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ADDENDUM TO: "BLOCKS OF CONSECUTIVE INTEGERS IN SUMSETS $(A+B)_t$"

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Abstract: Let $A, B \subseteq \{1, \cdots, n\}$. For $m \in \mathbf{Z}$, let $r_{A,B}(m)$ be the cardinality of the set of ordered pairs $(a, b) \in A \times B$ such that $a+b = m$. For $t \geq 1$, denote by $(A+B)_t$ the set of the elements m for which $r_{A,B}(m) \geq t$. Recently Guo and Chen proved that for any subsets $A, B \subseteq \{1, \cdots, n\}$ such that $|A|+|B| \geq (4n+4t-4)/3$, the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$ unless $3|(n+t-1)$ and $A = B = [1, (n+t-1)/3] \cup [\,(2n-t+4)/3, n]$. In this paper we give an addendum to this result by dealing with the case when $|A|+|B| = (4n+4t-5)/3$.

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1. Introduction

Let G be an Abelian group written additively. For two subsets A and B of G, the sumset and the restricted sumset of A and B are defined as

$$A+B = \{a+b : a \in A, b \in B\}, \quad A \hat{+} B = \{a+b : a \in A, b \in B, a \neq b\},$$

respectively. We abbreviate $2A = A+A$, $2^\wedge A = A \hat{+} A$, and we use the standard notation $|A|$ for the cardinality of the set A. For $m \in G$, let $r_{A,B}(m)$ be the cardinality of the set of ordered pairs $(a,\, b) \in A \times B$ such that $a+b = m$. For $t \geq 1$, denote by $(A+B)_t$ the set of the elements $m \in A+B$ for which $r_{A,B}(m) \geq t$. Obviously, $(A+B)_1 = A+B$ and $(A+A)_2 = 2^\wedge A$ for $A \subseteq \mathbf{Z}$.

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Pollard [6] first studied the sumset $(A+B)_t$ and extended the Cauchy-Davenport Theorem by showing that for $A, B \subseteq \mathbf{Z}/p\mathbf{Z}$ and $t \le \min(|A|, |B|)$,

$$\sum_{i=1}^{t} |(A+B)_i| \ge \min(tp,\, t(|A|+|B|-t)).$$

Caldeira and Dias da Silva gave in [2] an extension of the Pollard's Theorem to an arbitrary field, and in [1] an analogue for restricted sums.

For a positive integer n, let $[1,n] = \{1,2,\cdots,n\}$. Many problems in combinatorial number theory have the following character: for a given arithmetic property P, find $f(n)$ such that if $A \subseteq [1,n]$ with $|A| > f(n)$, then A has property P. For related research, one may refer to [3-5, 7] and the references therein. Guo and Chen [3] studied the blocks of consecutive integers in $(A+B)_t$ and proved the following result.

Theorem 1. *Let n, t be positive integers. For any $A, B \subseteq [1,n]$ such that $|A|+|B| \ge (4n+4t-4)/3$, the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$ unless $3|(n+t-1)$ and*

$$A = B = [1,(n+t-1)/3\,] \cup [\,(2n-t+4)/3, n\,].$$

In this paper, we give an addendum to Theorem 1 by proving the following theorem.

Theorem 2. *Let n, t be positive integers. Let $A, B \subseteq [\,1,n\,]$ such that $|A|+|B| = (4n+4t-5)/3$ and $|A| \ge |B|$. Then the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$ unless one of the following cases occurs.*

(i) $A = [1,(n+t+1)/3\,] \cup [\,(2n-t+5)/3, n\,]$, $B = ([1,(n+t+1)/3\,] \cup [\,(2n-t+5)/3, n\,]) \setminus \{\,(n-2t+4)/3+i\,\}$, *where* $i = 0,1,\cdots,t-1$;

(ii) $A = [1,(n+t-2)/3\,] \cup [\,(2n-t+2)/3, n\,]$, $B = [1,(n+t-2)/3\,] \cup ([\,(2n-t+2)/3, n\,] \setminus \{(2n-t+2)/3+i\})$, *where* $i = 0,1,\cdots,t-1$;

(iii) $A = [1,(n+t-2)/3\,] \cup [\,(2n-t+5)/3, n\,] \cup \{(n+t-2)/3+i\}$, $B = [1,(n+t-2)/3\,] \cup [\,(2n-t+5)/3, n\,]$, *where* $i = 1,2,\cdots,(n-2t+4)/3$.

2. Proof

In order to complete the proof of Theorem 2, we need the following lemmas which were proved in [3].

