\cdots,p_n(a,\tau),\beta_1(\tau),\cdots,\beta_n(\tau))\,\mathrm{d}a\mathrm{d}\tau, \quad (9)$$

where $(p(a,\tau),\beta(\tau),v(\tau))$ is subject to (7)-(8).

Let $\beta(\tau)$ be fixed as $\beta^*(\tau)$, then $(p^*(a,\tau),v^*(\tau))$ solves the problem

$$\text{Minimize } J(p,\beta^*,v) = \int_0^1\int_0^{a_+} v(\tau)L(p_1(a,\tau),\cdots,p_n(a,\tau),\beta_1^*(\tau),\cdots,\beta_n^*(\tau))\,\mathrm{d}a\mathrm{d}\tau, \quad (10)$$

where $(p(a,\tau),\beta^*(\tau),v(\tau))$ satisfies (7)-(8).

Suppose that (p^*,v^*) is a solution of the problem (10), we seek the optimality conditions via Dubovitskii-Milyutin general extremal theory.

Let $X=C(0,1;L^2(0,a_+;R^n))\times L^\infty(0,1)$, define the inequality constraint

$$\Omega_1=\{(p,v)\in X:\ v(\tau)\ge 0,\ \forall\,\tau\in[0,1]\}$$

and the equality constraint

$$\Omega_2=\{(p,v)\in X:\ (p,\beta^*,v)\text{ is subject to (7)-(8)}\}.$$

It is clear that the problem (10) is equivalent to the problem below

$$\left\{\begin{array}{ll}\text{Minimize} & J(p,v)=\int_0^1\int_0^{a_+}v(\tau)\\ & L(p_1(a,\tau),\cdots,p_n(a,\tau),\beta_1^*(\tau),\cdots,\beta_n^*(\tau))\,\mathrm{d}a\mathrm{d}\tau,\\ & (p,v)\in\Omega_1\cap\Omega_2\subset X.\end{array}\right. \quad (11)$$

It is easy to see that the functional J is differentiable at every $(\bar p,\bar v)$, and

$$J'(\bar p,\bar v)(p,v)=\int_0^1\int_0^{a_+}\{\bar v(\tau)\sum_{i=1}^n p_i(a,\tau)\frac{\partial L}{\partial p_i}(\bar p,\beta^*)+v(\tau)L(\bar p,\beta^*)\}\,\mathrm{d}a\mathrm{d}\tau.$$

Hence J is regularly decreasing at (p^*,v^*) and its cone of directions of decrease is characterized by

$$K_0=\{(p,v)\in X:\ J'(p^*,v^*)(p,v)<0\}.$$

If $K_0\ne\emptyset$, then (see [3], Proposition 6.3.5) for any $f_0\in K_0^*$, there exists $\lambda_0\ge 0$ such that

$$\begin{array}{rl}f_0(p,v)= & -\lambda_0\int_0^1\int_0^{a_+}\{v(\tau)L(p^*,\beta^*)\\ & +v^*(\tau)\sum\limits_{i=1}^n p_i(a,\tau)\frac{\partial L}{\partial p_i}(p^*,\beta^*)\}\,\mathrm{d}a\mathrm{d}\tau.\end{array} \quad (12)$$

For the closed convex inequality constraint Ω_1, its interior is given by

$$\mathrm{int}(\Omega_1) = C(0,1; L^2(0,a_+;R^n)) \times \mathrm{int}(\hat{\Omega}_1) \neq \emptyset,$$

where $\hat{\Omega}_1 = \{v \in L^\infty(0,1) : v(\tau) > 0, \forall\, \tau \in [0,1]\}$. Consequently (see [3], Proposition 6.3.6) the cone of feasible directions of Ω_1 at (p^*, v^*) is as follows

$$K_1 = \{\lambda[(p,v) - (p^*,v^*)] : (p,v) \in \mathrm{int}(\Omega_1), \lambda > 0\}.$$

For every $f_1 \in K_1^*$, if there exists $c(\tau) \in L^1(0,1)$ such that

$$f_1(p,v) = \int_0^1 c(\tau) v(\tau)\,\mathrm{d}\tau,$$

then (see [1], p. 76, Example 10.5)

$$c(\tau)[v - v^*(\tau)] \geq 0,\ \forall\, v \in [0,+\infty),\ \tau \in [0,1] \text{ a.e.} \tag{13}$$

Next we determine the cone of tangent directions of Ω_2 at (p^*, v^*). Note that the solution of system (7) corresponding to $\beta = \beta^*$ satisfies

$$\begin{aligned}
u_1(a,\tau) :=\ & \int_0^a p_1(\theta,\tau)\,\mathrm{d}\theta - \int_0^a p_{10}(\theta)\,\mathrm{d}\theta \\
& + \int_0^\tau v(\sigma)[p_1(a,\sigma) - \beta_1^*(\sigma)\int_{a_1}^{a_2} m_1(\theta,\sigma)p_1(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
& + \int_0^a\int_0^\tau [\mu_1(\theta,\sigma) + \lambda_1(\theta,\sigma)P_2(\sigma)]v(\sigma)p_1(\theta,\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
=\ & 0, \\
u_k(a,\tau) :=\ & \int_0^a p_k(\theta,\tau)\,\mathrm{d}\theta - \int_0^a p_{k0}(\theta)\,\mathrm{d}\theta \\
& + \int_0^\tau v(\sigma)[p_k(a,\sigma) - \beta_k^*(\sigma)\int_{a_1}^{a_2} m_k(\theta,\sigma)p_k(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
& + \int_0^a\int_0^\tau \mu_k(\theta,\sigma)v(\sigma)p_k(\theta,\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
& + \int_0^a\int_0^\tau [\lambda_{2k-1}(\theta,\sigma)P_{k+1} - \lambda_{2k-2}(\theta,\sigma)P_{k-1}(\sigma)]v(\sigma) \\
& \times p_k(\theta,\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma = 0, \\
& k = 2,3,\cdots,n-1, \\
u_n(a,\tau) :=\ & \int_0^a p_n(\theta,\tau)\,\mathrm{d}\theta - \int_0^a p_{n0}(\theta)\,\mathrm{d}\theta \\
& + \int_0^\tau v(\sigma)[p_n(a,\sigma) - \beta_n^*(\sigma)\int_{a_1}^{a_2} m_n(\theta,\sigma)p_n(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
& + \int_0^a\int_0^\tau [\mu_n(\theta,\sigma) - \lambda_{2n-2}(\theta,\sigma)P_{n-1}(\sigma)]v(\sigma)p_n(\theta,\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
=\ & 0.
\end{aligned}$$

Define the operator $G : X \to C(0,1; L^2(0,a_+;R^n)) \times L^\infty(0,a_+;R^n)$,

$$\begin{aligned}
G(p_1(a,\tau),\cdots,p_n(a,\tau),v(\tau)) = (&u_1(a,\tau),\cdots,u_n(a,\tau), \\
& p_1(a,1) - p_1^0(a),\cdots,p_n(a,1) - p_n^0(a)).
\end{aligned}$$

So, $\Omega_2 = \{(p,v) \in X : G(p,v) = 0\}$.

