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WEAK RPP SEMIGROUPS SATISFYING PERMUTATION IDENTITIES

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Abstract: The aim of this paper is to study a class of wrpp semigroups which satisfy permutation identities, namely, PI-adequate wrpp semigroups. In particular, the spined product structure of PI-adequate wrpp semigroups is estabilished.

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1. Introduction

Completely regular semigroups form a very important class of regular semigroups. Strongly rpp semigroups are analogue of completely regular semigroups in the range of rpp semigroups. Strongly rpp semigroups are initially investigated by Y.Q. Guo, K.P. Shum and P.Y. Zhu in [10]. After then, several classes of strongly rpp semigroups are researched (see [3], [6]-[10] and their references). Du and Shum [1] generalized strongly rpp semigroups and defined so-called adequate wrpp semigroups. They investigated a subclass of such semigroups, named left C-wrpp semigroups.

Let

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

by a non-identity permutation on n objects. Then a semigroup S is said *to satisfy the permutation identity determined by* σ (in short, *to satisfy the per-*

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mutation identity if there is no ambiguity) if

$$(\forall x_1, x_2, \cdots x_n \in S)(x_1 x_2 \cdots x_n = x_{\sigma_{(1)}} x_{\sigma_{(2)}} \cdots x_{\sigma_{(n)}}),$$

where $x_1 x_2 \cdots x_n$ is the product of $x_1, x_2, \cdots x_n$ in S. For brevity, S is also called a *PI-semigroup determined by σ* or a *PI-semigroup*.

In 1967, Yamada [15] investigated regular semigroups satisfying permutation identities. He gave the classification of such regular semigroups. After then, X.J. Guo in [3] and [4] established the structure of PI-strongly rpp semigroups and that of PI-abundant semigroups, respectively. It was found that PI-strongly rpp semigroups are abundant. Recently, X.J. Guo [5] studied general rpp semigroups satisfying permutation identities. This paper continues this work. We shall consider PI-adequate wrpp semigroups.

2. Main Results

Throughout this paper we shall use the notations of Howie [11] and Tang [13]. A semigroup S is called *left* [*right*] *κ-cancellative* if for all $a, x, y \in S$, $ax \ \kappa \ ay$ [$xa \ \kappa \ ya$] implies that $x \kappa y$. S is called *κ-cancellative* if it is both left κ-cancellative and right κ-cancellative.

We define relations on S given by: for $a, b \in S$:

(1) $a\mathcal{L}^{**}b$ if and only if for all $x, y \in S^1$, $(ax, ay) \in \mathcal{R} \Leftrightarrow (bx, by) \in \mathcal{R}$.

(2) $a\mathcal{R}^{**}b$ if and only if for all $x, y \in S^1$, $(ax, ay) \in \mathcal{L} \Leftrightarrow (bx, by) \in \mathcal{L}$.

As in [1] and [13], we call S a *wrpp semigroup* if the following conditions are satisfied:

(1) Each $\mathcal{L}^{**}$-class of S contains at least one idempotent.

(2) If $e\mathcal{L}^{**}a$, then $a = ae$.

A wrpp semigroup S is called a *C-wrpp semigroup* if its idempotents are central. Tang [13] noted that a semigroup is a C-wrpp semigroup if and only if it is a strong semilattice of left $\mathcal{R}$-cancellative monoids. We call S is a C-$\mathcal{R}$-ca if it is a strong semilattice of commute $\mathcal{R}$-cancellative monoids. A wrpp semigroup S is called an *adequate wrpp semigroup* if for every $a \in S$, there exists a unique idempotent $a^\dagger$ satisfying that $a\mathcal{L}^{**}a^\dagger$ and $a = a^\dagger a$.

By a *band*, we mean a semigroup B in which every element is an idempotent. Call a band B a [*left; right*] *normal band* if B satisfies the identity [$abc = acb$; $abc = bac$] $abcd = acbd$.

Let $S = (Y; S_\alpha)$ be a semilattice decomposition of S into $\mathcal{N}$-clases S_α, and $T = (Z; T_\alpha)$ a semilattice decomposition of T into $\mathcal{N}$-classes T_α. Assume that that there exists an isomorphism φ of Y onto Z. The set $\cup_{\alpha \in Y}(S_\alpha \times T_{(\alpha)\varphi})$

is a subdirect product of S and T. We call it the *spined product of S and T relative to φ*, or simply a *spined product of S and T* (for detail, see [12]).

