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EQUIENERGETIC DIGRAPHS

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Abstract: The energy of a graph G is defined as $\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. Recently, many authors have considered the problem of generating pairs of non-cospectral equienergetic graphs. Since every graph G can be identified with a symmetric digraph G^*, and the concept of the energy was recently generalized to digraphs, it is natural to consider the problem of generating pairs of non-symmetric, non-cospectral and equienergetic digraphs.

AMS Subject Classification: 05C25
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1. Introduction

Let G be a graph with set of vertices $V_G = \{v_1, \ldots, v_n\}$ and set of edges E_G. The adjacency matrix $A = (a_{ij})$ of G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_i v_j \in E_G\,, \\ 0 & \text{if} \quad v_i v_j \notin E_G\,. \end{cases}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then the energy of G is defined as

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$$\mathbb{E}(G) = \sum_{i=1}^{n} |\lambda_i| .$$

The concept of energy has its origin in chemistry: for those graphs which in Hückel's orbital molecular theory represent the scheleton of carbon atoms, the energy is related with the total π-electron energy. For more details on this theory we refer the reader to [3] and the survey [4].

Several articles have recently appeared in the mathematical literature in relation to the problem of constructing non-cospectral equienergetic graphs (see [1], [7], [8], [9] and [11]). More precisely, the authors in (see [7] and [8]) give a systematic construction of pairs of non-cospectral connected graphs of the same order and having equal energies. In another direction, R. Balakrishnan [1] shows that for a given integer $n \geq 3$, there exists two non-cospectral graphs of order $4n$ which have equal energies. This result was later generalized by H.S. Ramane and H.B. Walikar [9], showing that it is possible to construct pairs of non-cospectral connected equienergetic graphs of order n, for every integer $n \geq 9$.

On the other hand, the concept of energy was recently generalized to digraphs [6] as follows: if D is a digraph with eigenvalues $\lambda_1, \ldots, \lambda_n$ then

$$\mathbb{E}(D) = \sum_{i=1}^{n} |\mathrm{Re}\lambda_i| .$$

Note that in the case of digraphs, the eigenvalues $\lambda_1, \ldots, \lambda_n$ can be complex numbers. This generalization was done in such a way that well-known results in the theory of energy for graphs still hold for digraphs (for example, the Coulson's integral formula). Since the set of graphs can be identified with the set of symmetric digraphs, the results on the construction of non-cospectral equienergetic graphs translates into the construction of non-cospectral equienergetic symmetric digraphs. Naturally, the following problems arise.

Problem 1.1. *Is it possible to give a method to construct pairs of non-cospectral non-symmetric digraphs of the same order and having equal energies?*

Problem 1.2. *Given a positive integer n, is it possible to construct pairs of non-cospectral non-symmetric equienergetic digraphs with n vertices?*

The aim of this work is to study these two problems. We do this using similar techniques as in graphs. However, we need to define the complete product of digraphs in a way that the well-known formulas for the characteristic polynomial of the complete product of (regular) graphs [2] can be generalized to digraphs. Also, we consider the line digraph introduced by F. Harary and

R.Z. Norman [5] and the results obtained by Rosenfeld [10] on the characteristic polynomial of the line digraph, in order to generate pairs of non-symmetric non-cospectral equienergetic digraphs.

2. Preliminaries

A directed graph (or digraph for short) D, is a nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of D called arcs. The vertex set is denoted by V_D and the set of arcs by E_D. If $a = (u, v)$ is an arc of D then u is said to be adjacent to v and v is adjacent from u. The outdegree of a vertex u is denoted by $od\,(u)$ and defined as the number of vertices of D that are adjacent from u. Similarly, the indegree $id\,(u)$ of u is the number of vertices of D adjacent to u. A digraph D is r-regular if $id\,(u) = od\,(u) = r$ for every $u \in V_D$.