It is easy to get that

$$G'(p^*,v^*)(p_1,\cdots,p_n,v) = (v_1(a,\tau),\cdots,v_n(a,\tau),p_1(a,1),\cdots,p_n(a,1)),$$

where

$$\begin{aligned}
v_1(a,\tau) &= \textstyle\int_0^a p_1(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau [v^*(\sigma)p_1(a,\sigma) + v(\sigma)p_1^*(a,\sigma)]\,\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau \beta_1^*(\sigma)\int_{a_1}^{a_2} m_1(\theta,\sigma)[v^*(\sigma)p_1(\theta,\sigma) + v(\sigma)p_1^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \mu_1(\theta,\sigma)[v^*(\sigma)p_1(\theta,\sigma) + v(\sigma)p_1^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \lambda_1(\theta,\sigma)p_1^*(\theta,\sigma)P_2(\sigma)v^*(\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \lambda_1(\theta,\sigma)P_2^*(\sigma)[v^*(\sigma)p_1(\theta,\sigma) + v(\sigma)p_1^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma,
\end{aligned} \tag{14}$$

$$\begin{aligned}
v_k(a,\tau) &= \textstyle\int_0^a p_k(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau [v^*(\sigma)p_k(a,\sigma) + v(\sigma)p_k^*(a,\sigma)]\,\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau \beta_k^*(\sigma)\int_{a_1}^{a_2} m_k(\theta,\sigma)[v^*(\sigma)p_k(\theta,\sigma) + v(\sigma)p_k^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \mu_k(\theta,\sigma)[v^*(\sigma)p_k(\theta,\sigma) + v(\sigma)p_k^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau\int_0^a \lambda_{2k-2}(\theta,\sigma)p_k^*(\theta,\sigma)P_{k-1}(\sigma)v^*(\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau\int_0^a \lambda_{2k-2}(\theta,\sigma)P_{k-1}^*(\sigma)[v^*(\sigma)p_k(\theta,\sigma) + v(\sigma)p_k^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma, \\
&+ \textstyle\int_0^\tau\int_0^a \lambda_{2k-1}(\theta,\sigma)p_k^*(\theta,\sigma)P_{k+1}(\sigma)v^*(\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \lambda_{2k-1}(\theta,\sigma)P_{k+1}^*(\sigma)[v^*(\sigma)p_k(\theta,\sigma) + v(\sigma)p_k^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma, \\
&k = 2,3,\cdots,n-1.
\end{aligned} \tag{15}$$

$$\begin{aligned}
v_n(a,\tau) &= \textstyle\int_0^a p_n(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau [v^*(\sigma)p_n(a,\sigma) + v(\sigma)p_n^*(a,\sigma)]\,\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau \beta_n^*(\sigma)\int_{a_1}^{a_2} m_n(\theta,\sigma)[v^*(\sigma)p_n(\theta,\sigma) + v(\sigma)p_n^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&+ \textstyle\int_0^\tau\int_0^a \mu_n(\theta,\sigma)[v^*(\sigma)p_n(\theta,\sigma) + v(\sigma)p_n^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau\int_0^a \lambda_{2n-2}(\theta,\sigma)p_n^*(\theta,\sigma)P_{n-1}(\sigma)v^*(\sigma)\,\mathrm{d}\theta\mathrm{d}\sigma \\
&- \textstyle\int_0^\tau\int_0^a \lambda_{2n-2}(\theta,\sigma)P_{n-1}^*(\sigma)[v^*(\sigma)p_n(\theta,\sigma) + v(\sigma)p_n^*(\theta,\sigma)]\,\mathrm{d}\theta\mathrm{d}\sigma.
\end{aligned} \tag{16}$$

To show that $G'(p^*,v^*)$ is an onto mapping, we solve the equation

$$G'(p^*,v^*)(p_1,\cdots,p_n,v) = (w_1,w_2,\cdots,w_{2n}),$$

that is, finding $(p_1,\cdots,p_n,v)$ such that

$$\begin{cases} v_i(a,\tau) = w_i(a,\tau), \ \ i = 1,2,\cdots,n\,, \\ p_i(a,1) = w_{i+n}(a), \ \ i = 1,2,\cdots,n\,, \end{cases} \tag{17}$$

in which v_i is given by (14)-(16).

It can be proved that if the linearized system around (p^*,v^*) of system (7) corresponding to $\beta = \beta^*$

$$\begin{aligned}
\frac{\partial p_1}{\partial \tau} + v^*(\tau)\frac{\partial p_1}{\partial a} &= -[\mu_1(a,\tau) + \lambda_1(a,\tau)P_2^*(\tau)][v^*(\tau)p_1(a,\tau) + v(\tau)p_1^*(a,\tau)] \\
&\quad -\lambda_1(a,\tau)P_2(\tau)v^*(\tau)p_1^*(a,\tau) - v(\tau)\frac{\partial p_1^*}{\partial a},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p_k}{\partial \tau} + v^*(\tau)\frac{\partial p_k}{\partial a} &= -[\mu_k(a,\tau) + \lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau) - \lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)] \\
&\quad \times[v^*(\tau)p_k(a,\tau) + v(\tau)p_k^*(a,\tau)] \\
&\quad +\lambda_{2k-2}(a,\tau)P_{k-1}(\tau)v^*(\tau)p_k^*(a,\tau) \\
&\quad -\lambda_{2k-1}(a,\tau)P_{k+1}(\tau)v^*(\tau)p_k^*(a,\tau) - v(\tau)\frac{\partial p_k^*}{\partial a}, \\
&\quad k = 2,3,\cdots,n-1,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial p_n}{\partial \tau} + v^*(\tau)\frac{\partial p_n}{\partial a} = -[\mu_n(a,\tau) - \lambda_{2n-2}(a,\tau)P^*_{n-1}(\tau)] \\
&\qquad\qquad \times[v^*(\tau)p_n(a,\tau) + v(\tau)p^*_n(a,\tau)] \\
&\qquad\qquad +\lambda_{2n-2}(a,\tau)P_{n-1}(\tau)v^*(\tau)p^*_n(a,\tau) - v(\tau)\frac{\partial p^*_n}{\partial a}, \\
&v^*(\tau)p_i(0,\tau) + v(\tau)p^*_i(0,\tau) = \beta^*_i(\tau)\int_{a_1}^{a_2} m_i(a,\tau) \\
&\qquad\qquad \times[v^*(\tau)p_i(a,\tau) + v(\tau)p^*_i(a,\tau)]\,\mathrm{d}a, \\
&p_i(a,0) = 0, \quad i = 1,2,\cdots,n.
\end{aligned} \tag{18}$$

is exactly controllable at $\tau = 1$, then there must be a solution to the system (17). In fact, there exists $\hat{v}(\tau)$ such that the solution of the system (18) satisfies

$$\hat{p}_i(a,1) = w_{i+n}(a) - \gamma_i(a,1), \quad i = 1,2,\cdots,n\,,$$

where γ_i, $i = 1,2,\cdots,n$, is the unique solution to the following integral equations