The following are our main results.

Theorem 2.1. *The following conditions on a semigroup S are equivalent:*

(1) *S is a PI-adequate wrpp semigroup;*

(2) *S satisfies the permutation identity: $x_1x_2x_3x_4 = x_1x_3x_2x_4$;*

(3) *S is isomorphic to the spined product of a C-$\mathcal{R}$-ca semigroup and a normal band.*

Theorem 2.2. *Let S be PI-adequate wrpp semigroup.*

(1) *If $E(S)$ is a left normal band, then S is isomorphic to the spined of a left normal band and a C-$\mathcal{R}$-ca semigroup. In this case, S satisfies the identity $xyz = xzy$.*

(2) *If $E(S)$ is a right normal band, then S is isomorphic to the spined of a right normal band and a C-$\mathcal{R}$-ca semigroup. In this case, S satisfies the identity $xyz = yxz$.*

(3) *If $E(S)$ is a semilattice, then S is isomorphic to a C-$\mathcal{R}$-ca semigroup. In this case, S is a commutative semigroup.*

3. Proofs

In this section, we shall give the proofs of Theorem 2.1 and Theorem 2.2. To begin with, we recall some known results.

Lemma 3.1. (see [13]) (1) *$\mathcal{L}^{**}$ is a right congruence on S and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^{**}$.*

(2) *$\mathcal{R}^{**}$ is a left congruence on S and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^{**}$.*

Lemma 3.2. (see [11]) *The following statements are equivalent for a band B:*

(1) *B is normal.*

(2) *B is a strong semilattice of rectangular bands.*

(3) *$\mathcal{L}$ and $\mathcal{R}$ are a left normal band congruence and a right normal band congruence on B, respectively.*

Lemma 3.3. (see [15]) *Let B be a band. If B satisfies permutation identities, then B is a normal band.*

Lemma 3.4. *If S is a wrpp semigroup, then $\mathcal{L}^{**}|_{E(S)} = \mathcal{L}|_{E(S)}$, where $E(S)$ is the set of idempotents of S.*

Proof. We need only to prove that $e\mathcal{L}^{**}f$ implies that $e\mathcal{L}f$ for all $e, f \in E(S)$. Now let $e\mathcal{L}^{**}f$. By the definition of wrpp semigroups, $ef = e$ and $fe = f$. Thus $e\mathcal{L}f$. $\square$

Lemma 3.5. *If S is a PI-adequate wrpp semigroup, then $E(S)$ is a normal band.*

Proof. By Lemma 3.3, we only need to show that $E(S)$ is a band since each subsemigroup of S still satisfy the same permutation identity. Let k be the smallest positive integers such that $\sigma(k) \neq k$. Let $\sigma(m) = k$. Obviously, $k < m$. Now let $e, f \in E(S)$. Put $x_i = f$ for all i such that $m \leq i \leq n$, and $x_i = e$ otherwise. Then $\prod_{i=1}^{n} x_i = ef$ and $\prod_{i=1}^{n} x_{\sigma(i)} = (ef)^2, efe$ or ef. Each of these cases implies that $ef = (ef)^2$. So, $E(S)$ is a band. □

It is well known that any band is a semilattice of rectangular bands. If $B = \bigcup_{\alpha \in Y} B_\alpha$ is the semilattice decomposition of a band B into rectangular bands B_α with $\alpha \in Y$, then we shall write $B_\alpha = E(e)$ for $e \in B_\alpha$, and $B_\alpha \geq B_\beta$ when $\alpha \geq \beta$ on the indexed semilattice Y.

Lemma 3.6. *Let S be an PI-adequate wrpp semigroup. If $a = af$ for all $f \in E(S)$, then we have $a^\dagger = a^\dagger f$.*

