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A GEOMETRIC INTERPRETATION FOR SPHERE DISTORTION OF SURFACES IN 3-SPACE

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Abstract: Let Σ be a 2-dimensional Jordan surface in $\overline{R}^3$ which contains ∞, in this paper, the authors prove that Σ has sphere distortion c if and only if there exists a constant b, $2 \leq b < \infty$, such that for each point w_1 in one component of $\overline{R}^3 \setminus \Sigma$ there exists a point w_2 in the other component with $b(w_j, \Sigma) \geq |w_1 - w_2|$, $j = 1, 2$.

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1. Introduction

Let α be a Jordan curve in $\overline{R}^2$, we say that α has circular distortion c, $1 \leq c < \infty$, if for each Möbius transformation φ of $\overline{R}^2$, either $\varphi(\alpha)$ separates the boundary circles of an annulus

$$R = R(x_0; r, s) = \{x \in R^2 : r \leq |x - x_0| \leq s\}$$

with radii ratio $\frac{s}{r} = c$ or $\varphi(\alpha)$ contains the point ∞. The circular distortion is

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a Möbius invariant which measures how far a Jordan curve differs from being a circle or line. In particular, α has circular distortion 1 if and only if it is a circle or line.

The concept of circular distortion was first put forward by R. Kühnau in [3]. R. Kühnau established the following relation between circular distortion curves and quasicircles.

Theorem A. *If α is a $K-$quasicircle in $\overline{R}^2$, then α has circular distortion c, where c depends only on K.*

R. Kühnau found sharp bounds for the constant c in terms of K and then asked if the converse of Theorem A is true, that is, if each Jordan curve α of $\overline{R}^2$ with circular distortion c is a $K-$quasicircle where K depends only on c. F.W. Gehring and Ch. Pommerenke answered this question in [2] as follows.

Theorem B. *If α is a Jordan curve in $\overline{R}^2$ with circular distortion $c < \sqrt{2}$, then α is a K-quasicircle where k depends only on c. Also there is a Jordan curve α in $\overline{R}^2$ with circular distortion $c = 5$ which is not a quasicircle (that is, a curve with circular distortion $c \geq 5$ need not be a quasicircle).*

In this paper, we shall extend the concept of circular distortion curves in $\overline{R}^2$ to sphere distortion surfaces in $\overline{R}^3$, and give a geometric property for sphere distortion surfaces.

Suppose that Σ is a 2-dimensional Jordan surface in $\overline{R}^3$, we say that Σ has sphere distortion c, $1 \leq c < \infty$, if for each Möbius transformation φ of $\overline{R}^3$, either $\varphi(\Sigma)$ separates the boundary spheres of a spherical ring

$$A = A(x_0; r, s) = \{x \in R^3 : r \leq |x - x_0| \leq s\} \tag{1.1}$$

with radii ratio $\frac{s}{r} = c$ or $\varphi(\Sigma)$ contains the point ∞.

The main purpose of this paper is to prove the following theorem.

Theorem. *Suppose that Σ is a 2-dimensional Jordan surface in $\overline{R}^3$ which contains ∞. Then Σ has sphere distortion c if and only if there exists a contant b, $2 \leq b < \infty$, such that for each point w_1 in one component of $\overline{R}^3 \setminus \Sigma$ there exists a point w_2 in the other component with*

$$b(w_j, \Sigma) \geq |w_1 - w_2|. \tag{1.2}$$

Here $b = 2c$ in the necessity and $c = b^2 + b - 1$ in the sufficiency.

2. Proof of Theorem

Proof. ($\Rightarrow$) Choose w_1 in a component of $\overline{R}^3 \setminus \Sigma$ and let φ be a Möbius transformation of $\overline{R}^3$ for which $\varphi(w_1) = \infty$. Since Σ has sphere distortion c, hence $\varphi(\Sigma)$ separates the boundary spheres of a spherical ring $A = A(x_0; r, cr)$. By a preliminary change of variables we may assume that $x_0 = 0$. Then $w_2 = \varphi^{-1}(0)$ lies in the other component of $\overline{R}^3 \setminus \Sigma$.

Let Σ_1 and Σ_2 denote the images under φ^{-1} of the outer and inner boundary spheres of A, respectively. Next for $j = 1, 2$, let z_j and z'_j denote the points where Σ_j meets the extended line L through w_1 and w_2, labeled so that z_j lies in the segment $[w_1, w_2]$, and set $r_j = |z_j - w_j|$. Then by the Möbius invariance of the cross ratio,

$$\frac{|z - w_1|}{|z - w_2|} = \frac{|z_j - w_1|}{|z_j - w_2|} \qquad \text{for} \quad z \in \Sigma_j, \quad j = 1, 2. \tag{2.1}$$

If $\infty \notin \Sigma_1$, then Σ_1 is a sphere which does not separate w_2 from ∞,

$$|z'_1 - w_1| \le |z'_1 - w_2|,$$

and we obtain

$$|z_1 - w_1| \le |z_1 - w_2| \tag{2.2}$$

from (2.1) with $j = 1$ and $z = z'_1$. If $\infty \in \Sigma_1$, then $z'_1 = \infty$ and (2.2) again follows from (2.1). Interchanging the roles of Σ_1 and Σ_2 in the above discussion then shows that

$$|z_2 - w_2| \le |z_2 - w_1|. \tag{2.3}$$

Next

$$\frac{|z_1 - w_2||z_2 - w_1|}{|z_1 - w_1||z_2 - w_2|} = \frac{|\varphi(z_1)|}{|\varphi(z_2)|} = c, \tag{2.4}$$

and with (2.2) and (2.3) we obtain

$$|z_1 - w_2| \le c|z_1 - w_1|, \quad |z_2 - w_1| \le c|z_2 - w_2|, \tag{2.5}$$

whence

$$|w_1 - w_2| \le |z_j - w_1| + |z_j - w_2| \le (c+1)|z_j - w_j| = (c+1)r_j \tag{2.6}$$

for $j = 1, 2$.