$$\begin{aligned}
&\int_0^a \gamma_1(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau v^*(\sigma)[\gamma_1(a,\sigma) - \beta^*_1(\sigma)\int_{a_1}^{a_2}(m_1\gamma_1)(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
&+\int_0^a\int_0^\tau \mu_1(\theta,\sigma)\gamma_1(\theta,\sigma)v^*(\sigma)\,\mathrm{d}\sigma\mathrm{d}\theta \\
&+\int_0^a\int_0^\tau v^*(\sigma)\lambda_1(\theta,\sigma)[p^*_1(\theta,\sigma)\Gamma_2(\sigma) + P^*_2(\sigma)\gamma_1(\theta,\sigma)]\,\mathrm{d}\sigma\mathrm{d}\theta = w_1(a,\tau), \\
&\int_0^a \gamma_k(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau v^*(\sigma)[\gamma_k(a,\sigma) - \beta^*_k(\sigma)\int_{a_1}^{a_2}(m_k\gamma_k)(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
&+\int_0^a\int_0^\tau \mu_k(\theta,\sigma)\gamma_k(\theta,\sigma)v^*(\sigma)\,\mathrm{d}\sigma\mathrm{d}\theta \\
&-\int_0^a\int_0^\tau v^*(\sigma)\lambda_{2k-2}(\theta,\sigma)[p^*_k(\theta,\sigma)\Gamma_{k-1}(\sigma) + P^*_{k-1}(\sigma)\gamma_k(\theta,\sigma)]\,\mathrm{d}\sigma\mathrm{d}\theta \\
&+\int_0^a\int_0^\tau v^*(\sigma)\lambda_{2k-1}(\theta,\sigma)[p^*_k(\theta,\sigma)\Gamma_{k+1}(\sigma) + P^*_{k+1}(\sigma)\gamma_k(\theta,\sigma)]\,\mathrm{d}\sigma\mathrm{d}\theta \\
&= w_k(a,\tau), \quad k = 2,3,\cdots,n-1, \\
&\int_0^a \gamma_n(\theta,\tau)\,\mathrm{d}\theta + \int_0^\tau v^*(\sigma)[\gamma_n(a,\sigma) - \beta^*_n(\sigma)\int_{a_1}^{a_2}(m_n\gamma_n)(\theta,\sigma)\,\mathrm{d}\theta]\mathrm{d}\sigma \\
&+\int_0^a\int_0^\tau \mu_n(\theta,\sigma)\gamma_n(\theta,\sigma)v^*(\sigma)\,\mathrm{d}\sigma\mathrm{d}\theta \\
&-\int_0^a\int_0^\tau v^*(\sigma)\lambda_{2n-2}(\theta,\sigma)[p^*_n(\theta,\sigma)\Gamma_{n-1}(\sigma) + P^*_{n-1}(\sigma)\gamma_n(\theta,\sigma)]\,\mathrm{d}\sigma\mathrm{d}\theta \\
&= w_n(a,\tau), \\
&\Gamma_i(\sigma) = \int_0^{a_+}\gamma_i(a,\sigma)\,\mathrm{d}\sigma, \quad i = 1,2,\cdots,n.
\end{aligned} \tag{19}$$

Note that the solution of the system (18) satisfies $v_1(a,\tau) = v_2(a,\tau) = \cdots = v_n(a,\tau) = 0$. From (19) it is easy to show that $(\hat{p}_1 + \gamma_1, \cdots, \hat{p}_n + \gamma_n, \hat{v})$ is a solution to the system (17). Thus, the tangent directions cone of Ω_2 at (p^*, v^*) is given by

$$K_2 = \{(p,v) \in X : G'(p^*,v^*)(p,v) = 0\}.$$

Let

$$\begin{aligned}
K_{11} &= \{(p,v) \in X : (p,v) \text{ is subject to (18)}\}, \\
K_{12} &= \{(p,v) \in X : p_i(a,1) = 0,\ i = 1,2,\cdots,n\}.
\end{aligned} \tag{20}$$

Then $K_2 = K_{11} \cap K_{12}$. Since K_{11} and K_{12} are subspaces, so $K^*_2 = K^*_{11} + K^*_{12}$.

For any $f_2 \in K_2^*$, $f_2 = f_{11} + f_{12}$, $f_{1i} \in K_{1i}^*$, $i = 1, 2$, there exists

$$\alpha(a) = (\alpha_1(a), \cdots, \alpha_n(a)) \in L^2(0, a_+; R^n),$$

such that

$$\begin{aligned} & f_{12}(p, v) \\ & = \int_0^{a_+} \alpha(a) \cdot p(a, 1)\, \mathrm{d}a \\ & = \sum_{i=1}^{n} \int_0^{a_+} \alpha_i(a) p_i(a, 1)\, \mathrm{d}a. \end{aligned} \tag{21}$$

According to Dubovitskii-Milyutin Theorem, there are functionals $f_i \in K_i^*$, $i = 0, 1, 2$, not all zero and such that

$$f_0 + f_1 + f_{11} + f_{12} = 0. \tag{22}$$

In particular, $f_{11}(p, v) = 0$ if (p, v) satisfies (18). From (12) and (21)-(22) it follows that

$$\begin{aligned} f_1(p, v) = -f_0(p, v) - f_{12}(p, v) = \textstyle\int_0^1 \int_0^{a_+} \lambda_0 v(\tau) L(p^*(a, \tau), \beta^*(\tau))\, \mathrm{d}a \mathrm{d}\tau \\ + \sum_{i=1}^{n} [\textstyle\int_0^1 \int_0^{a_+} \lambda_0 v^*(\tau) p_i(a, \tau) \frac{\partial L}{\partial p_i}(p^*(a, \tau), \beta^*(\tau))\, \mathrm{d}a \mathrm{d}\tau \\ - \textstyle\int_0^{a_+} \alpha_i(a) p_i(a, 1)\, \mathrm{d}a]. \end{aligned} \tag{23}$$

Define the adjoint system

$$\begin{aligned} & \tfrac{\partial q_1}{\partial \tau} + v^*(\tau) \tfrac{\partial q_1}{\partial a} = [\mu_1(a, \tau) + \lambda_1(a, \tau) P_2^*(\tau)] q_1(a, \tau) v^*(\tau) + [\lambda_0 \tfrac{\partial L}{\partial p_1}(p^*(a, \tau), \\ & \beta^*(\tau)) - \beta_1^*(\tau) m_1(a, \tau) q_1(0, \tau) - \textstyle\int_0^{a_+} (\lambda_2 p_2^* q_2)(\theta, \tau)\, \mathrm{d}\theta] v^*(\tau), \\ & \tfrac{\partial q_k}{\partial \tau} + v^*(\tau) \tfrac{\partial q_k}{\partial a} = [\mu_k(a, \tau) + \lambda_{2k-1}(a, \tau) P_{k+1}^*(\tau) - \lambda_{2k-2}(a, \tau) P_{k-1}^*(\tau)] \\ & \times q_k(a, \tau) v^*(\tau) + [\lambda_0 \tfrac{\partial L}{\partial p_k}(p^*(a, \tau), \beta^*(\tau)) - \beta_k^*(\tau) m_k(a, \tau) q_k(0, \tau) \\ & \qquad + \textstyle\int_0^{a_+} (\lambda_{2k-3} p_{k-1}^* q_{k-1} - \lambda_{2k} p_{k+1}^* q_{k+1})(\theta, \tau)\, \mathrm{d}\theta] v^*(\tau), \\ & \qquad k = 2, 3, \cdots, n-1, \\ & \tfrac{\partial q_n}{\partial \tau} + v^*(\tau) \tfrac{\partial q_n}{\partial a} = [\mu_n(a, \tau) - \lambda_{2n-2}(a, \tau) P_{n-1}^*(\tau)] q_n(a, \tau) v^*(\tau) \\ & \qquad + [\lambda_0 \tfrac{\partial L}{\partial p_n}(p^*(a, \tau), \beta^*(\tau)) - \beta_n^*(\tau) m_n(a, \tau) q_n(0, \tau) \\ & \qquad + \textstyle\int_0^{a_+} (\lambda_{2n-3} p_{n-1}^* q_{n-1})(\theta, \tau)\, \mathrm{d}\theta] v^*(\tau), \\ & q_i(a, 1) = \alpha_i(a), \quad q_i(a_+, \tau) = 0,\ i = 1, 2, \cdots, n. \end{aligned} \tag{24}$$

