International Journal of Pure and Applied Mathematics
Volume 36 No. 3 2007, 303-311

ON A REVERSE HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract: This paper gives a reverse Hardy-Hilbert's integral inequality with some parameters and the equivalent forms. Some particular cases are considered.

AMS Subject Classification: 26D15
Key Words: Hardy-Hilbert's integral inequality, weight function, reverse inequality

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If $p>1, \frac{1}{p}+\frac{1}{q}=1,\ f,g\geq 0, 0<\int_0^\infty f^p(t)dt<\infty$ and $0<\int_0^\infty g^q(t)dt<\infty$, then (see [4])

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\frac{\pi}{p})}(\int_0^\infty f^p(t)dt)^{\frac{1}{p}}(\int_0^\infty g^q(t)dt)^{\frac{1}{q}}, \qquad (1)$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1) is named Hardy-Hilbert's integral inequality, which is important in analysis and its applications (see [6]). Recently, by introducing a parameter $\lambda>2-\min\{p,q\}$, Yang [8] gave a extension of (1) as:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy<k_\lambda(p)(\int_0^\infty t^{1-\lambda}f^p(t)dt)^{\frac{1}{p}}(\int_0^\infty t^{1-\lambda}g^q(t)dt)^{\frac{1}{q}}, \qquad (2)$$

where the constant $k_\lambda(p):=B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q})$ is the best possible, and the Beta function $B(u,v)$ is defined by (see [7])

Received: May 18, 2006

$$B(u,v)=\int_0^\infty \frac{1}{(1+t)^{u+v}}t^{u-1}dt\ \ (u,v>0). \tag{3}$$

Another new extensions of (1) are given in [10], [1], [2], [3].

In 2004, Yang [9] gave a reverse (2) as: for $0<p<1, \frac{1}{p}+\frac{1}{q}=1, 2-p<\lambda<2-q$,

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy > k_\lambda(p)(\int_0^\infty t^{1-\lambda}f^p(t)dt)^{\frac{1}{p}}(\int_0^\infty t^{1-\lambda}g^q(t)dt)^{\frac{1}{q}}, \tag{4}$$

where the constant $k_\lambda(p)$ is the best possible. In particular, for $\lambda=2$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^2}dxdy > (\int_0^\infty \frac{1}{t}f^p(t)dt)^{\frac{1}{p}}(\int_0^\infty \frac{1}{t}g^q(t)dt)^{\frac{1}{q}}. \tag{5}$$

If we replace $x^{\frac{1-\lambda}{p}}f(x)$ by $f(x)$, $y^{\frac{1-\lambda}{q}}g(y)$ by $g(y)$ in (4), setting $||f||_p=(\int_0^\infty f^p(t)dt)^{\frac{1}{p}}$ and $||g||_q=(\int_0^\infty g^q(t)dt)^{\frac{1}{q}}$, we have the equivalent form as:

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{\lambda-1}{p}}y^{\frac{\lambda-1}{q}}f(x)g(y)}{(x+y)^\lambda}dxdy > k_\lambda(p)||f||_p||g||_q. \tag{6}$$

In this paper, a best extension of (6) is given by introducing some parameters and obtaining the weight function. The equivalent forms and some particular cases are considered.

Theorem 1. *Suppose* $0<p<1, \frac{1}{p}+\frac{1}{q}=1, f,g\geq 0, 0<||f||_p<\infty$ *and* $0<||g||_q<\infty$. *If the set* $H=\{\lambda|\varphi+\psi=\lambda,\varphi,\psi>0\}\neq\Phi$, *then for* $\lambda\in H$, *we have*

$$\int_0^\infty\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda}f(x)g(y)dxdy > B(\varphi,\psi)||f||_p||g||_q, \tag{7}$$

where the constant factor $B(\varphi,\psi)$ *is the best possible.*

Proof. By the reverse $H\ddot{o}lder's$ inequality with weight (see [5]), we have

$$\begin{aligned} I &:= \int_0^\infty\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda}f(x)g(y)dxdy \\ &= \int_0^\infty\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda}[(\frac{x}{y})^{\frac{1}{pq}}f(x)][(\frac{y}{x})^{\frac{1}{pq}}g(y)]dxdy \\ &\geq \left\{\int_0^\infty[\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda}(\frac{x}{y})^{\frac{1}{q}}dy]f^p(x)dx\right\}^{\frac{1}{p}} \end{aligned}$$

$$\times \left\{ \int_0^\infty [\int_0^\infty \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} (\frac{y}{x})^{\frac{1}{p}} dx] g^q(y) dy \right\}^{\frac{1}{q}} . \quad (8)$$