Lemma 1. *Let n, t be positive integers, and let $A, B \subseteq [1, n]$ such that $|A| + |B| \geq n + t$. Then*

$$[\,2n + 1 - |A| - |B| + t,\ \ |A| + |B| + 1 - t\,] \subseteq (A + B)_t.$$

Lemma 2. *Let A and B be two sets of integers, and let α, β be two integers such that $\alpha \leq \beta$. If $|[\alpha, \beta] \cap A| + |[\alpha, \beta] \cap B| > \beta - \alpha + t$, then $\alpha + \beta \in (A + B)_t$.*

Lemma 3. *Let n, t be positive integers, and let $A, B \subseteq [1, n]$ such that $|A| + |B| \geq n + 2t - 1$. Let $m \in [1, n]$ be the largest integer such that $m \notin (A + B)_t$ and $l \in [n + 2, 2n + 1]$ be the least integer such that $l \notin (A + B)_t$. Then*

$$[l - 1 + 2t - |A| - |B|,\, l + 1 - 2t + |A| + |B| - 2n] \subseteq (A + B)_t,$$

$$[m - 1 + 2t + 2n - |A| - |B|,\, m + 1 - 2t + |A| + |B|] \subseteq (A + B)_t.$$

Proof of Theorem 2. Since $(4n + 4t - 5)/3 = |A| + |B| \leq 2n$, it follows that $n \geq 2t - 5/2$ and $3 \mid (n + t - 2)$. Thus $n \geq 2t - 1$. In the case when $n = 2t - 1$, we have $|A| + |B| = 2n - 1$. Therefore there exists $i \in [\,1,\, n\,]$ such that

$$A = [\,1,\, n\,], \quad B = [\,1,\, n\,] \setminus \{i\}.$$

If $i > (n + 1)/2$, then, by direct calculation, we have

$$(A + B)_t = [\,(n + 1)/2 + 1,\, 2n - (n + 1)/2\,].$$

If $i < (n + 1)/2$, then, by direct calculation, we have

$$(A + B)_t = [\,(n + 1)/2 + 2,\, 2n + 1 - (n + 1)/2\,].$$

If $i = (n + 1)/2$, then we have

$$A = [\,1,\, n\,], \quad B = [\,1,\, n\,] \setminus \{(n + 1)/2\}.$$

This is (iii). Combining the above arguments, we get the proof of Theorem 2 in this case.

In the case when $n > 2t - 1$, by $3 \mid (n + t - 2)$ we have $n \geq 2t + 2$. It follows that $|A| + |B| \geq n + 2t - 1$. Let $l \in [\,n + 2, 2n + 1\,]$ be the least integer such that $l \notin (A + B)_t$. By Lemma 1 and Lemma 3, we have $l \geq |A| + |B| + 2 - t$, and

$$[\,l - 1 + 2t - |A| - |B|,\, l + 1 - 2t + |A| + |B| - 2n\,] \cup [\,2n + 1 - |A| - |B| + t,\ l - 1\,] \subseteq (A + B)_t. \quad (1)$$

It is sufficient to prove that

$$[\, l-1+2t-|A|-|B|,\, l-1\,] \subseteq (A+B)_t.$$

By $3 \mid (n+t-2)$, we may denote $n+t-2=3u$. Then $|A|+|B|=4u+1$.

Case 1. $l \ge |A|+|B|+4-t$. Then

$$l+1-2t+|A|+|B|-2n+1 \ge 2n+1-|A|-|B|+t.$$

Hence

$$[\, l-1+2t-|A|-|B|,\, l-1\,] \subseteq (A+B)_t.$$

Case 2. $l=|A|+|B|+3-t$. Then $l=4u+4-t$ and (1) becomes

$$[\, t+2,\, 2u-t+2\,] \cup [\, 2u-t+4,\, 4u+3-t\,] \subseteq (A+B)_t.$$

It is enough to prove that $2u-t+3 \in (A+B)_t$ unless (i) occurs.

Since $4u+4-t=l \notin (A+B)_t$, by Lemma 2 we have

$$|A \cap [\, u+2, 3u+2-t\,]| + |B \cap [\, u+2, 3u+2-t\,]| \le 2u.$$

By $|A|+|B|=4u+1$, we have

$$|A \cap [\, 1, u+1\,]| + |B \cap [\, 1, u+1\,]| \ge 2u+1.$$

Subcase 2.1. $|A\cap[\,1,u+1\,]|+|B\cap[\,1,u+1\,]|=2u+1$ and $|A\cap[\,1,u+1\,]|=u+1$. Then there exists $k \in [\,1,\ u+1\,]$ such that

$$[\,1,\ u+1\,] \subseteq A, \quad [\,1,\ u+1\,] \setminus \{k\} \subseteq B.$$

When $k<u-t+2$, we have

$$|[\,u-t+2,u+1\,]\cap A|+|[\,u-t+2,u+1\,]\cap B| = 2t > (u+1)-(u-t+2)+t.$$

It follows from Lemma 2 that $2u-t+3 \in (A+B)_t$.