After some calculations by using (24), we can obtain that the solution of (18) and the solution of (24) have the following relation

$$\begin{aligned}
&\sum_{i=1}^{n}[\int_0^1\int_0^{a_+}\lambda_0 v^*(\tau)p_i(a,\tau)\tfrac{\partial L}{\partial p_i}(p^*(a,\tau),\beta^*(\tau))\,\mathrm{d}a\mathrm{d}\tau\\
&-\int_0^{a_+}\alpha_i(a)p_i(a,1)\,\mathrm{d}a]\\
=\;&\int_0^1\int_0^{a_+}v(\tau)\{(q_1p_1^*)(a,\tau)[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2^*(\tau)]\\
&-p_1^*(a,\tau)[\tfrac{\partial q_1}{\partial a}+\beta_1^*(\tau)m_1(a,\tau)q_1(0,\tau)]\\
&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,\tau)[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau)-\lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)]\\
&-\sum_{k=2}^{n-1}p_k^*(a,\tau)[\tfrac{\partial q_k}{\partial a}+\beta_k^*(\tau)m_k(a,\tau)q_k(0,\tau)]\,\mathrm{d}a\mathrm{d}\tau\\
&+(q_np_n^*)(a,\tau)[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}^*(\tau)]\\
&-p_n^*(a,\tau)[\tfrac{\partial q_n}{\partial a}+\beta_n^*(\tau)m_n(a,\tau)q_n(0,\tau)]\}\,\mathrm{d}a\mathrm{d}\tau.
\end{aligned} \tag{25}$$

which holds for every λ_0, $\alpha(a)$.

Combining (23) with (25) derives

$$\begin{aligned}
&f_1(p,v)=\int_0^1\int_0^{a_+}v(\tau)\{(q_1p_1^*)(a,\tau)[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2^*(\tau)]\\
&-p_1^*(a,\tau)[\tfrac{\partial q_1}{\partial a}+\beta_1^*(\tau)m_1(a,\tau)q_1(0,\tau)]\\
&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,\tau)[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau)-\lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)]\\
&-\sum_{k=2}^{n-1}p_k^*(a,\tau)[\tfrac{\partial q_k}{\partial a}+\beta_k^*(\tau)m_k(a,\tau)q_k(0,\tau)]\\
&+(q_np_n^*)(a,\tau)[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}^*(\tau)]\\
&-p_n^*(a,\tau)[\tfrac{\partial q_n}{\partial a}+\beta_n^*(\tau)m_n(a,\tau)q_n(0,\tau)]\\
&+\lambda_0L(p^*(a,\tau),\beta^*(\tau))\}\,\mathrm{d}a\mathrm{d}\tau.
\end{aligned} \tag{26}$$

Let

$$\begin{aligned}
S(\tau)=\;&\int_0^{a_+}\{(q_1p_1^*)(a,\tau)[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2^*(\tau)]\\
&-p_1^*(a,\tau)[\tfrac{\partial q_1}{\partial a}+\beta_1^*(\tau)m_1(a,\tau)q_1(0,\tau)]\\
&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,\tau)[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau)\\
&\qquad-\lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)]\\
&-\sum_{k=2}^{n-1}p_k^*(a,\tau)[\tfrac{\partial q_k}{\partial a}+\beta_k^*(\tau)m_k(a,\tau)q_k(0,\tau)]\\
&+(q_np_n^*)(a,\tau)[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}^*(\tau)]\\
&-p_n^*(a,\tau)[\tfrac{\partial q_n}{\partial a}+\beta_n^*(\tau)m_n(a,\tau)q_n(0,\tau)]\\
&+\lambda_0L(p^*(a,\tau),\beta^*(\tau))\}\,\mathrm{d}a.
\end{aligned} \tag{27}$$

From (26) and (13) it follows that

$$S(\tau)[v - v^*(\tau)] \geq 0, \ \forall\ v \in [0, \infty),\ \tau \in [0,1] \text{ a.e.} \tag{28}$$

Define the sets

$$S_1 = \{\tau \in [0,1] : v^*(\tau) > 0\}, \quad S_2 = \{\tau \in [0,1] : v^*(\tau) = 0\}.$$

We can see from (28) that

$$S(\tau) = 0, \text{if } \tau \in S_1, \quad S(\tau) \geq 0, \text{if } \tau \in S_2. \tag{29}$$

We claim that λ_0 and $\alpha(\cdot)$ are not both zero. Otherwise, it follows from (12) and (21) that $f_0 = 0, f_{12} = 0$. Then (24) implies $q_i = 0,\ i = 1, 2, \cdots, n$; consequently (26) and (22) lead to $f_1 = 0,\ f_{11} = 0$, which is a contradiction.

Besides, if $K_0 = \emptyset$, that is, for any $(p, v) \in X$

$$\begin{aligned}&\sum_{i=1}^{n} \int_0^1 \int_0^{a_+} v^*(\tau) p_i(a,\tau) \tfrac{\partial L}{\partial p_i}(p^*(a,\tau), \beta^*(\tau))\, \mathrm{d}a\mathrm{d}\tau \\ &+ \int_0^1 \int_0^{a_+} v(\tau) L(p^*(a,\tau), \beta^*(\tau))\, \mathrm{d}a\mathrm{d}\tau \geq 0,\end{aligned} \tag{30}$$