Let D be a digraph. A walk π of length l from vertex u to vertex v is a sequence of vertices

$$\pi : u = u_0, u_1, \ldots, u_l = v\,,$$

where (u_{t-1}, u_t) is an arc of D for every $1 \leq t \leq l$. If $u = v$ then π is a closed walk. If $u = v$ but $u_i \neq u_j$ for $i \neq j$ $(i, j = 1, \ldots, l)$ then π is a cycle of D. D is strongly connected if for every pair of distinct vertices u, v of D, there exists a walk from u to v and a walk from v to u. A strong component of a digraph D is a maximal subdigraph with respect to the property of being strongly connected.

A digraph is called symmetric if whenever (u, v) is an arc of D then (v, u) is also an arc of D. The set of graphs can be identified with the set of symmetric digraphs. Indeed, if G is a graph then G^* denotes the symmetric digraph obtained by replacing each edge of G by a symmetric pair of arcs. Then the correspondence $G \rightsquigarrow G^*$ is one-to-one between the set of graphs and symmetric digraphs. It is easy to see that if G is a connected graph then G^* is strongly connected. Also, if G is r-regular then G^* is r-regular.

Let D be a digraph with set of vertices $V_D = \{v_1, \ldots, v_n\}$. The adjacency matrix of D is the matrix $A = (a_{ij})$ defined as

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_i v_j \in E_D\,, \\ 0 & \text{if} \quad v_i v_j \notin E_D. \end{cases}$$

The characteristic polynomial of D, denoted by $\Phi_D\,(x)$, is the characteristic polynomial of A. In other words,

$$\Phi_D\,(x) = |xI - A|\ ,$$

where I is the $n \times n$ identity matriz and $|M|$ is the determinant of the matriz M. The eigenvalues of D are, by definition, the eigenvalues of A. An important difference between the spectrum of a graph and a digraph is that in general, the eigenvalues of a digraph can be complex numbers, since in this case the adjacency matrix is not necessarily symmetric.

As we mentioned in Introduction, the energy of a digraph D, denoted by $\mathbb{E}(D)$, is defined as

$$\mathbb{E}(D) = \sum_{i=1}^{n} |\mathrm{Re}\lambda_i| ,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D.

The following result is a consequence of the so called coefficient theorem for digraphs (see [2, Theorem 1.2]). Recall that the direct sum $D_1 \oplus D_2$ of the digraphs D_1 and D_2 is the digraph

$$V_{D_1 \oplus D_2} = V_{D_1} \cup V_{D_2} \text{ and } E_{D_1 \oplus D_2} = E_{D_1} \cup E_{D_2}.$$

Theorem 2.1. *Let D be a digraph and $D_1, \ldots, D_s$ its strong componentes. Then*

$$\mathbb{E}(D) = \sum_{i=1}^{s} \mathbb{E}(D_i).$$

Proof. Let $W = \{a \in E_D : a \text{ does not belong to a cycle of } D\}$. Then by the coefficient theorem for digraphs, $\Phi_D(x) = \Phi_{D-W}(x)$, where $D - W$ is the digraph obtained from D by deleting every arc in W. Since $D - W = D_1 \oplus \cdots \oplus D_s$ and

$$\Phi_{D-W}(x) = \Phi_{D_1}(x) \cdot \cdots \cdot \Phi_{D_s}(x)$$

it follows that

$$\mathbb{E}(D) = \mathbb{E}(D - W) = \sum_{i=1}^{s} \mathbb{E}(D_i). \qquad \square$$

Example 2.2. In Figure 1, the strong components D_1 and D_2 of D are shown.

Then

$$\mathbb{E}(D) = \mathbb{E}(D_1) + \mathbb{E}(D_2) = 2 + 2\sqrt[4]{2}.$$

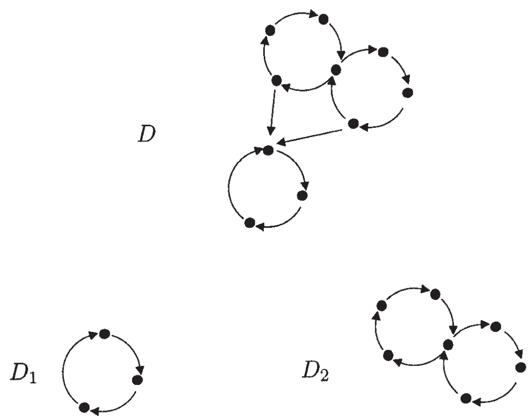

Figure 1:

