International Journal of Pure and Applied Mathematics
Volume 36 No. 3 2007, 321-336

A PROJECTION-TYPE METHOD FOR THE GENERALIZED LINEAR COMPLEMENTARITY PROBLEM OVER A POLYHEDRAL CONE

Hongchun Sun[1 §], Houchun Zhou[2], Qingjun Ren[3]
[1,2,3]Department of Mathematics
Linyi Teachers University
Linyi, Shandong, 276005, P.R. CHINA
[1]e-mail: sunhc68@126.com

Abstract: In this paper, we propose a projection-type method for the generalized linear complementarity problem (GLCP) over a polyhedral cone, which ensures that the corrector stepsizes and predictor stepsizes both have a uniformly positive bound from below, under the suitable conditions, we prove its global convergence. Furthermore, the error bound for GLCP is also given, based on which we prove that the method has a Q-linear convergence rate. Some numerical experiments of the method are also reported in this paper.

AMS Subject Classification: 65H10, 90C33, 90C30
Key Words: GLCP, the projection-type method, error bound, the global Q-linear convergence

1. Introduction

Let $F(x) = Mx + p, G(x) = Nx + q$, where $M, N \in R^{m\times n}$, $p, q \in R^m$. The generalized linear complementarity problem, abbreviated as GLCP, is to find a vector $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \quad G(x^*) \in \mathcal{K}^0, \quad F(x^*)^\top G(x^*) = 0, \tag{1.1}$$

where $\mathcal{K}$ is a polyhedral cone in R^m, that is, there exists $A \in R^{s\times m}, B \in R^{t\times m}$, such that $\mathcal{K} = \{v \in R^m \mid Av \geq 0,\ Bv = 0\}$. It is easy to verify that its polar

Received: June 2, 2006

§Correspondence author

cone $\mathcal{K}^\circ$ assumes the following from (see [1, 2, 15])

$$\mathcal{K}^\circ = \{u \in R^m \mid u = A^\top \lambda_1 + B^\top \lambda_2, \lambda_1 \in R^s_+, \lambda_2 \in R^t\}.$$

Throughout this paper, the solution set of the GLCP is denoted by X^*, which is always assumed to be nonempty.

The GLCP is a special case of the generalized nonlinear complementarity over a polyhedral cone (GCP) which was firstly considered in [1]. The GCP plays a significant role in economics, operation research, and nonlinear analysis, etc. and has been received much attention of researchers (see [1, 2, 3, 7, 8, 13, 15, 17, 19]).

In recent years, many effective methods have been proposed for solving GLCP. The basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem (see [1, 2, 7, 13, 15, 17, 19]). Different from the algorithms listed above, we propose the projection-type method in this paper, to solve GLCP based on the algorithm given by Noor and Wang in [9] for solving the general variational inequalities, and we establish the global convergence under suitable condition. Furthermore, using the technique of error bound for the variational inequalities developed by Xiu (see [16]) and Solodov (see [12]), we present a error bound for GLCP under the same condition, and apply these new error bound in linear convergence rate analysis of the proposed algorithm, and give some numerical experiments report.

Some notations used in this paper are in order. The norm $\|\cdot\|$ denotes the Euclidean 2-norm; R^n_+ denotes the nonnegative orthant in R^n; we denote the transpose of a matrix M by $M^\top$.

2. Preliminary

In this section, we will establish an equivalent reformulation of the GLCP, i.e., convert the GLCP into a variational inequality problem, and state some well known properties of the projection operator. Now, we give the following assumptions which is crucial to our method.

Assumption 2.1. (i) The matrix M is column-full-rank; (ii) The matrix NM_L^{-1} is positive semi-definite in X; here M and N are the matrices defined in (1.1) and $M_L^{-1} = (M^\top M)^{-1}M^\top$, $X = \{x \in R^n | A(Mx+p) \geq 0,\ B(Mx+p) = 0\}$.

By Assumption 2.1 (i), it is easy to deduce that $F(x)$ is invertible and

$F^{-1}(x) = M_L^{-1}x - M_L^{-1}p$, and we have

$$(1/\|M\|)\|F(u) - F(v)\| \leq \|u - v\| \leq \|M_L^{-1}\|\|F(u) - F(v)\|. \tag{2.1}$$

By Assumption 2.1 $(i) - (ii)$ again, we know that there exist $\beta > 0$ such that

$$\begin{aligned}\langle G(u) - G(v), F(u) - F(v)\rangle &= \langle N(u - v), M(u - v)\rangle = [M(u - v)]^\top NM_L^{-1} \\ \times [M(u - v)] &= \frac{1}{2}[M(u - v)]^\top((NM_L^{-1}) + (NM_L^{-1})^\top)[M(u - v)] \\ &\geq \beta\|[M(u - v)]\|^2 = \beta\|F(u) - F(v)\|^2. \end{aligned} \tag{2.2}$$

In the following, we give the equivalent reformulation of the GLCP.