Next for each $z \in \Sigma_1$, making use of the triangle inequality and (2.1) we get

$$|z - w_2| \ge |w_1 - w_2| - |z - w_1| = |w_1 - w_2| - |z - w_2|\frac{|z_1 - w_1|}{|z_1 - w_2|},$$

$$|z - w_2| \ge \frac{|w_1 - w_2|}{1 + \frac{|z_1 - w_1|}{|z_1 - w_2|}}. \tag{2.7}$$

(2.7) and (2.2) imply

$$|z - w_2| \geq \frac{1}{2}|w_1 - w_2|. \tag{2.8}$$

Combining (2.1), (2.5) and (2.8) we can obtain

$$\begin{aligned}|z - w_1| = |z_1 - w_1|\frac{|z - w_2|}{|z_1 - w_2|} &\geq \frac{1}{2}r_1\frac{|w_1 - w_2|}{|z_1 - w_2|}\\ &= \frac{1}{2}r_1\left(1 + \frac{|z_1 - w_1|}{|z_1 - w_2|}\right) \geq \frac{1}{2}\left(1 + \frac{1}{c}\right)r_1. \end{aligned}\tag{2.9}$$

(2.9) implies

$$d(w_1, \Sigma_1) \geq \frac{1+c}{2c}r_1, \tag{2.10}$$

(2.6) and (2.10) yield

$$d(w_1, \Sigma) \geq d(w_1, \Sigma_1) \geq \frac{1+c}{2c}\frac{1}{1+c}|w_1 - w_2| = \frac{1}{2c}|w_1 - w_2|,$$

$$|w_1 - w_2| \leq 2cd(w_1, \Sigma). \tag{2.11}$$

A similar argument as above shows that

$$|w_1 - w_2| \leq 2cd(w_2, \Sigma). \tag{2.12}$$

(2.11) and (2.12) imply (1.2).

($\Leftarrow$) Suppose that φ is any Möbius transformation of $\overline{R}^3$ for which $\varphi(\Sigma)$ does not contain ∞ and let $w_1 = \varphi^{-1}(\infty)$. Then w_1 lies in a component of $\overline{R}^3 \setminus \Sigma$. Let w_2 denotes the point in the other component of $\overline{R}^3 \setminus \Sigma$ for which (1.2) holds and set $\psi(x) = \frac{|w_1 - w_2|}{|x - w_1|^2}(x - w_1)$. Then $\varphi \circ \psi^{-1}$ is a Möbius transformation of $\overline{R}^3$ which fixes the infinity, by [3, Theorem 3.5.1(ii)] we know that $\varphi \circ \psi^{-1}$ is a Euclidean similarity. In order to show that $\varphi(\Sigma)$ separates the boundary spheres of a spherical ring of radii ratio c, it suffices to consider the case where $\varphi = \psi$.

Now let $r = \frac{|w_1 - w_2|}{b}$. For any $x \in \partial B^3(w_1, r)$ we have

$$|\varphi(x)| = |\psi(x)| = \frac{|w_1 - w_2|}{|x - w_1|^2}|x - w_1| = \frac{|w_1 - w_2|}{r} = b,$$

hence we can get

$$\varphi(B^3(w_1, r)) = \overline{R}^3 \setminus \overline{B}^3(0, b). \tag{2.13}$$

Next suppose that $\varphi(B^3(w_2,r)) = B^3(x_0,t)$, then we have

$$\begin{aligned} x_0 &= \frac{1}{2}\left[\varphi\left(w_2 + r\frac{w_2-w_1}{|w_2-w_1|}\right) + \varphi\left(w_2 - r\frac{w_2-w_1}{|w_2-w_1|}\right)\right] \\ &= \frac{b^2}{b^2-1}\frac{w_2-w_1}{|w_2-w_1|} \end{aligned}$$

and

$$t = \frac{1}{2}\left|\varphi\left(w_2 + r\frac{w_2-w_1}{|w_2-w_1|}\right) - \varphi\left(w_2 - r\frac{w_2-w_1}{|w_2-w_1|}\right)\right| = \frac{b}{b^2-1}.$$

Hence

$$\varphi(B^3(w_2,r)) = B^3\left(\frac{b^2}{b^2-1}\frac{w_2-w_1}{|w_2-w_1|}, \frac{b}{b^2-1}\right). \tag{2.14}$$

Since (1.2) implies that

$$\Sigma \subset \overline{R}^3 \setminus (B^3(w_1,r) \cup B^3(w_2,r)), \tag{2.15}$$

hence (2.13), (2.14) and (2.15) yield

$$\begin{aligned} \varphi(\Sigma) &\subset \varphi(\overline{R}^3 \setminus (B^3(w_1,r) \cup B^3(w_2,r))) \\ &\subset A\left(\frac{b^2}{b^2-1}\frac{w_2-w_1}{|w_2-w_1|}; \frac{b}{b^2-1}, \frac{b(b^2+b-1)}{b^2-1}\right), \end{aligned}$$

a spherical ring with radii ratio b^2+b-1. □

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