If (8) takes the form of equality, then (see [5]), there exists constants A and B, such that they are not all zero, and $A(\frac{x}{y})^{\frac{1}{q}} f^p(x) = B(\frac{y}{x})^{\frac{1}{p}} g^q(y)$ a.e. in $(0,\infty)\times(0,\infty)$. Hence we find $Axf^p(x) = Byg^q(y)$ a.e. in $(0,\infty)\times(0,\infty)$, and there exists a constant C, such that $Axf^p(x) = C = Byg^q(y)$ a.e. in $(0,\infty)$. Without lose generality, suppose $A \neq 0$. It follows that $f^p(x) = \frac{C}{Ax}$ a.e. in $(0,\infty)$, which contradicts the fact that $0 < ||f||_p < \infty$. Setting the weight functions $\omega(q,\varphi,x)$ and $\omega(p,\psi,y)$ as

$$\omega(q,\varphi,x) := \int_0^\infty \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} (\frac{x}{y})^{\frac{1}{q}} dy\,,$$

$$\omega(p,\psi,y) := \int_0^\infty \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} (\frac{y}{x})^{\frac{1}{p}} dx \quad x,y \in (0,\infty), \quad (9)$$

then we may reduce (8) as

$$I > \left\{ \int_0^\infty \omega(q,\varphi,x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(p,\psi,y) g^q(y) dy \right\}^{\frac{1}{q}} . \quad (10)$$

Since we find

$$\omega(q,\varphi,x) = \int_0^\infty \frac{u^{\psi-1}}{(1+u)^\lambda} du = B(\varphi,\psi) = \varpi(p,\psi,y), \quad (11)$$

in view of (10), we have (7).

For $0 < \varepsilon < p\varphi$, setting $\widetilde{f}, \widetilde{g}$ as: $\widetilde{f}(x) = \widetilde{g}(x) = 0$, $x \in (0,1)$, $\widetilde{f}(x) = x^{\frac{-1-\varepsilon}{p}}$, $\widetilde{g}(x) = x^{\frac{-1-\varepsilon}{q}}$, $x \in [1,\infty)$, we obtain $\varepsilon||\widetilde{f}||_p||\widetilde{g}||_q = 1$ and

$$\begin{aligned} \widetilde{I} \quad : \quad &= \int_0^\infty \int_0^\infty \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} \widetilde{f}(x)\widetilde{g}(y) dx dy < \int_1^\infty [\int_0^\infty \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} dx] dy \\ &= \int_1^\infty y^{-1-\varepsilon} [\int_0^\infty \frac{u^{\frac{-\varepsilon}{p}+\varphi-1}}{(1+u)^\lambda} du] dy = \frac{1}{\varepsilon} B(\varphi - \frac{\varepsilon}{p}, \psi + \frac{\varepsilon}{p}). \quad (12) \end{aligned}$$

If there exists a real number $k \geq B(\varphi,\psi)$, such that (7) is still valid if we replace $B(\varphi,\psi)$ by k, then by (12), we have

$$B(\varphi - \frac{\varepsilon}{p}, \psi + \frac{\varepsilon}{p}) > \varepsilon\widetilde{I} > \varepsilon k ||\widetilde{f}||_p ||\widetilde{g}||_q = k,$$

and then $B(\varphi,\psi) \geq k\ \ (\varepsilon \to 0^+)$. It follows that the constant factor $k = B(\varphi,\psi)$ in (7) is the best possible. The theorem is proved. □

Theorem 2. *Suppose* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0, 0 < ||f||_p < \infty$ *and* $0 < ||g||_q < \infty$*. If the set* $H = \{\lambda | \varphi + \psi = \lambda, \varphi, \psi > 0\} \neq \Phi$*, then for* $\lambda \in H$*, we have*

$$\left\{\int_0^\infty y^{p\psi-1}[\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > B(\varphi,\psi)||f||_p, \tag{13}$$

$$\left\{\int_0^\infty x^{q\varphi-1}[\int_0^\infty \frac{y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > B(\varphi,\psi)||g||_q, \tag{14}$$

where the same constant factor $B(\varphi,\psi)$ *is the best possible. Inequalities (7), (13) and (14) are equivalent.*