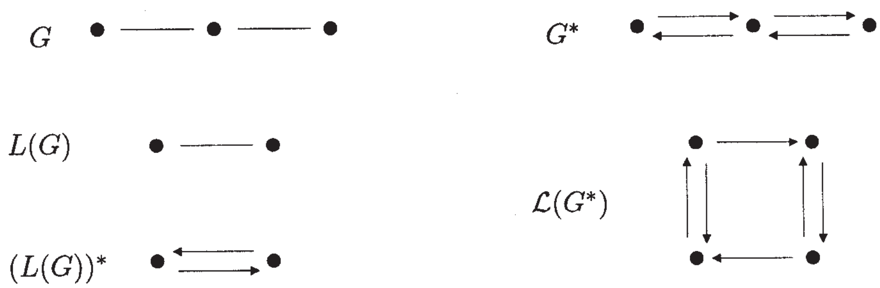

Figure 2:

3. Line Digraphs

The line digraph was introduced by Harary and Norman [5] as an adaptation of the line graph to digraphs. Recall that if D is a digraph then the line digraph of D, which we will denote by $\mathcal{L}(D)$, is the digraph with set of vertices E_D (the arcs of D); there is an arc in $\mathcal{L}(D)$ from (u,v) to (w,z) if $v=w$. The iterated line digraphs of D are defined recursively as $\mathcal{L}^1(D)=\mathcal{L}(D)$ and for $k\geq 2$, $\mathcal{L}^k(D)=\mathcal{L}\left(\mathcal{L}^{k-1}(D)\right)$.

Remark 3.1. Let G be a graph and $L(G)$ the line graph of G. In general (see Figure 2),

$$(L(G))^* \neq \mathcal{L}(G^*).$$

Proposition 3.2. *Let D be a digraph. Then:*

1. If D is strongly connected then $\mathcal{L}(D)$ is strongly connected;

2. If D is r-regular then $\mathcal{L}(D)$ is r-regular;

3. If D has a non-closed walk of length 2 then $\mathcal{L}(D)$ is a non-symmetric digraph;

4. If D has $n \geq 3$ vertices and is strongly connected then $\mathcal{L}^k(D)$ is a non-symmetric digraph for every $k \geq 1$.

Proof. 1. Let (u,v) and (w,z) be vertices of $\mathcal{L}(D)$. Since D is strongly connected, there exists a walk from v to w : $v = u_0, u_1, \ldots, u_l = w$, where (u_{t-1}, u_t) is an arc of D for every $1 \leq t \leq l$. Then

$$(u,v), (u_0,u_1), \ldots, (u_{l-1},u_l), (w,z)$$

is a walk in $\mathcal{L}(D)$ from (u,v) to (w,z). Similarly, there exists a walk from (w,z) to (u,v).

2. Let (u,v) be a vertex of $\mathcal{L}(D)$. Since D is r-regular, there exists exactly r arcs of the form (v,w). Hence there are exactly r arcs in $\mathcal{L}(D)$ of the form $((u,v),(v,w))$ and so the outdegree of (u,v) is r. Similarly, the indegree of (u,v) is r.

3. Let u,v,w be a non-closed walk of D of length 2. Then $((u,v),(v,w))$ is an arc in $\mathcal{L}(D)$. Since $u \neq w$ then $((v,w),(u,v))$ is not an arc in $\mathcal{L}(D)$. Hence $\mathcal{L}(D)$ is a non-symmetric digraph.

4. It is easy to see that if D has $n \geq 3$ vertices and is strongly connected then $\mathcal{L}(D)$ has a non-closed walk of length 2. Since $\mathcal{L}(D)$ has also $m \geq 3$ vertices and is strongly connected then the result follows from part 3. □

The relation between the characteristic polynomial of a digraph D and the line digraph $\mathcal{L}(D)$ was obtained by Rosenfeld [10]: let D be a digraph with n vertices and l arcs. Then

$$\Phi_{\mathcal{L}(D)}(x) = x^{l-n}\Phi_D(x). \tag{1}$$

Proposition 3.3. *Let D be a digraph. Then $\mathbb{E}(D) = \mathbb{E}\left(\mathcal{L}^k(D)\right)$ for every integer $k \geq 1$.*