Theorem 2.1. *$\mathcal{K}^0 = \bar{\mathcal{K}}$, where $\bar{\mathcal{K}} := \{u \in R^n \mid u^\top v \geq 0, \forall v \in \mathcal{K}\}$.*

Proof. For any $u \in \mathcal{K}^0$, there exist $\lambda_1 \in R_+^s, \lambda_2 \in R^t$ such that $u = A^\top\lambda_1 + B^\top\lambda_2$. For any $v \in \mathcal{K}$, $u^\top v = (A^\top\lambda_1 + B^\top\lambda_2)^\top v \geq 0$, therefore $u \in \bar{\mathcal{K}}$. On the other hand, for any $u \in \bar{\mathcal{K}}$ and $v \in \mathcal{K}$, we have $u^\top v \geq 0$, it is easy to know that there exist $\lambda_1 \in R_+^s, \lambda_2 \in R^t$ such that $u = A^\top\lambda_1 + B^\top\lambda_2$. Thus $u \in \mathcal{K}^0$. □

Hence, the GLCP can be equivalently converted into the following problem: find $x^* \in R^n$, such that

$$F(x^*) \in \mathcal{K}, G(x^*) \in \bar{\mathcal{K}}, F(x^*)^\top G(x^*) = 0, \tag{2.3}$$

in the sense that $x^* \in R^n$ is a solution of the GLCP if and only if x^* is a solution of (2.3).

Theorem 2.2. *Under Assumption 2.1 (i), we have that x^* is a solution of (1.1) if and only if x^* is a solution of the following problem*

$$G(x^*)^\top((F(x) - F(x^*)) \geq 0, \quad \forall F(x) \in \mathcal{K}. \tag{2.4}$$

Proof. Suppose that x^* is a solution of (2.4). Since vector $0 \in \mathcal{K}$, by substituting $F(x) = 0$ into (2.4), we have $G(x^*)^\top F(x^*) \leq 0$. On the other hand, since $F(x^*) \in \mathcal{K}$, then $2F(x^*) \in \mathcal{K}$. By substituting $F(x) = 2F(x^*)$ into (2.4), we obtain $G(x^*)^\top F(x^*) \geq 0$. Consequently, $G(x^*)^\top F(x^*) = 0$. For any $F(x) \in \mathcal{K}$, we have $G(x^*)^\top F(x) = G(x^*)^\top[F(x) - F(x^*)] \geq 0$, i.e., $G(x^*) \in \bar{\mathcal{K}}$. Thus, x^* solves (2.3), then x^* is a solution of (1.1).

On the contrary, suppose that x^* is a solution of (1.1), then x^* solves (2.3), since $G(x^*) \in \bar{\mathcal{K}}$, for any $F(x) \in \mathcal{K}$, we have $G(x^*)^\top F(x) \geq 0$, combining $G(x^*)^\top F(x^*) = 0$, we have $G(x^*)^\top[F(x) - F(x^*)] \geq 0$, therefore, x^* is a solution of (2.4). □

Now, we give the definition of projection operator and some relate properties. For nonempty closed convex set $\Omega \subset R^n$ and any vector $x \in R^n$, the orthogonal projection of x onto Ω, i.e., $\arg\min\{\|y-x\| | y \in \Omega\}$, is denoted by $P_\Omega(x)$.

Lemma 2.3. *For any $u \in R^n, v \in \Omega$, then:*

$$\begin{array}{ll}(i) & \langle P_\Omega(u)-u, v-P_\Omega(u)\rangle \geq 0,\\ (ii) & \|P_\Omega(u)-P_\Omega(v)\| \leq \|u-v\|.\end{array}$$

Invoking Lemma 2.3 and Theorem 2.2, one can prove that (2.4) is equivalent to the fixed-point problem, this result is due to Noor (see [10]). For convenience, throughout this paper, we define the projection residue vector

$$R(u,\rho) := F(u) - P_{\mathcal{K}}[F(u)-\rho G(u)], \quad \rho > 0.$$

Lemma 2.4. *x^* is a solution of the GLCP if and only if $R(x^*,\rho)=0$, for some $\rho > 0$.*

Based on this fixed-point formulation, various projection type iterative method for solving variational inequalities have been suggested and analyzed, see [6], [9], [10], [11].

3. Existence and Uniqueness of the Solutions

In this section, we discuss existence and uniqueness of the solutions to the GLCP. First, we prove that the GLCP has at most one solution under Assumption 2.1. Using (2.1) and (2.2), we have

$$(F(x)-F(y))^\top (G(x)-G(y)) \geq \beta \|M_L^{-1}\|^{-2}\|x-y\|^2, \quad \forall\, x,y \in R^n. \tag{3.1}$$

Lemma 3.1. *Assume that Assumption 2.1 holds, then the GLCP has at most one solution.*

Proof. Suppose that x^* and y^* are two different solutions of the GLCP, then there exist nonnegative vectors $\lambda_1^{x^*}, \lambda_1^{y^*} \in R^s$ and vectors $\lambda_2^{x^*}, \lambda_2^{y^*} \in R^t$ such that

$$G(x^*) = A^\top \lambda_1^{x^*} + B^\top \lambda_2^{x^*},\ G(y^*) = A^\top \lambda_1^{y^*} + B^\top \lambda_2^{y^*}.$$