Proof. Since $||f||_p > 0$, there exists $T > 0$, such that $\int_0^T f^p(x)dx > 0$, and

$$g(y,T) := [\int_0^T \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)dx]^{p-1} > 0.$$

By (7), we obtain

$$\begin{aligned}\int_0^T g^q(y,T)dy &= \int_0^T [\int_0^T \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)dx]^p dy \\ &= \int_0^T \int_0^T \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)g(y,T)dxdy \\ &> B(\varphi,\psi)(\int_0^T f^p(x)dx)^{\frac{1}{p}} \{\int_0^T g^q(y,T)dy\}^{\frac{1}{q}}, \end{aligned} \tag{15}$$

$$\begin{aligned}\{\int_0^T g^q(y,T)dy\}^{\frac{1}{p}} &= \left\{\int_0^T [\int_0^T \frac{x^{\varphi-\frac{1}{q}} y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} \\ &> B(\varphi,\psi)(\int_0^T f^p(x)dx)^{\frac{1}{p}} > 0. \end{aligned} \tag{16}$$

Setting $T \to \infty$ in (15) and (16), if $\int_0^\infty g^q(y,\infty)dy = \infty$, (16) takes the form of strict inequality since $||f||_p < \infty$, if $0 < \int_0^\infty g^q(y,\infty)dy < \infty$, (15) takes the form of strict inequality by using (7), so does (16). Hence (13) is valid.

On the other-hand, if (13) is valid, then by the reverse $H\ddot{o}lder's$ inequality (see [5]), we have

$$\begin{aligned} I &= \int_0^\infty [\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)dx][g(y)]dy \\ &\geq \left\{\int_0^\infty [\int_0^\infty \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}. \quad (17) \end{aligned}$$

Hence by (13), we have (7). It follows that (7) and (13) are equivalent. If the constant factor in (13) is not the best possible, then by (17), we can make a contradiction that the constant factor in (7) is the best possible.

By the same way, setting

$$f(x,T) := [\int_0^T \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} g(y)dy]^{q-1} > 0,$$

by (7), we can get the following inequalities similar to (15) and (16):

$$\begin{aligned} \int_0^T f^p(x,T)dx &= \int_0^T [\int_0^T \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} g(y)dy]^q dy \\ &> B(\varphi,\psi)(\int_0^T f^p(x,T)dx)^{\frac{1}{p}}\{\int_0^T g^q(y)dy\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \{\int_0^T f^p(x,T)dx\}^{\frac{1}{q}} &= \left\{\int_0^T [\int_0^T \frac{x^{\varphi-\frac{1}{q}}y^{\psi-\frac{1}{p}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} \\ &> B(\varphi,\psi)(\int_0^T g^q(y)dy)^{\frac{1}{p}} > 0, \end{aligned}$$

and then for $T \to \infty$, we have (17). On the other-hand, by using the reverse $H\ddot{o}lder's$ inequality, we can conclude that (17) educes to (7). Hence (17) is equivalent to (7). We still can show that the constant factor in (17) is the best possible from the equivalence of (17) and (7). It follows that inequalities (7), (16) and (17) are equivalent. The theorem is proved. □

In the following, we set $k_\lambda(p,r) := B(\frac{\lambda-1}{p}+\frac{1}{r}, \frac{\lambda-1}{q}+\frac{1}{s})(= k_\lambda(q,s))$, and omit the assumption that $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, f,g \geq 0, 0 < ||f||_p < \infty$, $0 < ||g||_q < \infty$ and $r > 1, \frac{1}{r}+\frac{1}{s} = 1$, and the result that the constant factors in the new inequalities are all the best possible.

(a) Setting $\varphi = \frac{\lambda}{r}, \psi = \frac{\lambda}{s}$, then $H = \{\lambda | \lambda > 0\} \neq \Phi$, we have

Corollary 1. *If $\lambda > 0$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\lambda}{r}-\frac{1}{q}} y^{\frac{\lambda}{s}-\frac{1}{p}}}{(x+y)^\lambda} f(x)g(y)dxdy > B(\frac{\lambda}{r}, \frac{\lambda}{s})||f||_p||g||_q \,, \tag{18}$$

$$\left\{\int_0^\infty y^{\frac{p\lambda}{s}-1}[\int_0^\infty \frac{x^{\frac{\lambda}{r}-\frac{1}{q}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > B(\frac{\lambda}{r}, \frac{\lambda}{s})||f||_p \,, \tag{19}$$

$$\left\{\int_0^\infty x^{\frac{q\lambda}{r}-1}[\int_0^\infty \frac{y^{\frac{\lambda}{s}-\frac{1}{p}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > B(\frac{\lambda}{r}, \frac{\lambda}{s})||g||_q. \tag{20}$$