choosing $\lambda_0 = 1$, $\alpha(a) = 0$ in (25) gives

$$\begin{aligned}&\sum_{i=1}^{n} \int_0^1 \int_0^{a_+} \{v^*(\tau) p_i(a,\tau) \tfrac{\partial L}{\partial p_i}(p^*(a,\tau), \beta^*(\tau))\, \mathrm{d}a\mathrm{d}\tau \\ = \ & \int_0^1 \int_0^{a_+} v(\tau) \{(q_1 p_1^*)(a,\tau)[\mu_1(a,\tau) + \lambda_1(a,\tau) P_2^*(\tau)] \\ & - p_1^*(a,\tau)[\tfrac{\partial q_1}{\partial a} + \beta_1^*(\tau) m_1(a,\tau) q_1(0,\tau)] \\ & + \sum_{k=2}^{n-1} (q_k p_k^*)(a,\tau)[\mu_k(a,\tau) + \lambda_{2k-1}(a,\tau) P_{k+1}^*(\tau) - \lambda_{2k-2}(a,\tau) P_{k-1}^*(\tau)] \\ & - \sum_{k=2}^{n-1} p_k^*(a,\tau)[\tfrac{\partial q_k}{\partial a} + \beta_k^*(\tau) m_k(a,\tau) q_k(0,\tau)]\, \mathrm{d}a\mathrm{d}\tau \\ & + (q_n p_n^*)(a,\tau)[\mu_n(a,\tau) - \lambda_{2n-2}(a,\tau) P_{n-1}^*(\tau)] \\ & - p_n^*(a,\tau)[\tfrac{\partial q_n}{\partial a} + \beta_n^*(\tau) m_n(a,\tau) q_n(0,\tau)]\}\, \mathrm{d}a\mathrm{d}\tau.\end{aligned} \tag{31}$$

From (30)-(31) we know that

$$\int_0^1 S(\tau) v(\tau)\, \mathrm{d}\tau \geq 0$$

for any $(p, v) \in X$, in which $S(\tau)$ is given by (27). So $S(\tau) \in K_1^*$. Again from (13) we see that (28) and (29) still hold.

Finally if the adjoint system (24) has a nonzero solution $q_i, i = 1, 2, \cdots, n$, such that

$$\begin{aligned}&\int_0^{a_+}\{(q_1p_1^*)(a,\tau)[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2^*(\tau)]\\&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,\tau)[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau)-\lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)]\\&+(q_np_n^*)(a,\tau)[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}^*(\tau)]\\&-\sum_{i=1}^{n}p_i^*(a,\tau)[\tfrac{\partial q_i}{\partial a}+\beta_i^*(\tau)m_i(a,\tau)q_i(0,\tau)]\}\,\mathrm{d}a=0.\end{aligned}\tag{32}$$

Then choosing $\lambda_0=0$ in (27) enables (28) to be correct. If for every nonzero solution of the adjoint system the following relation holds

$$\begin{aligned}&\int_0^{a_+}\{(q_1p_1^*)(a,\tau)[\mu_1(a,\tau)+\lambda_1(a,\tau)P_2^*(\tau)]\\&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,\tau)[\mu_k(a,\tau)+\lambda_{2k-1}(a,\tau)P_{k+1}^*(\tau)-\lambda_{2k-2}(a,\tau)P_{k-1}^*(\tau)]\\&+(q_np_n^*)(a,\tau)[\mu_n(a,\tau)-\lambda_{2n-2}(a,\tau)P_{n-1}^*(\tau)]\\&-\sum_{i=1}^{n}p_i^*(a,\tau)[\tfrac{\partial q_i}{\partial a}+\beta_i^*(\tau)m_i(a,\tau)q_i(0,\tau)]\}\,\mathrm{d}a\neq 0.\end{aligned}$$

Then the linearized system (18) must be exactly controllable at $\tau=1$. Otherwise there exists $\alpha(a)\in L^2(0,a_+;R^n)$, $\alpha\neq 0$ such that $\int_0^{a_+}\alpha(a)\cdot p(a,1)\,\mathrm{d}a=0$. Taking $\lambda_0=0$ in (25), we arrive at (32), a contradiction.

In all cases, (29) is always true.

Define the time transformation

$$\tau(t)=\inf\{\tau\in[0,1]:\,t(\tau)=t\},$$

and

$$\begin{aligned}&q_i(a,t)=q_i(a,\tau(t)),\;q_i(0,t)=q_i(0,\tau(t)),\;i=1,2,\cdots,n.\\&S(t)=S(\tau(t)),\end{aligned}$$

where $S(\tau)$ is given by (27).

Because $\{t:\,t=t(\tau),\tau\in S_2\}$ is at most measurable (see [1], p. 99), it follows from the first part of (29) that

$$\begin{aligned}S(t):=\;&\int_0^{a_+}\{(q_1p_1^*)(a,t)[\mu_1(a,t)+\lambda_1(a,t)P_2^*(t)]\\&+\sum_{k=2}^{n-1}(q_kp_k^*)(a,t)[\mu_k(a,t)+\lambda_{2k-1}(a,t)P_{k+1}^*(t)\\&\qquad-\lambda_{2k-2}(a,t)P_{k-1}^*(t)]\\&-\sum_{i=1}^{n}p_i^*(a,t)[\tfrac{\partial q_i}{\partial a}+\beta_i^*(t)m_i(a,t)q_i(0,t)]\\&+(q_np_n^*)(a,t)[\mu_n(a,t)-\lambda_{2n-2}(a,t)P_{n-1}^*(t)]\\&+\lambda_0L(p^*(a,t),\beta^*(t))\}\,\mathrm{d}a=0.\end{aligned}\tag{33}$$

holds for almost every $t\in[0,t_1^*]$.

Let S_1 be a perfect nowhere dense subset of $[0,1]$, define

$$v^*(\tau)=\begin{cases} t_1^*/\mu(S_1), & \tau\in S_1,\\ 0, & \tau\in S_2:=[0,1]-S_1.\end{cases}$$

In a similar manner as that in [1], we can define $\beta^*(\tau)$ on S_2, and an analysis of the second part of (29) shows that

$$\begin{aligned}&\textstyle\int_0^{a_+}\{(q_1p_1^*)(a,t)[\mu_1(a,t)+\lambda_1(a,t)P_2^*(t)]\\ &+\sum_{k=2}^{n-1}(q_kp_k^*)(a,t)[\mu_k(a,t)+\lambda_{2k-1}(a,t)P_{k+1}^*(t)\\ &\qquad-\lambda_{2k-2}(a,t)P_{k-1}^*(t)]\\ &-\sum_{i=1}^{n}p_i^*(a,t)[\tfrac{\partial q_i}{\partial a}+\beta_i^*(t)m_i(a,\tau)q_i(0,t)]\\ &+(q_np_n^*)(a,t)[\mu_n(a,t)-\lambda_{2n-2}(a,t)P_{n-1}^*(t)]\\ &+\lambda_0L(p^*(a,t),\beta(t))\}\,\mathrm{d}a\ge 0.\end{aligned}\tag{34}$$

hold for every $\beta\in[\beta_0,\beta^0]$ and every $t\in[0,t_1^*]$.