By (3.1) and the definition of the GLCP, we have

$$
\begin{aligned}
0 < \beta \|M_L^{-1}\|^{-2} ||x^* - y^*||^2 &\leq (F(x^*) - F(y^*))^\top (G(x^*) - G(y^*)) \\
&= -[F(x^*)]^\top G(y^*) - [F(y^*)]^\top G(x^*) \\
&= -[F(x^*)]^\top \left(A^\top \lambda_1^{y^*} + B^\top \lambda_2^{y^*}\right) - [F(y^*)]^\top \left(A^\top \lambda_1^{x^*} + B^\top \lambda_2^{x^*}\right) \\
&= -[AF(x^*)]^\top \lambda_1^{y^*} - [AF(y^*)]^\top \lambda_1^{x^*} \leq 0.
\end{aligned}
$$

This contradiction implies that the GLCP has at most one solution. □

The following result is concerned with existence and uniqueness of the solutions to the variational inequalities (see Corollary 3.2 [3]).

Theorem 3.2. *Suppose the continuous mapping $H : R^n \to R^n$ is strongly monotone, i.e., there exists a constant $\rho > 0$ such that*

$$(H(x) - H(y))^\top (x - y) \geq \rho ||x - y||^2, \quad \forall\, x, y \in R^n.$$

Then the variational inequalities has a unique solution.

Using Assumption 2.1, (3.1), and Theorem 3.2, we can easily obtain existence and uniqueness of the solutions to the GLCP.

Theorem 3.3. *Assume that Assumption 2.1 holds, then the GLCP has a unique solution.*

4. Algorithm and Convergence

Now, we formally describe our method for solving the GLCP.

Algorithm 4.1.

Initial Step. Choose $u^0 \in R^n$ such that $F(u^0) \in \mathcal{K}$, select any $\sigma \in (0, 1), \rho_{-1} = 1, \varphi \in (0, 2)$, set $k := 0$.

Iterative Step. For $F(u^k) \in \mathcal{K}$, take $v^{k-1} \in R^n$ such that

$$F(v^{k-1}) = P_{\mathcal{K}}\{F(u^k) - \rho_{k-1}[NM_L^{-1}F(u^k) - NM_L^{-1}p + q]\}.$$

If $R(u^k, \rho_{k-1}) = 0$, then stop. Otherwise, Let $\rho_k = \gamma^{m_k}$, where m_k being the smallest nonnegative integer m satisfying

$$\gamma^m \|NM_L^{-1}F(u^k) - NM_L^{-1}F(v(m))\| \leq \sigma \|R(u^k, \gamma^m)\|, \tag{4.1}$$

where

$$F(v(m)) = P_{\mathcal{K}}\{F(u^k) - \gamma^m [NM_L^{-1}F(u^k) - NM_L^{-1}p + q]\}. \tag{4.2}$$

Compute u^{k+1} by solving the following equation

$$F(u^{k+1}) = P_{\mathcal{K}}[F(u^k) + \varphi\alpha_k d_k],$$

where

$$d_k = -\{R(u^k, \rho_k) - \rho_k(NM_L^{-1}F(u^k) - NM_L^{-1}F(v^k))\}, \tag{4.3}$$
$$F(v^k) = P_{\mathcal{K}}\{F(u^k) - \rho_k[NM_L^{-1}F(u^k) - NM_L^{-1}p + q]\},$$
$$\alpha_k = (1-\sigma)\|R(u^k, \rho_k)\|^2/\|d_k\|^2.$$

Remark 4.1. In Algorithm 4.1, several implicit equation of F need not to be solved at each iteration. ρ_k, α_k are said to be predictor stepsizes and the corrector stepsizes, respectively.

Remark 4.2. We recall the searching direction $-\{\eta_k R(u^k, \rho) + \eta_k T(u^k) + \rho T(v^k\}$ appears in [9] for solving general variational inequalities by Noor, Wang and Xiu, and differs from the direction in our algorithm.

Remark 4.3. In this paper, we do not require the matrices M and N to be square.

Now, we discuss the feasibility of stepsize rule of (4.1).

Lemma 4.1. *If u^k is not a solution of GLCP, then for any $\sigma \in (0,1)$, there exists $\hat{\rho}(u^k) \in (0,1]$, for any $\rho \in (0, \hat{\rho}(u^k)]$, we have*

$$\rho\|NM_L^{-1}F(u^k) - NM_L^{-1}F(v(\rho))\| \le \sigma\|R(u^k, \rho)\|\,, \tag{4.4}$$

where $u^k \in R^n$ and $F(u^k) \in \mathcal{K}$, $F(v(\rho))$ be defined in (4.2).