(b) Setting $\varphi = \frac{\lambda-1}{p} + \frac{1}{r}, \psi = \frac{\lambda-1}{q} + \frac{1}{s}$, we have $H = \{\lambda | 1 - \frac{p}{r} < \lambda < 1 - \frac{q}{s}\} \neq \Phi$, and

Corollary 2. *If $1 - \frac{p}{r} < \lambda < 1 - \frac{q}{s}$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\lambda}{p}-\frac{1}{s}} y^{\frac{\lambda}{q}-\frac{1}{r}}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(p,r)||f||_p||g||_q \,, \tag{21}$$

$$\left\{\int_0^\infty y^{p(\frac{\lambda}{q}-\frac{1}{r})}[\int_0^\infty \frac{x^{\frac{\lambda}{p}-\frac{1}{s}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > k_\lambda(p,r)||f||_p \,, \tag{22}$$

$$\left\{\int_0^\infty x^{q(\frac{\lambda}{p}-\frac{1}{s})}[\int_0^\infty \frac{y^{\frac{\lambda}{q}-\frac{1}{r}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > k_\lambda(p,r)||g||_q. \tag{23}$$

Setting $\varphi = \frac{\lambda-1}{q} + \frac{1}{s}, \psi = \frac{\lambda-1}{p} + \frac{1}{r}$, we have $H = \{\lambda | 1 - \frac{p}{r} < \lambda < 1 - \frac{q}{s}\} \neq \Phi$ and

Corollary 3. *If $1 - \frac{p}{r} < \lambda < 1 - \frac{q}{s}$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{1}{s}+\frac{\lambda-2}{q}} y^{\frac{1}{r}+\frac{\lambda-2}{p}}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(p,r)||f||_p||g||_q \,, \tag{24}$$

$$\left\{\int_0^\infty y^{\frac{p}{r}+\lambda-2}[\int_0^\infty \frac{x^{\frac{1}{s}+\frac{\lambda-2}{q}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > k_\lambda(p,r)||f||_p \,, \tag{25}$$

$$\left\{\int_0^\infty x^{\frac{q}{s}+\lambda-2}[\int_0^\infty \frac{y^{\frac{1}{r}+\frac{\lambda-2}{p}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > k_\lambda(p,r)||g||_q. \qquad (26)$$

Setting $\varphi = \frac{\lambda-1}{r} + \frac{1}{p}, \psi = \frac{\lambda-1}{s} + \frac{1}{q}$, we have $H = \{\lambda|\lambda > 1 - \frac{s}{q}\} \neq \Phi$, and
Corollary 4. *If $\lambda > 1 - \frac{s}{q}$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\lambda-1}{r}+1-\frac{2}{q}} y^{\frac{\lambda-1}{s}+1-\frac{2}{p}}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(r,p)||f||_p||g||_q\,, \qquad (27)$$

$$\left\{\int_0^\infty y^{p(\frac{\lambda-1}{s}+1)-2}[\int_0^\infty \frac{x^{\frac{\lambda-1}{r}+1-\frac{2}{q}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > k_\lambda(r,p)||f||_p\,, \qquad (28)$$

$$\left\{\int_0^\infty x^{q(\frac{\lambda-1}{r}+1)-2}[\int_0^\infty \frac{y^{\frac{\lambda-1}{s}+1-\frac{2}{p}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > k_\lambda(r,p)||g||_q. \qquad (29)$$

Setting $\varphi = \frac{\lambda-1}{s} + \frac{1}{q}, \psi = \frac{\lambda-1}{r} + \frac{1}{p}$, we have $H = \{\lambda|\lambda > 1 - \frac{s}{q}\} \neq \Phi$ and
Corollary 5. *If $\lambda > 1 - \frac{s}{q}$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\lambda-1}{s}} y^{\frac{\lambda-1}{r}}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(r,p)||f||_p||g||_q\,, \qquad (30)$$

$$\left\{\int_0^\infty y^{p\frac{\lambda-1}{r}}[\int_0^\infty \frac{x^{\frac{\lambda-1}{s}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > k_\lambda(r,p)||f||_p\,, \qquad (31)$$

$$\left\{\int_0^\infty x^{q\frac{\lambda-1}{s}}[\int_0^\infty \frac{y^{\frac{\lambda-1}{r}}}{(x+y)^\lambda} g(y)dy]^q dx\right\}^{\frac{1}{q}} > k_\lambda(r,p)||g||_q. \qquad (32)$$