We have so far proved that

Theorem 1. *If (p^*,β^*,t_1^*) is a solution of problem (1)-(3), then there exist a number $\lambda_0\ge 0$ and a function $\alpha(a)\in L^2(0,a_+;R^n)$ such that (33) and (34) hold, in which q_i, $i=1,2,\cdots,n$, solves the following adjoint system*

$$\begin{aligned}\tfrac{\partial q_1}{\partial t}+\tfrac{\partial q_1}{\partial a}&=[\mu_1(a,t)+\lambda_1(a,t)P_2^*(t)]q_1(a,t)\\ &\quad+\lambda_0\tfrac{\partial L}{\partial p_1}(p^*(a,t),\beta^*(t))-\beta_1^*(\tau)m_1(a,t)q_1(0,t)\\ &\quad-\textstyle\int_0^{a_+}(\lambda_2p_2^*q_2)(a,t)\,da,\\ \tfrac{\partial q_k}{\partial t}+\tfrac{\partial q_k}{\partial a}&=[\mu_k(a,t)+\lambda_{2k-1}(a,t)P_{k+1}^*(t)-\lambda_{2k-2}(a,t)P_{k-1}^*(t]q_k(a,t)\\ &\quad+\lambda_0\tfrac{\partial L}{\partial p_k}(p^*(a,t),\beta^*(t))-\beta_k^*(t)m_k(a,t)q_k(0,t)\\ &\quad+\textstyle\int_0^{a_+}(\lambda_{2k-3}p_{k-1}^*q_{k-1}-\lambda_{2k}p_{k+1}^*q_{k+1})(a,t)\,da,\\ &\quad k=2,3,\cdots,n-1,\\ \tfrac{\partial q_n}{\partial \tau}+\tfrac{\partial q_n}{\partial a}&=[\mu_n(a,t)-\lambda_{2n-2}(a,t)P_{n-1}^*(t)]q_n(a,t)\\ &\quad+\lambda_0\tfrac{\partial L}{\partial p_n}(p^*(a,t),\beta^*(t))-\beta_n^*(t)m_n(a,t)q_n(0,t)\\ &\quad+\textstyle\int_0^{a_+}(\lambda_{2n-3}p_{n-1}^*q_{n-1})(a,t)\,da,\\ q_i(a,t_1^*)=\alpha_i(a),&\quad q_i(a_+,t)=0,\ i=1,2,\cdots,n.\end{aligned}$$

Remark 1. If the phase constraint (3) is replaced with

$$p(a,t_1)\in\{h(a):\|h-p^0\|<\varepsilon\},$$

then the corresponding optimality conditions can be obtained by choosing $\lambda_0=1$, $\alpha(a)=p^*(a,t_1^*)-p^0(a)$ in Theorem 1.

Remark 2. (Time-Optimal Control) Let $L(p,\beta)\equiv 1$, one can readily deduce the maximum principle for the time-optimal problem.

3. Problems with Fixed Horizon and Integral Phase Constraint

Consider the control problem

Minimize

$$J(p,\beta)=\int_0^T\int_0^{a_+}L(p_1(a,t),\cdots,p_n(a,t),\beta_1(t),\cdots,\beta_n(t))\,\mathrm{d}a\mathrm{d}t,\quad (35)$$

where $T>0$ is fixed, (p,β) is subject to (2) and

$$p_i(\cdot,T)=p_i^0,\ i=1,2,\cdots,n. \quad (36)$$

and

$$\int_0^{a_+}G(p_1(a,t),\cdots,p_n(a,t),t)\,\mathrm{d}a\le 0,\ \forall\, t\in[0,T]. \quad (37)$$

Let the state space be $X=C(0,T;L^2(0,a_+;R^n))\times L^\infty(0,T;R^n)$, define

$\Omega_1=\{(p,\beta)\in X:\ \beta_i(t)\in[\beta_0,\beta^0],\ t\in[0,T]\,\text{a.e.},i=1,2,\cdots,n\}$,
$\Omega_2=\{(p,\beta)\in X:\ (p,\beta)$ is subject to (2) and (36)$\}$,
$\Omega_3=\{(p,\beta)\in X:\ (p,\beta)$ is subject to (37)$\}$.

So the problem (35)-(37) is equivalent to the problemfinding $(p^*,\beta^*)\in\Omega_1\cap\Omega_2\cap\Omega_3$, such that

$$\begin{cases} J(p^*,\beta^*)=\min J(p,\beta),\\ (p,\beta)\in\Omega_1\cap\Omega_2\cap\Omega_3.\end{cases}\quad (38)$$

We have discussed the cones corresponding to the functional J, inequality constraint Ω_1 and equality constraint Ω_2. Now we need only to analyze the inequality constraint Ω_3.

It is clear that Ω_3 can be rewritten as

$$\Omega_3=\{(p,\beta)\in X:\ F(p)\le 0\},$$

where

$$F(p)=\max_{0\le t\le T}\int_0^{a_+}G(p_1(a,t),\cdots,p_n(a,t),t)\,\mathrm{d}a. \quad (39)$$

We assume that the following conditions hold:

(1) $\int_0^{a_+}G(p_1(a),\cdots,p_n(a),t)\mathrm{d}a$ is a continuous functional on $L^2(0,a_+;R^n)\times[0,\infty)$;

(2) $\int_0^{a_+}G(p_{10}(a),\cdots,p_{n0}(a),0)\,\mathrm{d}a<0$, $\int_0^{a_+}G(p_1^0(a),\cdots,p_n^0(a),T)\,\mathrm{d}a<0$;

(3) $\int_0^{a_+}G'_{p_i}(p_1(a),\cdots,p_n(a),t)\,\mathrm{d}a$ is continuous on $L^2(0,a_+;R^n)\times[0,\infty)$, and $\int_0^{a_+}G'_{p_i}(p_1(a),\cdots,p_n(a),t)\,\mathrm{d}a\ne 0$, $i=1,2,\cdots,n$, if $\int_0^{a_+}G(p_1(a),\cdots,p_n(a),t)\,\mathrm{d}a=0$.

Let (p^*,β^*) be a solution of the problem (35)-(37). Without loss of generality, we need only to consider the case of $F(p^*) = 0$. In fact, if $F(p^*) < 0$, then it follows from (1) that the cone of feasible directions of Ω_3 at (p^*, β^*) is $K_3 = X$, which implies $K_3^* = \{0\}$. This situation is equivalent to the absence of the constraint Ω_3. Therefore

$$\Omega_3 = \{(p, \beta) \in X : F(p) \leq F(p^*)\}.$$

By means of Example 7.5 in [1], p. 52, we can state

Lemma 1. *$F(p)$ is differentiable at every $\hat{p}$ in every direction p, and*

$$F'(\hat{p}, p) = \max_{t \in S} \sum_{i=1}^{n} \int_0^{a_+} G'_{p_i}(\hat{p}_1(a,t), \cdots, \hat{p}_n(a,t), t) p_i(a,t)\, da,$$

where

$$S = \{t \in [0, T] : \int_0^{a_+} G(\hat{p}_1(a,t), \cdots, \hat{p}_n(a,t), t)\, da = F(\hat{p})\}. \tag{40}$$

In addition, $F(p)$ is of Lipschitz in any ball.