Proof. Since $\mathcal{L}^{**}$ is a right congruence, we have $a^\dagger f \mathcal{L}^{**} af = a \mathcal{L}^{**} a^\dagger$, and so by Lemma 3.4, $a^\dagger f \mathcal{L} a^\dagger$. Thus $E(a^\dagger f) = E(a^\dagger)$ and so $E(a^\dagger) = E(a^\dagger f) \leq E(f)$. Because $E(S)$ is a normal band, we know that $E(S)$ can be expressed by a strong semilattice $[Y; E_\alpha, \psi_{\alpha,\beta}]$ of the rectangular bands E_α. Suppose that $E_\alpha = E(f)$ and $E_\beta = E(a^\dagger)$. Then $\alpha \geq \beta$. Let $a^\dagger = (i, \lambda) \in E_\beta$ and $f\psi_{\alpha,\beta} = (j, \mu) \in E_\beta$. Then $a^\dagger f = a^\dagger(f\psi_{\alpha,\beta}) = (i, \lambda)(j, \mu) = (i, \mu)$. From above, $(i, \mu)\mathcal{L}(i, \lambda)$. This implies that $\lambda = \mu$. Thus $a^\dagger = a^\dagger f$. □

Let S be a strongly rpp semigroup whose idempotents form a band. We define a relation ε on S by:

$a\varepsilon b$ if and only if $a = ebf$ where $a, b \in S$ and $e, f \in E(b^\dagger)$.

Lemma 3.7 *Let S be a PI-adequate wrpp semigroup. Then ε is a congruence on S.*

Proof. Because S is an adequate wrpp semigroup, there exists $a^\dagger \in E(S)$ such that $a\mathcal{L}^{**}a^\dagger$ and $a = a^\dagger a$. Thereby $a = a^\dagger a a^\dagger$. Clearly, $a^\dagger \in E(a^\dagger)$. Thus ε is reflexive.

To see that ε is symmetric. First we proof that if $a\varepsilon b$, then $E(a^\dagger) = E(b^\dagger)$. Let $a\varepsilon b$. Then, by the definition of ε, we have $a = ebf$ for some $e, f \in E(b^\dagger)$. Hence $af = a$, and by Lemma 3.6, we have $a^\dagger = a^\dagger f$. This leads to $E(a^\dagger) \leq E(f) = E(b^\dagger)$. Again by $a = ebf$, we immediately get

$$b^\dagger a b^\dagger = b^\dagger ebfb^\dagger = (b^\dagger eb^\dagger)b(b^\dagger fb^\dagger) = b^\dagger bb^\dagger = b\,,$$

since $E(e)$ $(e \in E(S))$ is a rectangular band. Hence, by using the above argument, we have $E(b^\dagger) \leq E(a^\dagger)$. Thus $E(a^\dagger) = E(b^\dagger)$. Now, by $b^\dagger ab^\dagger = b$, we have $b\varepsilon a$.

We show here that ε is transitive. Suppose that $a\varepsilon b$ and $b\varepsilon c$. Then using the above arguments, we immediately have $E(a^\dagger) = E(b^\dagger) = E(c^\dagger)$, $a^\dagger ba^\dagger = a$, and $b^\dagger cb^\dagger = b$. The first equality can imply that $E(a^\dagger b^\dagger) = E(c^\dagger)$. At the same time, the latter two formulae yield that $a^\dagger b^\dagger cb^\dagger a^\dagger = a$. We have now proved that $a\varepsilon c$. So, ε is indeed an equivalence relation on S.

We show here that ε is right compatible. Suppose that $a, b, c \in S$ and $a\varepsilon b$. Then by the definition of ε, we have $a = ebf$ for some $e, f \in E(b^\dagger)$. Obviously, $b^\dagger eb^\dagger = b^\dagger$ and $b^\dagger fb^\dagger = b^\dagger$. By the facts $b^\dagger b = b = bb^\dagger, c^\dagger c = c$ and E is a normal band, we have

$$ac = ebfc = ebb^\dagger b^\dagger fc^\dagger c = ebb^\dagger fb^\dagger c^\dagger c = ebc.$$

Thereby,

$$ac = (ac)^\dagger ac = (ac)^\dagger ebc = (ac)^\dagger eb^\dagger b^\dagger bc = (ac)^\dagger b^\dagger eb^\dagger bc = (ac)^\dagger bc.$$

And from $ac = ebc$, we get $ac = ac(bc)^\dagger$, by Lemma 3.6, we have $(ac)^\dagger = (ac)^\dagger (bc)^\dagger$. This implies that $E((ac)^\dagger) \subseteq E((bc)^\dagger)$. Symmetrically, we also have $E((bc)^\dagger) \subseteq E((ac)^\dagger)$. Thus $E((ac)^\dagger) = E((bc)^\dagger)$. Since $ac = (ac)^\dagger bc = (ac)^\dagger bc(bc)^\dagger$ and $(ac)^\dagger \in E((ac)^\dagger) = E((bc)^\dagger)$, we obtain that $ac\varepsilon bc$. We can see that ε is right compatible.