Setting $\varphi = \frac{\lambda-1}{r} + \frac{1}{2}, \psi = \frac{\lambda-1}{s} + \frac{1}{2}$, we have $H = \{\lambda|\lambda > 1 - \frac{1}{2}\min\{r,s\}\} \neq \Phi$ and

Corollary 6. *If $\lambda > 1 - \frac{1}{2}\min\{r,s\}$, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{\lambda-1}{r}+\frac{q-2}{2q}} y^{\frac{\lambda-1}{s}+\frac{p-2}{2p}}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(r,2)||f||_p||g||_q\,, \qquad (33)$$

$$\left\{\int_0^\infty y^{\frac{p(\lambda-1)}{s}+\frac{p-2}{2}}[\int_0^\infty \frac{x^{\frac{\lambda-1}{r}+\frac{q-2}{2q}}}{(x+y)^\lambda} f(x)dx]^p dy\right\}^{\frac{1}{p}} > k_\lambda(r,2)||f||_p\,, \qquad (34)$$

$$\left\{\int_0^\infty x^{\frac{q(\lambda-1)}{r}+\frac{q-2}{2}}\left[\int_0^\infty \frac{y^{\frac{\lambda-1}{s}+\frac{p-2}{2p}}}{(x+y)^\lambda} g(y)dy\right]^q dx\right\}^{\frac{1}{q}} > k_\lambda(r,2)||g||_q. \tag{35}$$

Setting $\varphi = \frac{\lambda-1}{q}+\frac{1}{p}$, $\psi = \frac{\lambda-1}{p}+\frac{1}{q}$, and $k_\lambda(p) := k_\lambda(q,p)$, we have $H = \{\lambda|2-p<\lambda<2-q\} \neq \Phi$ and

Corollary 7. *If $2-p<\lambda<2-q$, we have the following equivalent inequalities:*

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{\lambda-3}{q}+1}y^{\frac{\lambda-3}{p}+1}}{(x+y)^\lambda} f(x)g(y)dxdy > k_\lambda(p)||f||_p||g||_q\,, \tag{36}$$

$$\left\{\int_0^\infty y^{\lambda+p-3}\left[\int_0^\infty \frac{x^{\frac{\lambda-3}{q}+1}}{(x+y)^\lambda} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} > k_\lambda(p)||f||_p\,, \tag{37}$$

$$\left\{\int_0^\infty x^{\lambda+q-3}\left[\int_0^\infty \frac{y^{\frac{\lambda-3}{p}+1}}{(x+y)^\lambda} g(y)dy\right]^q dx\right\}^{\frac{1}{q}} > k_\lambda(p)||g||_q. \tag{38}$$

Remarks. (a) For $\varphi = \frac{\lambda-1}{p}+\frac{1}{q}, \psi = \frac{\lambda-1}{q}+\frac{1}{p}$, we have $H = \{\lambda|2-p<\lambda<2-q\} \neq \Phi$. If $2-p<\lambda<2-q$, then by (7), we have (6), by (13) and (14), we have the equivalent forms of (6) as

$$\left\{\int_0^\infty y^{(p-1)(\lambda-1)}\left[\int_0^\infty \frac{x^{\frac{\lambda-1}{p}}}{(x+y)^\lambda} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} > k_\lambda(p)||f||_p\,, \tag{39}$$

$$\left\{\int_0^\infty x^{(q-1)(\lambda-1)}\left[\int_0^\infty \frac{y^{\frac{\lambda-1}{q}}}{(x+y)^\lambda} g(y)dy\right]^q dx\right\}^{\frac{1}{q}} > k_\lambda(p)||g||_q. \tag{40}$$

Hence inequality (7) is a best extension of (6).

(b) For $\lambda = 1$ in (21) and (24), we have two new reverse inequalities of (1) as follows:

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{p}-\frac{1}{s}}y^{\frac{1}{q}-\frac{1}{r}}}{x+y} f(x)g(y)dxdy > \frac{\pi}{\sin(\pi/r)}||f||_p||g||_q\,, \tag{41}$$

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{s}-\frac{1}{q}}y^{\frac{1}{r}-\frac{1}{p}}}{x+y} f(x)g(y)dxdy > \frac{\pi}{\sin(\pi/r)}||f||_p||g||_q. \tag{42}$$

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