Proof. By (1) D and $\mathcal{L}(D)$ have the same eigenvalues and the same multiplicities except possibly the multiplicity of 0, but this has no influence on the value of the energy. Hence $\mathbb{E}(D) = \mathbb{E}(\mathcal{L}(D))$, and consequently $\mathbb{E}(D) = \mathbb{E}\left(\mathcal{L}^k(D)\right)$ for all $k \geq 1$. □

Now we can give a method to construct pairs of non-symmetric, equienergetic but non-cospectral digraphs.

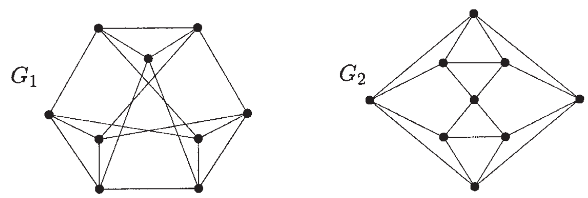

Figure 3:

Theorem 3.4. *Let D_1 and D_2 be symmetric, equienergetic and non-cospectral digraphs. Suppose that D_1 and D_2 are r-regular, strongly connected and have $n \geq 3$ vertices. Then $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$ are non-symmetric, equienergetic and non-cospectral digraphs for every integer $k \geq 1$. Furthermore, they are strongly connected, r-regular and have equal number of vertices and arcs.*

Proof. Let $k \geq 1$. By Proposition 3.2, $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$ are strongly connected, r-regular and non-symmetric. It is clear that if D is r-regular and has n vertices then D has nr arcs. Hence, $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$ have nr^k vertices. Since the vertices of $\mathcal{L}^{k+1}(D_1)$ and $\mathcal{L}^{k+1}(D_2)$ are the arcs of $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$, then $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$ have nr^{k+1} arcs. By Proposition 3.3, $\mathbb{E}\left(\mathcal{L}^k(D_1)\right) = \mathbb{E}\left(\mathcal{L}^k(D_2)\right)$ since D_1 and D_2 are equienergetic. Finally, by (1)

$$\Phi_{\mathcal{L}(D_1)}(x) = x^{nr-n}\Phi_{D_1}(x)$$

and

$$\Phi_{\mathcal{L}(D_2)}(x) = x^{nr-n}\Phi_{D_2}(x).$$

Since D_1 and D_2 are non-cospectral then $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ are non-cospectral. A recursive argument shows that $\mathcal{L}^k(D_1)$ and $\mathcal{L}^k(D_2)$ are non-cospectral. □

Note that there exists infinite pairs of digraphs which satisfy the hypothesis of Theorem 3.4 (see [8, Theorem 4]).

Example 3.5. Consider the graphs G_1 and G_2 shown in Figure 3.
The characteristic polynomials of G_1 and G_2 are

$$\Phi_{G_1}(x) = (x-4)(x-1)^4(x+2)^4$$

and

$$\Phi_{G_2}(x) = (x-4)(x-2)(x-1)^2(x+1)^2(x+2)^3.$$

Then G_1 and G_2 are non-cospectral and $\mathbb{E}(G_1) = \mathbb{E}(G_2) = 16$. Furthermore, G_1 and G_2 are connected, have 9 vertices and are 4-regular. Therefore, the associated symmetric digraphs G_1^* and G_2^* are equienergetic, non-cospectral, have 9 vertices, are strongly connected and 4-regular. Hence, G_1^* and G_2^* satisfy the hypothesis of Theorem 3.4.