Finally, we show that ε is left compatible. Let $a\varepsilon b$. Then, by the definition of ε, we have $a = ebf$ for $e, f \in E(b^\dagger)$. Hence

$$ca = cebf = cc^\dagger eb^\dagger b^\dagger bf = cc^\dagger b^\dagger eb^\dagger bf = cbf.$$

Consequently,

$$cab^\dagger = cbfb^\dagger = cbb^\dagger fb^\dagger = cb.$$

This leads to $(ca)^\dagger ca(ca)^\dagger b^\dagger = cb$. On the other hand, by $ca = cbf$, we have $ca = caf$, and by Lemma 3.6, we obtain $(ca)^\dagger = (ca)^\dagger f$. Hence, by $f \in E(b^\dagger)$, we obtain $E((ca)^\dagger) = E((ca)^\dagger f) = E((ca)^\dagger b^\dagger)$, that is, $(ca)^\dagger b^\dagger \in E((ca)^\dagger)$. So we immediately get $ca\varepsilon cb$, and ε is left compatible. All above, we have proved that ε is a congruence. $\square$

Lemma 3.8. *In the above lemma, if $a\mathcal{L}^{**}b$ for all $a, b \in S$, then $a\varepsilon \mathcal{L}^{**S/\varepsilon} b\varepsilon$. That is, the relation $\mathcal{L}^{**}$ on S holds hereditarily on the quotient semigroup S/ε. Moreover, S/ε is a wrpp semigroup.*

Proof. We need to verify that ε preserves the $\mathcal{L}^{**}$-class of S. For this purpose, we let $a,b \in S$ and $(a,b) \in \mathcal{L}^{**}$. If there are $x,y \in S^1$ such that $((ax)\varepsilon,(ay)\varepsilon) \in \mathcal{R}^{S/\varepsilon}$, then there exists $u,v \in S^1$ such that $(ax)\varepsilon\ (u)\varepsilon = (ay)\varepsilon$ and $(ay)\varepsilon\ (v)\varepsilon = (ax)\varepsilon$. Hence we can find $e,f \in E((ay)^\dagger)$ and $g,h \in E((ax)^\dagger)$ such that $axu = eayf$ and $ayv = gaxh$. Note that $E(S)$ is a normal band. We have

$$\begin{aligned} axu(ay)^\dagger = a^\dagger axu(ay)^\dagger = a^\dagger eayf(ay)^\dagger = a^\dagger e(ay)^\dagger(ay)^\dagger ay(ay)^\dagger f(ay)^\dagger \\ = a^\dagger(ay)^\dagger e(ay)^\dagger ay(ay)^\dagger = a^\dagger(ay)^\dagger ay = ay\,, \end{aligned}$$

and similarly $ayv(ax)^\dagger = ax$. Thus $(ax,ay) \in \mathcal{R}$. This implies that $(bx,by) \in \mathcal{R}$ since $(a,b) \in \mathcal{L}^{**}$. By the definition of $\mathcal{R}$, there exists $r,s \in S^1$ such that $bxr = by$ and $bys = bx$. It follows that $(bx)\varepsilon(r)\varepsilon = (by)\varepsilon$ and $(by)\varepsilon(s)\varepsilon = (bx)\varepsilon$. Thus, we have $((bx)\varepsilon,(by)\varepsilon) \in \mathcal{R}$. From this and its dual, we conclude that $(a\varepsilon,b\varepsilon) \in \mathcal{L}^{**S/\varepsilon}$. This shows that the relation $\mathcal{L}^{**}$ is preserved in the quotient semigroup S/ε and hence S/ε is a wrpp semigroup. □

Lemma 3.9. *ε is idempotent-pure, that is, $x\varepsilon \in E(S/\varepsilon)$ implies that $e \in E(S)$.*

Proof. Let $x\varepsilon \in E(S/\varepsilon)$. Then we have $x\varepsilon x^2$. By the definition of ε, $x^2 = exf$ for some $e,f \in E(x^\dagger)$. Thus $x^2 = x^\dagger exfx^\dagger = x^\dagger xx^\dagger = x$. Consequently, ε is idempotent-pure. □