Proof. Assume that there exists $\sigma \in (0,1)$, for any $0 < \hat{\rho} \le 1$, there exists $0 < \rho \le \hat{\rho}$ such that

$$\rho\|NM_L^{-1}F(u^k) - NM_L^{-1}F(v(\rho))\| > \sigma\|R(u^k, \rho)\|. \tag{4.5}$$

Let $\hat{\rho}$ goes to 0, then we have that ρ tends to 0, furthermore, for any $\varepsilon > 0$, we take $\bar{\delta} = \varepsilon$ such that

$$\begin{aligned}\|F(u^k) - F(v(\rho))\| &= \|F(u^k) - P_{\mathcal{K}}(F(u^k) - \rho G(u^k))\| \\ &= \|P_{\mathcal{K}}F(u^k) - P_{\mathcal{K}}(F(u^k) - \rho G(u^k))\| \le \|G(u^k)\|\rho \le \bar{\delta},\end{aligned}$$

by continuity of G, F and (4.5), we have

$$\sigma\|R(u,\rho)\| < \rho\|NM_L^{-1}\|\|F(u^k) - F(v(\rho))\| \le \rho\|NM_L^{-1}\|\varepsilon,$$

combining Lemma 2.4, we know that $u^k \in X^*$, it contradicts that u^k is not a solution of GLCP. □

In the following, we are in the position to show the global (linear) convergence of the Algorithm 4.1.

Lemma 4.2. *Under Assumption 2.1, for given $u^* \in X^*$, then*

$$\langle F(u^k) - F(u^*), -d_k \rangle \geq (1-\sigma)\|R(u^k, \rho_k)\|^2.$$

Proof. From the iterative procedure, for any positive integer k, we know that $F(u^k),\ F(v^k) \in \mathcal{K}$. By $u^* \in X^*$, Lemma 2.3(i), we have

$$\langle F(u^k) - \rho_k G(u^k) - F(v^k), F(v^k) - F(u^*) \rangle \geq 0,$$

combining definition of $R(u, \rho)$, we know that

$$\langle R(u^k, \rho_k) - \rho_k G(u^k), F(u^k) - F(u^*) - R(u^k, \rho_k) \rangle \geq 0,$$

i.e.,

$$\begin{aligned}\langle R(u^k, \rho_k), F(u^k) - F(u^*) \rangle - \|R(u^k, \rho_k)\|^2 \\ - \langle \rho_k G(u^k), F(u^k) - F(u^*) \rangle + \langle \rho_k G(u^k), R(u^k, \rho_k) \rangle \geq 0,\end{aligned}$$

we obtain

$$\begin{aligned}&\langle R(u^k, \rho_k) - \rho_k [NM_L^{-1} F(u^k) - NM_L^{-1} p + q], F(u^k) - F(u^*) \rangle \\ &\qquad \geq \|R(u^k, \rho_k)\|^2 - \langle \rho_k [NM_L^{-1} F(u^k) - NM_L^{-1} p + q], R(u^k, \rho_k) \rangle.\end{aligned}$$

On the other hand, using (2.2) and (2.4), we have

$$\langle G(v^k), F(v^k) - F(u^*) \rangle \geq 0, \tag{4.6}$$

by Algorithm 4.1 and definition of $R(u, \rho)$, we know that

$$\begin{aligned}0 \leq \langle F(v^k) - F(u^*), G(v^k) \rangle &= \langle F(u^k) - R(u^k, \rho_k) - F(u^*), G(v^k) \rangle \\ &= \langle F(u^k) - F(u^*), G(v^k) \rangle - \langle R(u^k, \rho_k), G(v^k) \rangle,\end{aligned}$$

i.e.,

$$\begin{aligned}&\langle F(u^k) - F(u^*), [NM_L^{-1} F(v^k) - NM_L^{-1} p + q] \rangle \\ &\qquad \geq \langle R(u^k, \rho_k), [NM_L^{-1} F(v^k) - NM_L^{-1} p + q] \rangle.\end{aligned}$$

Thus, combining (4.1), we have

$$\begin{aligned}&\langle F(u^k) - F(u^*), -d_k \rangle = \langle F(u^k) - F(u^*), R(u^k, \rho_k) \\ &\quad + \rho_k \{[NM_L^{-1} F(v^k) - NM_L^{-1} p + q] - [NM_L^{-1} F(u^k) - NM_L^{-1} p + q]\} \rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle F(u^k) - F(u^*), R(u^k, \rho_k) - \rho_k[NM_L^{-1}F(u^k) - NM_L^{-1}p + q]\rangle \\
&\quad + \langle F(u^k) - F(u^*), \rho_k[NM_L^{-1}F(v^k) - NM_L^{-1}p + q]\rangle \\
&\geq \|R(u^k, \rho_k)\|^2 - \langle \rho_k[NM_L^{-1}F(u^k) - NM_L^{-1}p + q], R(u^k, \rho_k)\rangle \\
&\quad + \langle \rho_k R(u^k, \rho_k), [NM_L^{-1}F(v^k) - NM_L^{-1}p + q]\rangle \\
&= \|R(u^k, \rho_k)\|^2 - \rho_k\langle NM_L^{-1}F(u^k) - NM_L^{-1}F(v^k), R(u^k, \rho_k)\rangle \\
&\geq \|R(u^k, \rho_k)\|^2 - \rho_k\|NM_L^{-1}F(u^k) - NM_L^{-1}F(v^k)\|\|R(u^k, \rho_k)\| \\
&\geq (1-\sigma)\|R(u^k, \rho_k)\|^2. \quad \square
\end{aligned}
$$