4. Complete Product of Digraphs

The complete product $G_1 \nabla G_2$ of the graphs G_1 and G_2 is the graph obtained from $G_1 \oplus G_2$ by joining every vertex of G_1 with every vertex of G_2. The characteristic polynomial of the complete product is given by the relation (see [2, Theorem 2.7])

$$\begin{aligned} \Phi_{G_1 \nabla G_2}(x) &= (-1)^{n_2} \Phi_{G_1}(x) \Phi_{\overline{G_2}}(x)(-x-1) \\ &\quad + (-1)^{n_1} \Phi_{G_2}(x) \Phi_{\overline{G_1}}(x)(-x-1) \\ &\quad - (-1)^{n_1+n_2} \Phi_{\overline{G_1}}(x)(-x-1) \Phi_{\overline{G_2}}(x)(-x-1), \end{aligned} \tag{2}$$

where n_1 and n_2 are the number of vertices of G_1 and G_2, respectively, and $\overline{G}$ denotes the complement of G. Furthermore, if G_i is r_i-regular ($i = 1, 2$) then relation (2) is simpler (see [2, Theorem 2.8]):

$$(x - r_1)(x - r_2) \Phi_{G_1 \nabla G_2}(x) = [(x - r_1)(x - r_2) - n_1 n_2] \Phi_{G_1}(x) \Phi_{G_2}(x). \tag{3}$$

We next generalize the concept of complete product to digraphs in such a way that relations (2) and (3) hold.

Definition 4.1. The complete product $D_1 \nabla D_2$ of the digraphs D_1 and D_2 is the digraph obtained from $D_1 \oplus D_2$ by joining by a pair of symmetric arcs each vertex of D_1 with each vertex of D_2.

Example 4.2. In Figure 4 we illustrate the complete product of digraphs. Note that the pair of symmetric arcs between two vertices are represented by a double headed arrow.

Recall that the complement of a digraph D, denoted by $\overline{D}$, is the digraph with set of vertices $V_{\overline{D}} = V_D$; if u, v are two different vertices of $\overline{D}$ then

$$(u, v) \in E_{\overline{D}} \Leftrightarrow (u, v) \notin E_D.$$

Proposition 4.3. *Let D_1 and D_2 be two digraphs. Then*

$$\overline{D_1 \oplus D_2} = \overline{D_1} \nabla \overline{D_2}.$$

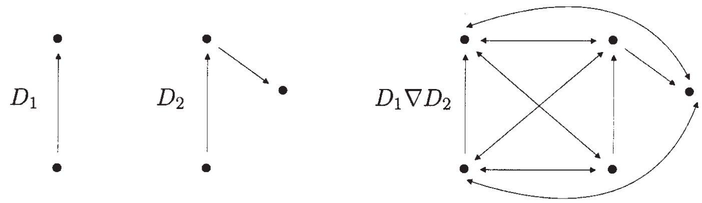

Figure 4:

Proof. It follows from

$$(u,v) \notin E_{\overline{D_1 \oplus D_2}} \Leftrightarrow (u,v) \in E_{D_1 \oplus D_2} \Leftrightarrow$$
$$(u,v) \in E_{D_1} \cup E_{D_2} \Leftrightarrow (u,v) \notin E_{\overline{D_1} \nabla \overline{D_2}}. \quad \square$$

Let D be a digraph with n vertices. If $H_D(x) = \sum_{k=0}^{\infty} N_k x^k$ is the generating function of the numbers N_k of walks of length k in D, then it is easy to adapt the proof of [2, Theorem 1.11] to obtain

$$H_D(x) = \frac{1}{x}\left[(-1)^n \frac{\Phi_{\overline{D}}\left(-\frac{x+1}{x}\right)}{\Phi_D\left(\frac{1}{x}\right)} - 1\right]. \tag{4}$$

Theorem 4.4. *Let D_1 and D_2 be digraphs with n_1 and n_2 vertices, respectively. Then*

$$\Phi_{D_1 \nabla D_2}(x) = (-1)^{n_2} \Phi_{D_1}(x) \Phi_{\overline{D_2}}(x)(-x-1) + (-1)^{n_1} \Phi_{D_2}(x) \Phi_{\overline{D_1}}(x)$$
$$\times (-x-1) - (-1)^{n_1+n_2} \Phi_{\overline{D_1}}(x)(-x-1) \Phi_{\overline{D_2}}(x)(-x-1).$$

Proof. Apply (4) on both sides of the relation

$$H_{D_1 \oplus D_2}(x) = H_{D_1}(x) + H_{D_2}(x).$$

Then use Proposition 4.3 and the fact that

$$\Phi_{D_1 \oplus D_2}(x) = \Phi_{D_1}(x) \Phi_{D_2}(x). \quad \square$$

If D is a r-regular digraph with n vertices then the proof of [2, Theorem 2.6] also works for digraphs:

$$\Phi_{\overline{D}}(x) = (-1)^n \frac{x-n-r+1}{x+r+1} \Phi_D(-x-1). \tag{5}$$

Hence, from Theorem 4.4 and (5) we generalize (2) and (3).