When $k \ge u-t+2$, by

$$\begin{aligned} 2u-t+3 &= u+1+(u-t+2) = u+(u-t+3) = \cdots \\ &= 2u-t-k+4+(k-1) = 2u-t-k+2+(k+1) \\ &= 2u-t-k+1+(k+2) = \cdots \\ &= u-t+3+u = u-t+2+(u+1), \end{aligned}$$

we have the number of representations of $2u-t+3$

$$r_{[\,1,u+1\,],[\,1,u+1\,]\setminus\{k\}}(2u-t+3) = t-1.$$

If $2u-t+3 \notin (A+B)_t$, then $u+2 \notin A\cup B$, $u+3 \notin A\cup B$, $\cdots$, $2u-t+2 \notin A\cup B$. By $|A|+|B| = 4u+1$, we have

$$\begin{aligned} A &= [\,1, u+1\,] \cup [\,2u-t+3, 3u+2-t\,], \\ B &= ([\,1, u+1\,] \cup [\,2u-t+3, 3u+2-t\,]) \setminus \{k\}. \end{aligned}$$

Let $k = u-t+2+i$. Then $i \in [\,0,\ t-1\,]$ and

$$\begin{aligned} A &= [\,1, (n+t+1)/3\,] \cup [\,(2n-t+5)/3, n\,], \\ B &= ([\,1, (n+t+1)/3\,] \cup [\,(2n-t+5)/3, n\,]) \setminus \{(n-2t+4)/3+i\}. \end{aligned}$$

This is (i). Hence, if (i) does not occur, then $2u-t+3 \in (A+B)_t$.

Subcase 2.2. $|A\cap[\,1,u+1\,]|+|B\cap[\,1,u+1\,]| = 2u+1$ and $|B\cap[\,1,u+1\,]| = u+1$. Then there exists $k \in [\,1,\ u+1\,]$ such that

$$[\,1,\ u+1\,] \setminus \{k\} \subseteq A, \quad [\,1,\ u+1\,] \subseteq B.$$

When $k < u-t+2$, similarly to Subcase 2.1, we have $2u-t+3 \in (A+B)_t$. When $k \geq u-t+2$, if $2u-t+3 \notin (A+B)_t$, then similarly to Subcase 2.1, we get $|A| < |B|$. This contradicts the condition $|A| \geq |B|$. Hence $2u-t+3 \in (A+B)_t$.

Subcase 2.3. $|A\cap[\,1,u+1\,]|+|B\cap[\,1,u+1\,]| = 2u+2$. Then $[\,1,\ u+1\,] \subseteq A$ and $[\,1,\ u+1\,] \subseteq B$. Therefore

$$|[\,u-t+2, u+1\,]\cap A| + |[\,u-t+2, u+1\,]\cap B| = 2t > (u+1)-(u-t+2)+t.$$

By Lemma 2, we have $2u-t+3 \in (A+B)_t$.

Case 3. $l = |A|+|B|+2-t$. Then $l = 4u+3-t$ and (1) becomes

$$[\,t+1,\ 2u-t+1\,] \cup [\,2u-t+4,\ 4u+2-t\,] \subseteq (A+B)_t.$$

It is enough to prove that $\{\,2u-t+2, 2u-t+3\,\} \subseteq (A+B)_t$ unless one of (ii) and (iii) occurs.

Since $4u+3-t = l \notin (A+B)_t$, it follows from Lemma 2 that

$$|A\cap[\,u+1, 3u+2-t\,]| + |B\cap[\,u+1, 3u+2-t\,]| \leq 2u+1-t+t = 2u+1.$$

By $|A|+|B| = 4u+1$, we have

$$|A\cap[\,1,u\,]| + |B\cap[\,1,u\,]| \geq 2u,$$

and so $[\,1,\ u\,] \subseteq A$ and $[\,1,\ u\,] \subseteq B$.