Lemma 3.10. *S/ε is a C-$\mathcal{R}$-ca semigroup.*

Proof. By Lemma 3.8 and Lemma 3.9, S/ε is an adequate wrpp semigroup whose idempotents is the semilattice $E(S)/\mathcal{D}^E$, and so S/ε is also an PI-adequate wrpp semigroup. Let $x,y \in S/\varepsilon$, and let k be the smallest positive integer such that $\sigma(k) \neq k$. Put $\sigma(m) = k$. Obviously, $k < m$. If $x_k = x, x_m = y$ and the other $x_i = x^\dagger$, then $\prod_{i=1}^n x_i = xy$ or $xyx^\dagger$, and $\prod_{i=1}^n x_{\sigma(i)} = yx$ or $x^\dagger yx$. Now take $y = e \in E(S/\varepsilon)$. Note that $E(S/\varepsilon)$ is a semilattice. We have $xe = ex$. This implies that $E(S/\varepsilon)$ is in the center of S/ε. Thus $\prod_{i=1}^n x_i = xy = xyx^\dagger$ and $\prod_{i=1}^n x_{\sigma(i)} = yx = x^\dagger yx$, and so we immediately get that $xy = yx$. That is, S/ε is a commute C-wrpp semigroup. It follows that S/ε is a strong semilattice of commute $\mathcal{R}$-cancellative monoids. Then S/ε is a C-$\mathcal{R}$-ca semigroup. □

Let S be a PI-adequate wrpp semigroup. Then $E(S)$ is a normal band. And by Lemma 3.2, $E(S)$ is a strong semilattice of the rectangular bands E_α, says $E(S) = [Y; E_\alpha, \psi_{\alpha,\beta}]$. It can be easily seen that $\varepsilon|E(S) = \mathcal{D}^{E(S)}$, where ε is the congruence on S. Since ε is idempotent-pure, we obtain that $E(S)/\varepsilon|_{E(S)} \cong Y$.

Therefore, we can identify $E(S)/\varepsilon|_{E(S)}$ with Y. By Lemma 3.10, S/ε is a C-$\mathcal{R}$-ca semigroup. Now let S/ε be a strong semilattice of commutative $\mathcal{R}$-cancellative monoids M_α, in notation, $[Y; M_\alpha, \varphi_{\alpha,\beta}]$. In fact, M_α is an $\mathcal{L}^{**}$-class of S/ε. Hence, the spined product of S/ε and the band $E(S)$ with respect to the semilattice Y is $M = \bigcup_{\alpha\in Y}(M_\alpha \times E_\alpha)$, where the multiplication on M is defined by $(m,n)\cdot(n,j) = (mn, ij)$, and mn and ij are the semigroup product of $m, n \in S/\varepsilon$ and $i, j \in E(S)$.

Lemma 3.11. *If S is a PI-adequate wrpp semigroup, then S is isomorphic to the spined product of a C-$\mathcal{R}$-ca semigroup and a normal band.*

Proof. We still need to verify that the mapping $\theta : S \to M$ defined by: for all $s \in S$,

$$s \mapsto (s\varepsilon, s^\dagger)$$

is an isomorphism. Now let $(s\varepsilon, s^\dagger) = (t\varepsilon, t^\dagger)$. Then $s\varepsilon = t\varepsilon$ and $s^\dagger = t^\dagger$. By the definition of ε, there exists $e, f \in E(t^\dagger)$ such that $s = etf$. Recall that $E(S)$ is a normal band and $E(t^\dagger)$ is a rectangular band, we have

$$s = s^\dagger s s^\dagger = s^\dagger etf s^\dagger = t^\dagger etf t^\dagger = t^\dagger e t^\dagger t t^\dagger f t^\dagger = t^\dagger t t^\dagger = t\,.$$

This shows that θ is an injective mapping.