Corollary 4.5. *Let D_i be r_i-regular digraphs with n_i vertices, $i = 1, 2$. Then*

$$(x - r_1)(x - r_2)\Phi_{D_1 \nabla D_2}(x) = [(x - r_1)(x - r_2) - n_1 n_2]\Phi_{D_1}(x)\Phi_{D_2}(x). \tag{6}$$

Now we extend (see [9, Lemma 1]) to digraphs.

Corollary 4.6. *Let D_i be r_i-regular digraphs with n_i vertices ($i = 1, 2$). Then*

$$\mathbb{E}(D_1 \nabla D_2) = \mathbb{E}(D_1) + \mathbb{E}(D_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2). \tag{7}$$

Proof. The roots of the polynomial on the left side of (6) are r_1, r_2 and the eigenvalues of $D_1 \nabla D_2$. Hence, the sum of the absolute values of the real parts of these roots is

$$\mathbb{E}(D_1 \nabla D_2) + r_1 + r_2. \tag{8}$$

On the other hand, the roots of the polynomial on the right side of (6) are the eigenvalues of D_1, D_2 and $\frac{1}{2}\left(r_1 + r_2 \pm \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}\right)$. Similarly, the sum of the absolute values of the real parts of these roots is

$$\mathbb{E}(D_1) + \mathbb{E}(D_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}. \tag{9}$$

Clearly (8) and (9) must be equal and the result follows. □

Let $D_1 = \mathcal{L}(G_1^*)$ and $D_2 = \mathcal{L}(G_2^*)$, where G_1^* and G_2^* are given in Example 3.5. By Theorem 3.4, D_1 and D_2 are non-symmetric digraphs which are strongly connected, equienergetic, non-cospectral, 4-regular, with equal number of vertices and arcs. Furthermore, since G_1^* and G_2^* have 9 vertices and are 4-regular then D_1 and D_2 have 36 vertices.

Theorem 4.7. *Let D_1 and D_2 be the digraphs with 36 vertices we defined above and K_p the complete digraph with p vertices. Then $D_1 \nabla K_p$ and $D_2 \nabla K_p$ are non-symmetric, equienergetic and non-cospectral digraphs. Also, they are strongly connected and have $36 + p$ vertices.*

Proof. It is clear that $D_1 \nabla K_p$ and $D_2 \nabla K_p$ are non-symmetric. Since D_1 and D_2 are non-cospectral, 4-regular, and have 36 vertices, and K_p is $(p-1)$-regular with p vertices, it follows from (6) that $D_1 \nabla K_p$ and $D_2 \nabla K_p$ are non-cospectral. Since the characteristic polynomial of K_p is

$$\Phi_{K_p}(x) = (x - p + 1)(x + 1)^{p-1},$$

then $\mathbb{E}(K_p) = 2(p-1)$. On the other hand, since D_1 and D_2 are 4-regular digraphs such that $\mathbb{E}(D_1) = \mathbb{E}(D_2) = 16$, it follows from (7) that

$$\mathbb{E}(D_1 \nabla K_p) = 16 + 2(p-1) + \sqrt{(4+p-1)^2 + 4(9p - 4(p-1))} - (4+p-1) = \mathbb{E}(D_2 \nabla K_p).$$

Therfore, $D_1 \nabla K_p$ and $D_2 \nabla K_p$ are equienergetic digraphs. It is clear that they are also strongly connected and have $36+p$ vertices. □

Corollary 4.8. *For each integer $n \geq 36$, there exists a pair of non-symmetric, equienergetic, non-cospectral and strongly connected digraphs with n vertices.*

Proof. This is an immediate consequence of Theorem 4.7. □

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