Lemma 4.3. *Under Assumption 2.1, the sequence $\{\alpha_k\}$ and $\{\rho_k\}$ generated by Algorithm 4.1 both have a uniformly positive bound from below, respectively.*

Proof. Firstly, we shall show that α_k have a uniformly positive bound from below. Using representation of d_k and (4.1), we know that

$$
\begin{aligned}
\|d(u^k, \beta_k)\|^2 &\leq 2\|R(u^k, \rho_k)\|^2 + 2\rho_k^2\|NM_L^{-1}F(u^k) - NM_L^{-1}F(v^k)\|^2 \\
&\leq 2(1+\sigma^2)\|R(u^k, \rho_k)\|^2.
\end{aligned}
$$

By representation of α_k again in Algorithm 4.1, we have that there exists a constant $\eta > 0$ such that

$$\alpha_k \geq \frac{1-\sigma}{2(1+\sigma^2)} =: \eta.$$

In the following, we prove that ρ_k also have a uniformly positive bound from below. By stepsize rule of Algorithm 4.1, we have

$$
\begin{aligned}
\sigma\|R(u^k, \rho)\| &< \rho\|NM_L^{-1}F(u^k) - NM_L^{-1}F(v(\rho))\| \\
&\leq \rho\|NM_L^{-1}\|\|F(u^k) - F(v(\rho))\| = \rho\|NM_L^{-1}\|\|R(u^k, \rho)\|,
\end{aligned}
$$

i.e., $\rho > \sigma/\|NM_L^{-1}\|$, therefore, $\rho_k > \min\{1, \sigma/\|NM_L^{-1}\|\} =: \tau$. $\square$

Lemma 4.4. *Under Assumption 2.1, the sequence $\{u^k\}$ generated by Algorithm 4.1 is bounded.*

Proof. Given $u^* \in \mathcal{K}^*$, then we have

$$
\begin{aligned}
\|u^{k+1} - u^*\|^2 &\leq \|M_L^{-1}\|^2\|F(u^{k+1}) - F(u^*)\|^2 \\
&= \|M_L^{-1}\|^2\|P_{\mathcal{K}}[F(u^k) + \varphi\alpha_k d_k] - P_{\mathcal{K}}F(u^*)\|
\end{aligned}
$$

$$\begin{aligned}
&\le \|M_L^{-1}\|^2\|F(u^k) - F(u^*) + \varphi\alpha_k d_k\|^2 = \|M_L^{-1}\|^2\|F(u^k) - F(u^*)\|^2 \\
&\quad + 2\|M_L^{-1}\|^2\varphi\alpha_k(F(u^k) - F(u^*))^\top d_k + \|M_L^{-1}\|^2\varphi^2\alpha_k^2\|d_k\|^2 \\
&\le \|M_L^{-1}\|^2\|F(u^k) - F(u^*)\|^2 \\
&\quad - 2\|M_L^{-1}\|^2\varphi\alpha_k(1-\sigma)\|R(u^k,\rho_k)\|^2 + \varphi^2\alpha_k^2\|M_L^{-1}\|^2\|d_k\|^2 \\
&= \|M_L^{-1}\|^2\|F(u^k) - F(u^*)\|^2 \\
&\quad - \varphi(2-\varphi)\alpha_k(1-\sigma)\|M_L^{-1}\|^2\|R(u^k,\rho_k)\|^2 \\
&\le \|u^k - u^*\|^2 - \varphi(2-\varphi)\alpha_k(1-\sigma)\|M_L^{-1}\|^2\|R(u^k,\rho_k)\|^2\,,
\end{aligned}$$

where the first inequality uses (2.1), the third inequality is derived from Lemma 4.2, the third equation uses Algorithm 4.1 representation of α_k in Algorithm 4.1. Based on the above analysis, we show that the sequence $\|u^k - u^*\|$ is decreasing and nonnegative, it is bounded, and so is also $\{u^k\}$. □

Theorem 4.5. *Under Assumption 2.1, the sequence $\{u^k\}$ are generated by Algorithm 4.1 converges globally to a solution of GLCP.*

Proof. Using Lemma 4.3 and Lemma 4.4, we know that

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \eta\varphi(2-\varphi)(1-\sigma)\|M_L^{-1}\|^2\|R(u^k,\rho_k)\|^2.$$

Since the sequence $\|u^k - u^*\|$ is decreasing and nonnegative, it is bounded, it must converges, and we have

$$\sum_{k=0}^{\infty}\|R(u^k,\rho_k)\|^2 \le \infty,$$

i.e.,

$$\lim_{k\to\infty}\|R(u^k,\rho_k)\| = 0. \tag{4.7}$$

Thus, we know that any cluster $\bar{u}$ of the sequence $\{u^k\}$ is a solution of GLCP. Since the sequence $\|u^k - u^*\|$ is non-increasing and nonnegative, it is bounded, if we take $u^* = \bar{u}$, then $\{u^k\}$ converges globally to $\bar{u}$. □

The following conclusion discusses the convergence rate of Algorithm 4.1.