In the case when $t=1$, if $2u+1 \notin (A+B)_1$, then $u+1 \notin A\cup B$, $u+2 \notin A\cup B$, $\cdots$, $2u \notin A \cup B$. Since $l = 4u+2 \notin (A+B)_1$, it follows that $2u+1 \notin A \cap B$. By $|A|+|B| = 4u+1$ and $|A| \geq |B|$, we have

$$A = [\,1,u\,] \cup [\,2u+1, 3u+1\,], \quad B = [\,1,u\,] \cup [\,2u+2, 3u+1\,].$$

This is (ii). Hence, if (ii) does not occur, then $2u+1 \in (A+B)_1$.

If $2u+2 \notin (A+B)_1$, then $u+2 \notin A \cup B$, $\cdots$, $2u+1 \notin A \cup B$. Since $l = 4u+2 \notin (A+B)_1$, it follows that

$$|A \cap \{\,u+1, 3u+1\,\}| + |B \cap \{\,u+1, 3u+1\,\}| \leq 2.$$

By $|A|+|B| = 4u+1$, we have

$$|A \cap [\,2u+2, 3u\,]| + |A \cap [\,2u+2, 3u\,]| \geq 4u+1-2u-2 = 2u-1,$$

a contradiction. Hence $2u+2 \in (A+B)_1$.

In the case when $t \geq 2$, by direct calculation, we have $r_{[\,1,u\,],[\,1,u\,]}(2u-t+2) = t-1$ and $r_{[\,1,u\,],[\,1,u\,]}(2u-t+3) = t-2$.

Assume that $2u-t+2 \notin (A+B)_t$. By $[\,1,\ u\,] \subseteq A$ and $[\,1,\ u\,] \subseteq B$, we have $u+1 \notin A\cup B$, $u+2 \notin A\cup B$, $\cdots$, $2u-t+1 \notin A\cup B$. Since $l = 4u+3-t \notin (A+B)_t$, it follows from Lemma 2 that

$$|[\,2u-t+2, 2u+1\,] \cap A| + |[\,2u-t+2, 2u+1\,] \cap B| \leq 2u+1-(2u-t+2)+t = 2t-1.$$

By $|A|+|B| = 4u+1$, we have

$$\begin{aligned} &|[\,2u+2, 3u+2-t\,] \cap A| + |[\,2u+2, 3u+2-t\,] \cap B| \\ &\qquad\qquad \geq 4u+1-2u-(2t-1) = 2u-2t+2. \end{aligned}$$

It follows that $[\,2u+2, 3u+2-t\,] \subseteq A$, $[\,2u+2, 3u+2-t\,] \subseteq A$ and

$$|[\,2u-t+2, 2u+1\,] \cap A| + |[\,2u-t+2, 2u+1\,] \cap B| = 2t-1.$$

By $|A| \geq |B|$, there exists $i \in [\,0,\ t-1]$ such that

$$\begin{aligned} A &= [\,1,u\,] \cup [\,2u-t+2, 3u+2-t\,], \\ B &= [\,1,u\,] \cup ([\,2u-t+2, 3u+2-t\,] \backslash \{2u-t+2+i\}). \end{aligned}$$

Namely

$$A = [\,1, (n+t-2)/3\,] \cup [\,(2n-t+2)/3, n\,],$$

$$B = [1,(n+t-2)/3] \cup ([(2n-t+2)/3, n] \setminus \{(2n-t+2)/3+i\}).$$

This is (ii). Hence if (ii) does not occur, then $2u-t+2 \in (A+B)_t$.

Assume that $2u-t+3 \notin (A+B)_t$. By $r_{[1,u],[1,u]}(2u-t+3) = t-2$ and $|A|+|B| = 4u+1$ we have

$$|A \cap [u+1, 2u+2-t]| + |B \cap [u+1, 2u+2-t]| = 1,$$

$$[2u+3-t,\, 3u+2-t] \subseteq A, \; [2u+3-t,\, 3u+2-t] \subseteq B.$$

Hence, there exists $i \in [1,\, u-t+2]$ such that

$$A = [1,u] \cup ([2u-t+3, 3u+2-t] \cup \{u+i\}),$$
$$B = [1,u] \cup [2u-t+3, 3u+2-t].$$

Namely

$$A = ([1,(n+t-2)/3] \cup [(2n-t+5)/3, n]) \cup \{(n+t-2)/3+i\},$$
$$B = [1,(n+t-2)/3] \cup [(2n-t+5)/3, n].$$

This is one of (iii). Hence if none of (iii) occurs, then $2u-t+3 \in (A+B)_t$. This completes the proof of Theorem 2. □

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