Notice that

$$F'(p^*, -G'_p(p^*, t)) < 0, \text{ where } G'_p(p^*, t) = (G'_{p_1}(p^*, t), \cdots, G'_{p_n}(p^*, t)).$$

According to the lemma in [1](p. 59), we have

$$K_3 = \{(p, \beta) \in X : F'(p^*, p) < 0\}.$$

Define the linear operator $B : X \to C[0, T]$,

$$B(p, \beta) = -\sum_{i=1}^{n} \int_0^{a_+} G'_{p_i}(p^*(a,t), t) p_i(a,t)\, \mathrm{d}a,$$

and the set

$$K = \{y(t) \in C[0, T] : y(t) \geq 0, \forall\, t \in S\},$$

where S is given by (40) corresponding to $\hat{p} = p^*$. It is easy to see

$$K_3 = \{(p, \beta) \in X : B(p, \beta) \in K\}.$$

Since $B(-G'_p(p^*, t)) \in \mathrm{int}(K)$, it follows from Theorem 10.4 in [1] (see p. 70), that $K_3^* = B^* K^*$, in which B^* denotes the adjoint operator of B. Thus, Riesz's Theorem implies that for any $f_3 \in K_3^*$, there exists a measure $\mathrm{d}m(t)$, which is supporting on S and

$$\begin{aligned} f_3(p, \beta) &= \int_0^T B(p(a,t), \beta(t)) \mathrm{d}m(t) \\ &= -\sum_{i=1}^{n} \int_0^T \int_0^{a_+} G'_{p_i}(p^*(a,t), t) p_i(a,t)\, \mathrm{d}a \mathrm{d}m(t). \end{aligned} \tag{41}$$

Combining (41) with the discussions for J, Ω_1, Ω_2, we assert that there exist

$\lambda_0 \geq 0$, $\alpha(a) = (\alpha_1(a), \cdots, \alpha_n(a)) \in L^2(0, a_+; R^n)$ such that

$$f_1(p,\beta) = \sum_{i=1}^{n} \{ \int_0^T \int_0^{a_+} \lambda_0 [p_i(a,t) \frac{\partial L}{\partial p_i}(p^*(a,t), \beta^*(t)) + \beta_i(t) \frac{\partial L}{\partial \beta_i}(p^*(a,t), \beta^*(t))] \, \mathrm{d}a\mathrm{d}t - \int_0^{a_+} p_i(a,T)\alpha(a)\,\mathrm{d}a \} + \sum_{i=1}^{n} \int_0^T \int_0^{a_+} G'_{p_i}(p^*(a,t),t) p_i(a,t)\,\mathrm{d}a\mathrm{d}m(t), \quad (42)$$

where (p, β) satisfies the following linearized system

$$\begin{aligned}
&\tfrac{\partial p_1}{\partial t} + \tfrac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1(a,t) - \lambda_1(a,t)[P_2^*(t)p_1(a,t) + P_2(t)p_1^*],\\
&\tfrac{\partial p_k}{\partial t} + \tfrac{\partial p_k}{\partial a} = -\mu_k(a,t)p_k(a,t) + \lambda_{2k-2}(a,t)[P_{k-1}^*(t)p_k(a,t)\\
&+P_{k-1}(t)p_k^*(a,t)]\\
&\qquad -\lambda_{2k-1}(a,t)[P_{k+1}^*(t)p_k(a,t) + P_{k+1}(t)p_k^*(a,t)],\\
&\qquad k = 2,3,\cdots,n-1,\\
&\tfrac{\partial p_n}{\partial t} + \tfrac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n(a,t) + \lambda_{2n-2}(a,t)[P_{n-1}^*(t)p_n(a,t)\\
&+P_{n-1}(t)p_n^*(a,t)],\\
&p_i(0,t) = \textstyle\int_{a_1}^{a_2} m_i(a,t)[\beta_i^*(t)p_i(a,t) + \beta_i(t)p_i^*(a,t)]\,\mathrm{d}a,\\
&p_i(a,0) = 0,\\
&P_i^*(t) = \textstyle\int_0^{a_+} p_i^*(a,t)\,\mathrm{d}a,\ i = 1,2,\cdots,n,\ (a,t) \in Q.
\end{aligned} \quad (43)$$

Equality (42) must be true as long as the cone of decrease directions of J is not empty and the system (43) is exactly controllable at T.

Define the adjoint system

$$\begin{aligned}
&\tfrac{\partial q_1}{\partial t} + \tfrac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1\beta_1^* q_1(0,t) + \lambda_1 q_1 P_2^*(t)\\
&\qquad +\lambda_0 \tfrac{\partial L}{\partial p_1}(\beta^*, p^*) - \textstyle\int_0^{a_+} (\lambda_2 p_2^* q_2)(a,t)\,\mathrm{d}a,\\
&\qquad +G'_{p_1}(p^*(a,t),t)\frac{\mathrm{d}m(t)}{\mathrm{d}t},\\
&\tfrac{\partial q_k}{\partial t} + \tfrac{\partial q_k}{\partial a} = \mu_k q_k - m_k\beta_k^* q_k(0,t) - \lambda_{2k-2} q_k P_{k-1}^*(t) + \lambda_{2k-1} q_k P_{k+1}^*(t)\\
&\qquad +\lambda_0 \tfrac{\partial L}{\partial p_k}(\beta^*, p^*) + \textstyle\int_0^{a_+} (\lambda_{2k-3} p_{k-1}^* q_{k-1} - \lambda_{2k} p_{k+1}^* q_{k+1})(a,t)\,\mathrm{d}a\\
&\qquad +G'_{p_k}(p^*(a,t),t)\frac{\mathrm{d}m(t)}{\mathrm{d}t},\\
&\tfrac{\partial q_n}{\partial t} + \tfrac{\partial q_n}{\partial a} = \mu_n q_n - m_n\beta_n^* q_n(0,t) - \lambda_{2n-2} q_n P_{n-1}^*(t)\\
&\qquad +\lambda_0 \tfrac{\partial L}{\partial p_n}(\beta^*, p^*) + \textstyle\int_0^{a_+} (\lambda_{2n-3} p_{n-1}^* q_{n-1})(a,t)\,\mathrm{d}a,\\
&\qquad +G'_{p_n}(p^*(a,t),t)\frac{\mathrm{d}m(t)}{\mathrm{d}t},\\
&\qquad k = 2,3,\cdots,n-1,\\
&q_i(a,T) = \alpha_i(a),\\
&q_i(a_+,t) = 0,\ (a,t) \in Q_T.
\end{aligned} \quad (44)$$

Some computations show the following

Lemma 2. *The solutions of the system (43) and of the system (44) are connected with the following relation*

$$\sum_{i=1}^{n}\{\int_0^T\int_0^{a_+}\lambda_0[p_i(a,t)\tfrac{\partial L}{\partial p_i}(p^*(a,t),\beta^*(t))$$
$$+\beta_i(t)\tfrac{\partial L}{\partial \beta_i}(p^*(a,t),\beta^*(t))]\,dadt-\int_0^{a_+}p_i(a,T)\alpha(a)\,da\}$$
$$+\sum_{i=1}^{n}\int_0^T\int_0^{a_+}G'_{p_i}(p^*(a,t),t)p_i(a,t)\,dadm(t)$$
$$=\sum_{i=1}^{n}\int_0^T\int_0^{a_+}[\lambda_0\tfrac{\partial L}{\partial \beta_i}(p^*(a,t),\beta^*(t))-p_i^*(a,t)m_i(a,t)q_i(0,t)]\,da\cdot\beta_i(t)dt.$$

Finally a slmilar analysis leads to

Theorem 2. *If (p^*,β^*) is a solution of the problem (35)-(37), then there exist $\lambda_0\geq 0$ and a function q_i, $i=1,2,\cdots,n$, not both zero and such that*

$$\sum_{i=1}^{n}\int_0^{a_+}[\lambda_0\frac{\partial L}{\partial \beta_i}(p^*(a,t),\beta^*(t))-p_i^*(a,t)m_i(a,t)q_i(0,t)]\,da\cdot[\beta_i-\beta_i^*(t)]\geq 0$$

holds for every $\beta_i\in[\beta_0,\beta^0]$ and for every $t\in[0,T]$, in which q_i is the solution of system (44).