Lemma 4.6. *Under Assumption 2.1, for any $u \in X$ and a constant $\rho > 0$, we have*

$$\frac{\|R(u,\rho)\|}{\|M\|(2+\rho\|NM_L^{-1}\|)} \le \|u - u^*\| \le \frac{\|M_L^{-1}\|(\rho\|NM_L^{-1}\|+1)}{\rho\beta}\|R(u,\rho)\|. \tag{4.8}$$

Proof. By using Theorem 3.3, we know that the GLCP has unique solution. We assume that u^* be a fixed solution of GLCP, and by Theorem 2.2, we have

$$\langle \rho G(u^*), P_{\mathcal{K}}(F(u) - \rho G(u)) - F(u^*) \rangle \geq 0\,. \tag{4.9}$$

By Lemma 2.3 (i) and $F(u^*) \in \mathcal{K}$, we know that

$$\langle P_{\mathcal{K}}(F(u) - \rho G(u)) - (F(u) - \rho G(u)), F(u^*) - P_{\mathcal{K}}(F(u) - \rho G(u)) \rangle \geq 0\,. \tag{4.10}$$

By (4.9), (4.10),

$$\langle P_{\mathcal{K}}(F(u) - \rho G(u)) - F(u) + \rho(G(u) - G(u^*)), F(u^*) - P_{\mathcal{K}}(F(u) - \rho G(u)) \rangle \geq 0,$$

i.e.,

$$\langle \rho(G(u) - G(u^*)) - R(u,\rho), F(u^*) - F(u) + R(u,\rho) \rangle \geq 0,$$

we have

$$\begin{aligned} &\langle \rho(G(u) - G(u^*)) + (F(u) - F(u^*)), R(u,\rho) \rangle \\ &\quad \geq \langle \rho(G(u) - G(u^*)), F(u) - F(u^*)\,, \rangle \end{aligned}$$

i.e.,

$$\begin{aligned} &\langle \rho(G(u) - G(u^*), F(u) - F(u^*) \rangle \\ &\quad \leq \|R(u,\rho)\| \cdot (\rho \|G(u) - G(u^*)\| + \|F(u) - F(u^*)\|). \end{aligned}$$

By (2.2), there exists $\beta > 0$ such that,

$$\begin{aligned} \rho\beta \|F(u) - F(u^*)\|^2 &\leq \langle \rho(G(u) - G(u^*)), F(u) - F(u^*) \rangle \\ &\leq \|R(u,\rho)\| \cdot (\rho \|NM_L^{-1}\| \|F(u) - F(u^*)\| + \|F(u) - F(u^*)\|) \\ &= \|R(u,\rho)\| \cdot (\rho \|NM_L^{-1}\| + 1) \|F(u) - F(u^*)\|, \end{aligned}$$

i.e.,

$$\rho\beta \|F(u) - F(u^*)\| \leq \|R(u,\rho)\| \cdot (\rho \|NM_L^{-1}\| + 1)\,. \tag{4.11}$$

Using (4.11) and the right-hand side of (2.1), we know that the right-hand side inequality of (4.8) holds. On the other hand, by Lemma 2.4 and Lemma 2.3(ii), we have

$$\begin{aligned} \|R(u,\rho)\| &= \|R(u,\rho) - R(u^*,\rho)\| \leq \|F(u) - F(u^*)\| \\ &\quad + \|P_{\mathcal{K}}(F(u) - \rho G(u)) - P_{\mathcal{K}}(F(u^*) - \rho G(u^*))\| \\ &\leq \|F(u) - F(u^*)\| + \|F(u) - \rho G(u) - F(u^*) + \rho G(u^*)\| \\ &\leq 2\|F(u) - F(u^*)\| + \rho \|NM_L^{-1}\| \|F(u) - F(u^*)\| \\ &\leq (2 + \rho \|NM_L^{-1}\|) \|F(u) - F(u^*)\| \leq (2 + \rho \|NM_L^{-1}\|) \|M\| \|u - u^*\|. \end{aligned}$$

Thus, the left-hand side inequality of (4.8) follows. □

Dimension	8	16	32	64	128
DAN iter.num.	9	20	72	208	>300
NAT iter.num.	13	12	18	99	99

Table 1: Numerical results by DNA, NTA for Example 5.1

Theorem 4.7. *Under Assumption 2.1, then the sequence* $\{u^k\}$ *generated by Algorithm 4.1 converges globally to a solution of GLCP at* $Q-$*linear rate, where* $\beta < \sqrt{2(1+\sigma^2)}(\|NM_L^{-1}\|+1)/[\sqrt{\varphi(2-\varphi)}(1-\sigma)\tau]$.