To see that θ is a surjective mapping, we let $(a,i) \in M$. Then there exists $x \in S$ such that $x\varepsilon = a$. Also, by definition of M, we have $x^\dagger \in E(i)$. We next prove that $(ixi)\theta = (a,i)$. For this, we need only to show that $(ixi)^\dagger = i$ and $(ixi)\varepsilon = a$. In fact, since $x^\dagger \in E(i)$, we have $(ixi)\varepsilon = x\varepsilon$. It follows that $(ixi)\varepsilon = a$. On the other hand, if $m, n \in S^1$ and $((ixi)m, (ixi)n) \in \mathcal{R}$, then $((ixi)im, (ixi)in) \in \mathcal{R}$, and so by $ixi\mathcal{L}^{**}(ixi)^\dagger$, $((ixi)^\dagger ix, (ixi)^\dagger iy) \in \mathcal{R}$ and $(i(ixi)^\dagger ix, i(ixi)^\dagger iy) \in \mathcal{R}$. This means that $(ix, iy) \in \mathcal{R}$. If $m, n \in S^1$ and $(im, in) \in \mathcal{R}$, then $((ixi)m, (ixi)n) \in \mathcal{R}$. We have now proved that $ixi\mathcal{L}^{**}i$. However, by $i(ixi) = ixi$, we obtain that $(ixi)^\dagger = i$ since S is an adequate wrpp semigroup. Therefore $(ixi)\theta = (a,i)$ and whence θ is a surjective mapping.

The final step is to show that θ is a homomorphism. Let $a, b \in S$. Then by Lemma 3.8 and Lemma 3.9,

$$\begin{aligned}(ab)^\dagger\varepsilon\mathcal{L}^{**S/\varepsilon}(ab)\varepsilon = (a\varepsilon)(b\varepsilon)\mathcal{L}^{**S/\varepsilon}(a^\dagger\varepsilon)(b\varepsilon)&\\ = (b\varepsilon)(a^\dagger\varepsilon)\mathcal{L}^{**S/\varepsilon}(b^\dagger\varepsilon)(a^\dagger\varepsilon) = (b^\dagger a^\dagger)\varepsilon\,,&\end{aligned}$$

since $\mathcal{L}^{**}$ is a right congruence. Hence

$$(ab)^\dagger\varepsilon = (ab)^\dagger\varepsilon(a^\dagger b^\dagger)\varepsilon = (a^\dagger b^\dagger)\varepsilon(ab)^\dagger\varepsilon = (a^\dagger b^\dagger)\varepsilon$$

and thus $E((ab)^\dagger) = E(a^\dagger b^\dagger)$. It follows that $E((ab)^\dagger) \leq E(a^\dagger)$ and $E((ab)^\dagger) \leq E(b^\dagger)$. Since $E(S)$ is a normal band, we have $a^\dagger(ab)^\dagger \mathcal{L}(ab)^\dagger$, and so $a^\dagger(ab)^\dagger \mathcal{L}^{**}(ab)^\dagger$ by Lemma 3.4. And by $a^\dagger(ab)^\dagger ab = ab$, we get $a^\dagger(ab)^\dagger = (ab)^\dagger$ since S is an adequate wrpp semigroup. By considering that $ab = (ab)b^\dagger$, by Lemma 3.6, we immediately have that $(ab)^\dagger = (ab)^\dagger b^\dagger$. Thus, $(ab)^\dagger = a^\dagger(ab)^\dagger b^\dagger$. Assume that $E(S)$ can be expressed by a strong semilattice $[Y; E_\alpha, \psi_{\alpha,\beta}]$ of the rectangular bands E_α. If we let $a^\dagger \in E_\alpha$ and $b^\dagger \in E_\beta$, then by $E((ab)^\dagger) = E(a^\dagger b^\dagger)$, we have $(ab)^\dagger \in E_{\alpha\beta}$ and also we have

$$(ab)^\dagger = a^\dagger(ab)^\dagger b^\dagger = (a^\dagger\psi_{\alpha,\alpha\beta})(ab)^\dagger(b^\dagger\psi_{\beta,\alpha\beta}) = (a^\dagger\psi_{\alpha,\alpha\beta})(b^\dagger\psi_{\beta,\alpha\beta}) = a^\dagger b^\dagger,$$

so that $(ab)^\dagger = a^\dagger b^\dagger$.

We have now proved that θ is an isomorphism. Whence S is isomorphic to a spined product of a C-$\mathcal{R}$-ca semigroup and a normal band. □

Proof of Theorem 2.1. (1) ⇒ (3) It follows from Lemma 3.11.

(3) ⇒ (2) Note that any normal band satisfies the identity: $xyzw = xzyw$. The rest of the proof is a routine calculation.

(2) ⇒ (1) It is trivial. □

Proof of Theorem 2.2. It is a routine calculation by Lemma 3.11. □

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