4. Problems with Free Horizon and Integral Phase Constraint

Consider further the optimal control problem

$$\text{Minimize}\quad J(p,\beta)=\int_0^{t_1}\int_0^{a_+}L(p_1(a,t),\cdots,p_n(a,t),\beta_1(t),\cdots,\beta_n(t))\,\mathrm{d}a\mathrm{d}t,$$

where $t_1>0$ is not fixed and (p,β) is subject to (2) and

$$\int_0^{a_+}G(p_1(a,t),\cdots,p_n(a,t),t)\,\mathrm{d}a\leq 0,\forall\, t\in[0,t_1],$$

$$p_i(a,t_1)=p_i^0(a),\quad a\in(0,a_+),\ i=1,2,\cdots,n.$$

Applying the approaches in the preceding two sections to the above problem, we obtain that

Theorem 3. *If (p^*,β^*,t_1^*) is a solution of the above problem, then there exist $\lambda_0\geq 0$, and a function $\alpha(a)\in L^2(0,a_+;R^n)$ which is supporting on*

$$S=\{t\in[0,t_1^*]:\int_0^{a_+}G(p^*(a,t),t)\,da=F(p^*),\ F\text{ is given by (39) }\},$$

and a measure $dm(t)$, such that (33)-(34) hold, but q_i, $i=1,2,\cdots,n$, is the *solution of (44) corresponding to* $T = t_1^*$.

5. Approximate Controllability of the State System

In what follows, we seek conditions for the approximate controllability of the state system.

Definition 1. The system (2) is said to be approximately controllable if for any $\varepsilon > 0$ and a prescribed age distribution $\bar{p}(a) \in L_n^\infty(0, a_+)$ (i.e., the space of all of the n-dimensional functions essentially bounded on $(0, a_+)$), there exist a finite time $T > 0$ and a continuous function

$$\beta(t) \in L_n^\infty(0,T),\ 0 \le \beta_0 \le \beta_i(t) \le \beta^0,\ i = 1, 2, \cdots, n;\ t \in [0, T],$$

such that the corresponding solution of system (2) satisfies

$$\| p(\cdot, T) - \bar{p} \|_\infty \le \varepsilon.$$

For given $v = (v_1, v_2, \cdots, v_n) \in L_n^\infty((0, a_+) \times (0, \infty))$, $v_i(a,t) \ge 0$, $i = 1, 2, \cdots, n$, consider the linear system

$$\begin{array}{l}
\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)V_2(t)p_1, \\
\frac{\partial p_k}{\partial t} + \frac{\partial p_k}{\partial a} = -\mu_k(a,t)p_k + \lambda_{2k-2}(a,t)V_{k-1}(t)p_k \\
\qquad\qquad -\lambda_{2k-1}(a,t)V_{k+1}(t)p_k,\ k = 2, 3, \cdots, n-1, \\
\frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial a} = -\mu_n(a,t)p_n + \lambda_{2n-2}(a,t)V_{n-1}(t)p_n, \\
p_i(0,t) = \beta_i(t) \int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\, \mathrm{d}a, \\
p_i(a, 0) = p_{i0}(a), i = 1, 2, \cdots, n. \\
V_i(t) = \int_0^{a_+} v_i(a,t)\, \mathrm{d}a,\ (a,t) \in Q.
\end{array} \tag{45}$$

It follows from Theorem 6.25 in [3] that the following result is true:

Theorem 4. *If the conditions below hold:*

(1) $p_{i0}(a) \ge c_i > 0$, $\forall\, a \in [0, a_2], c_i$ *are constants,* $i = 1, 2, \cdots, n;$

(2) For any $\varepsilon > 0$, $\exp\{\int_0^{a_2} \mu_i(\rho, t+\rho)\, d\rho\} = O(e^{\varepsilon t})$, $\quad i = 1, 2, \cdots, n;$

(3)

$$\beta^0 > \lim_{t\to\infty} \inf \int_{a_1}^{a_2} m_1(s,t) \exp[-\int_0^s \mu_1(\rho, \rho - s + t)\, d\rho] ds,$$

and $v_{k-1}(a,t) \le y_{k-1}(a,t), k = 2, 3,\ \cdots, n,$

$$\beta^0 > \lim_{t\to\infty} \inf \int_{a_1}^{a_2} m_k(s,t) \exp\{\int_0^s [\lambda_{2k-2}(\rho, \rho - s + t)$$
$$\times \int_0^{a_+} y_{k-1}(a, \rho - s + t)\, da - \mu_k(\rho, \rho - s + t)]\, d\rho\} ds, \quad k = 2, 3, \cdots, n.$$

where y_1 is the solution of the following system

$$\begin{cases} \frac{\partial y_1}{\partial t} + \frac{\partial y_1}{\partial a} = -\mu_1(a,t)y_1, \\ y_1(0,t) = \beta^0 \int_{a_1}^{a_2} m_1(a,t)y_1(a,t)\,da, \\ y_1(a,0) = p_{10}(a). \end{cases}$$

and $y_k(k = 2, 3, \cdots, n)$ is the solution of the following system

$$\begin{cases} \frac{\partial y_k}{\partial t} + \frac{\partial y_k}{\partial a} = -\mu_k(a,t)y_k + \lambda_{2k-2}(a,t)y_k \int_0^{a_+} y_{k-1}(a,t)\,da, \\ y_k(0,t) = \beta_k(t) \int_{a_1}^{a_2} m_k(a,t)y_k(a,t)\,da, \\ y_k(a,0) = p_{k0}(a), \ k = 2, 3, \cdots, n, \ (a,t) \in Q, \end{cases}$$

(4) For any $\delta > 0$, there exists $m_0 > 0$, and

$$\int_{a_2-\delta}^{a_2} m_i(a,t)\,da \geq m_0, \quad i = 1, 2, \cdots, n,$$

whenever $t > 0$, then the system (45) is approximately controllable.

Define the operator

$$D: L_n^2((0,a_+) \times (0,\infty)) \to L_n^2((0,a_+) \times (0,\infty)), \quad Dv = p^v,$$

where p^v is the solution of (2) corresponding to $\beta_i = \beta_i^v$, β_i^v is the control function determined by the approximate controllability of the system (45).

Treating in a similar manner as that in the analysis of well-posedness, we are able to prove that

Theorem 5. *If the assumptions in Theorem 4 are satisfied, then the system (2) is approximately controllable.*

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