Proof. Using Lemma 4.4-4.3, we have

$$\|u^{k+1}-u^*\|^2 \le \|u^k-u^*\|^2 - \eta\varphi(2-\varphi)(1-\sigma)\|M_L^{-1}\|^2\|R(u^k,\rho_k)\|^2.$$

Combining the right-hand side of (4.8), we get

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\le \|u^k-u^*\|^2 \\ &\quad - \varphi(2-\varphi)(1-\sigma)\eta\|M_L^{-1}\|^2\left(\frac{\rho_k\beta}{(\rho_k\|NM_L^{-1}\|+1)\|M_L^{-1}\|}\right)^2\|u^k-u^*\|^2,\end{aligned}$$

i.e.,

$$\frac{\|u^{k+1}-u^*\|^2}{\|u^k-u^*\|^2} \le 1-\varphi(2-\varphi)(1-\sigma)\eta\left(\frac{\rho_k\beta}{\rho_k\|NM_L^{-1}\|+1}\right)^2,$$

combining $\tau \le \rho_k \le 1$, we know that

$$\frac{\|u^{k+1}-u^*\|^2}{\|u^k-u^*\|^2} \le 1-\varphi(2-\varphi)(1-\sigma)\eta\left(\frac{\tau\beta}{\|NM_L^{-1}\|+1}\right)^2.$$

Since $\beta < \sqrt{2(1+\sigma^2)}(\|NM_L^{-1}\|+1)/[\sqrt{\varphi(2-\varphi)}(1-\sigma)\tau]$, then

$$0 < 1-\varphi(2-\varphi)(1-\sigma)\eta\left(\frac{\tau\beta}{\|NM_L^{-1}\|+1}\right)^2 < 1.$$

thus, the sequence $\{u^k\}$ converges globally to a solution of GLCP at $Q-$linear rate. □

Dimension	8	16	32	64	128
Iter.num.	7	11	21	45	81
$\mathbf{R}(\mathbf{u}^*)(1 \times 10^{-15})$	5.5612	0.11102	0.22204	0.22204	0.11102

Table 2: Numerical results of our algorithm for Example 5.1

Dimension	10	20	50	80	100	200
Iter.num.	4	4	4	4	4	4
$\mathbf{R}(\mathbf{u}^*)$	0	0	0	0	0	0

Table 3: Numerical results of Example 5.2

Starting point	**Iter.num.**	$\mathbf{R}(\mathbf{u}^*)$	**CPU time**
$(0,0,0)^\top$	2	3.1402×10^{-16}	0.0160
$(1,0,0)^\top$	8	2.2204×10^{-16}	0.0780
$(0,1,0)^\top$	6	2.2204×10^{-16}	0.0630
$(0,0,1)^\top$	3	2.2204×10^{-16}	0.0310

Table 4: Numerical results of Example 5.3

Trial	1	2	3	4	5
Iter.num.	2	2	2	2	2
$\mathbf{R}(\mathbf{u}^*)(1 \times 10^{-16})$	0	2.2204	2.2204	2.4825	1.1102
CPU time	0.0160	0.0160	0.0320	0.0150	0.0160
Trial	6	7	8	9	10
Iter.num.	4	2	2	2	2
$\mathbf{R}(\mathbf{u}^*)(1 \times 10^{-16})$	2.2204	0	1.5701	1.1102	2.4825
CPU time	0.0310	0.0150	0.0160	0.0150	0.0160

Table 5: Numerical results of our algorithm by random initial point for Example 5.3

5. Computational Experiments

In the following, we will implement Algorithm 4.1 in *Matlab* and run it on *Pentium IV* computer. Throughout our computation, **Iter.num.** denotes the number of iterations, $\mathbf{R}(\mathbf{u}^*)$ is the final value of $R(u^k, \rho_k)$ when the algorithm terminates.

Example 5.1. This problem is a linear complementarity problem (LCP) used by Harker and Pang (see [5]), in which $G(x) = Nx + q$, where

$$N = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

For this problem, Harker and Pang (see [5]) used the damped-Newton method (DNA), and Wang (see [19]) used the Newton-type method (NTA). The results for the above two methods and several values of the dimensions n are summarized in Table 1. In Table 2, we take parameter $\varepsilon = 10^{-8}, \sigma = 0.6, \gamma = 0.8, \varphi = 1.98$ and initial point $x^0 = (1, 1, \cdots, 1)^\top$, and summarize the results of our algorithm for several values of dimensions n. From Table 1 and Table 2, we can conclude that our algorithm excels the other two methods listed above.

Example 5.2. This example is LCP used by Wang (see [9]). Let $G(x) = Nx + q$, where

$$N = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix},$$

where the domain $C = \{x \in R^n \mid 0 \le x_i \le 1, \text{ for } i = 1, 2, \cdots, n\}$.

Table 3 list the results for this example with initial point $x^0 = -N^{-1}q$ for different dimensions n and parameter $\varepsilon = 10^{-8}, \sigma = 0.6, \gamma = 0.8, \varphi = 1.98$. Compared with the results of Table 4.2 in [9], we can conclude that our algorithm excels methods in [9].

Example 5.3. This example is a linear complementarity problem over a polyhedral cone used by Wang in [14]. Let

$$x \in \mathcal{K},\ G(x) = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} x + \begin{pmatrix} -8 \\ -6 \\ -4 \end{pmatrix} \in \mathcal{K}^0,$$

and $\mathcal{K} = \{x \in R^3 \mid x_1 + x_2 + x_3 \le 3, 0 \le x_i \le 1, i = 1, 2, 3\}$.

Table 4 lists the numerical results of Algorithm 4.1 with different starting points, and parameter $\varepsilon = 10^{-7}, \sigma = 0.9, \gamma = 0.8, \varphi = 1.98$. Compared with the results of Table 2 in [14], we can conclude that our algorithm excels methods in [14], and it has nice stability and high computation efficiency.

To illustrate the stability of our algorithm, under the initial point x^0 is produced randomly in (0,10), we use it to solve Example 5.3, and the results are listed in Table 5. Table 4 and Table 5 indicate that our algorithm is not sensitive to the change of initial point, thus, we can see Algorithm 4.1 performs well for this problem.

6. Acknowledgements

This work was supported by the Postdoctoral Fellowship of The Hong Kong Polytechnic University, Shandong provincial NSF (Y2003A02), the NSF of China (10231060), and the Linyi normal university priority academic discipline foundation.

References

[1] R. Andreani, A. Friedlander, S.A. Santos, On the resolution of the generalized nonlinear complementarity problem, *SIAM Journal on Optim.*, **12** (2001), 303-321.

[2] R. Andreani, A. Friedlander, S.A. Santos, Solving generalized nonlinear complementarity problems: Numerical experiments on polyhedral cones, *Technical Report*, IMECC, State University of Campinas, Campinas, Brazil (2001).

[3] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequality and Complementarity Problems*, Springer, New York (2003).

[4] C. Geiger, C. Kanzow, On the solution of monotone complementarity problems, *Comput. Optim. Appl.*, **5** (1996), 155-172.

[5] P.T. Harker, J.S. Pang, A damped-Newton method for the linear complementarity problem, *Lectures in Appl. Math.*, **26** (1990), 265-284.

[6] B.S. He, Inexact implicite methods for monotone general variational inequalities, *Math. Programming*, **86** (1999), 199-217.

[7] H. Jiang, M. Fukushima, L. Qi, D. Sun, A trust region method for solving generalized complementarity problems, *SIAM J.Optim.*, **8** (1998), 140-157.

[8] C. Kanzow, M. Fukushima, Equivalence of the generalized complementarity problem to differentiable unconstrained minimization, *J. Optim. Theory and Appl.*, **90**, No. 3 (1996), 581-603.

[9] M.A. Noor, Y.J. Wang, N.H. Xiu, Projection iterative schemes for general variational inequalities, *Journal of Inequalities in Pure and Applied Mathematics*, **3**, No. 3 (2002), article 34.

[10] M.A. Noor, General variational inequalities, *Appl. Math. Lett.*, **1**, No. 2 (1988), 119-121.

[11] M.A. Noor, Modified projection method for pseudomonotone variational inequalities, *Appl. Math. Lett.*, **15** (2002).

[12] M.V. Solodov, Convergence rate analysis of iteractive algorithms for solving variational inequality problems, *Math. Programming, Ser. A*, **96** (2003), 513-528.

[13] P. Tseng, N. Yamashita, M. Fukushima, Equivalence of complementarity problems to differentiable minimization: A unified approach, *SIAM J. Optim.*, **6** (1996), 446-460.

[14] Y.J. Wang, A new projection and contraction method for varitional inequalities, *Pure Math. and Appl.*, **13**, No. 4 (2002), 483-493.

[15] Y.J. Wang, F.M. Ma, J.Z. Zhang, A nonsmooth L-M method for solving the generalized nonlinear complementarity problem over a polyhedral cone, *Appl. Math. Optim.*, **52**, No. 1 (2005), 73-92.

[16] N.H. Xiu, J.Z. Zhang, Global projection-type error bound for general variational inequlities, *J. Optim. Theory and Appl.*, **112**, No. 1 (2002), 213-228.

[17] N. Yamashita, M. Fukushima, Modified Newton methods for solving semismooth reformulations of monotone complementarity problems, *Math. Programming*, **76** (1997), 469-491.

[18] J.Z. Zhang, N.H. Xiu, Global s-type error bound for extended linear complementarity problem and applications, *Math. Programming, Ser. B*, **88**, No. 2 (2000), 391-410.

[19] X.Z. Zhang, F.M. Ma, Y.J. Wang, A Newton-type algorithm for generalized linear complementarity problem over a polyhedral cone, *Appl. Math. Comput.* (2005), To Appear.