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ON THE PARTIAL SUMS OF WAVELET PACKET SERIES

Firdous Ahmad Shah[1], Khalil Ahmad[2 §]
[1,2]Department of Mathematics
Jamia Millia Islamia (Central University)
New Delhi, 110025, INDIA
[1]e-mail: fashah_jmi@yahoo.co.in
[2]e-mail: khalil_ahmad49@yahoo.com

Abstract: It is shown that for a given distribution f belonging to the Sobolev space $\mathcal{H}^{1/2}$, its partial sums of wavelet packet expansions behave like truncated versions of inverse Fourier transform of $\hat{f}$. Our result is sharp in the sense that such behavior no longer happens in general for $\mathcal{H}^s$ if $s < 1/2$.

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1. Introduction

A simple, but powerful extension of wavelets and multiresolution analysis is wavelet packets, pioneered by Coifman, Meyer, Wickerhauser and other researchers [3, 4, 5, 6, 15]. Wavelet packet functions comprise a rich family of building block functions. Wavelet packet functions are still localized in time, but offer more flexibility than wavelets in representing different types of signals. In particular, wavelet packets are better at representing signals that exhibit oscillatory or periodic behavior.

Wavelet packet functions are generated by scaling and translating a family of basic function shapes, which include father wavelet φ and mother wavelet ψ. In addition to φ and ψ there is a whole range of wavelet packet functions ω_n. These functions are parameterized by an oscillation or frequency index n. A father wavelet corresponds to $n = 0$, so $\varphi = \omega_0$. A mother wavelet corresponds to $n = 1$, so $\psi = \omega_1$. Larger values of n correspond to wavelet packets with more oscillations and higher frequency.

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§Correspondence author

The problem of convergence of the wavelet series has been studied by Meyer [11], Walter [13, 14], Tao [12] and Kelly et al [9, 10]. Meyer was amongst the first to study convergence results for wavelet expansions. He has shown that the regular wavelet expansions converge in L^p, $1 \leq p < \infty$ and also in L^∞ for expansions of uniformly continuous functions, the expansion of continuous functions converge everywhere. The results in [11] were based on the assumption of so called regularity for the basic wavelets and their derivatives. In addition, Walter [13, 14] established pointwise convergence results for regular wavelet expansions of continuous functions. Kelly et al [9, 10] have extended and obtained results analogous to those obtained by Carleson [2] and Hunt [8] for the Fourier series. In contrast, the results in [9, 10] assumed only that the wavelets being used be bounded by radial decreasing L^1-functions. In [10], it is shown that the wavelet expansions of a function belonging to L^p converges pointwise everywhere on the Lebesgue set of a given function, for $1 \leq p < \infty$.

Recently, Ahmad and Kumar have generalized the results of Kelly et al [9, 10], Walter [13, 14] and Hernández and Weiss [7] for wavelet packet setting in [1] and have shown that such expansions of $L^p(\mathbb{R})$ functions $(1 \leq p \leq \infty)$ converges pointwise almost everywhere on the Lebesgue set of the functions being expanded.

The main purpose of this paper is to study the behavior at points outside the Lebesgue set of a function being expanded which was not discussed in [1]. Our result describes the behavior of wavelet packet expansions at each point of the real line, and thus may be used to determine behavior at singularities.

2. Preliminaries

Throughout, the functions f, g, and ω_n will stand for $f(x)$, $g(x)$ and $\omega_n(x)$ respectively. Let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and real numbers, respectively. The inner product of two functions $f, g \in L^2(\mathbb{R})$ is denoted by $\langle f, g\rangle$ and is defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

The norm of $f \in L^2(\mathbb{R})$ is written as $\|f\|_2$. The Fourier transform of any function $f \in L^2(\mathbb{R})$ is denoted by $\hat{f}$ and is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$$

so that the inverse Fourier transform of $f \in L^2(\mathbb{R})$ becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi.$$

Definition 2.1. (see [7]) The Sobolev spaces $\mathcal{H}^s(\mathbb{R})$ consists of all functions (or tempered distributions in the case where $s < 0$) such that

$$\|f\|^2_{\mathcal{H}^s} = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi < \infty.$$

For basic ideas, results on wavelets, wavelet packets and multiresolution analysis, we refer to [3]-[7], and [15].

We construct wavelet packets from multiresolution analysis. In general, consider two sequences $\{\alpha_n\}_{n\in\mathbb{Z}}$ and $\{\beta_n\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$. Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$. Then, the sequences

$$f_{2n} = \sqrt{2} \sum_{k\in\mathbb{Z}} \alpha_{2n-k} e_k, \qquad f_{2n+1} = \sqrt{2} \sum_{k\in\mathbb{Z}} \beta_{2n-k} e_k$$

are orthonormal bases of two orthogonal closed subspaces $\mathbb{H}_1$ and $\mathbb{H}_0$, respectively, such that

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_0.$$

Using this "splitting trick" we now define the basic wavelet packets associated with the scaling function φ as defined in MRA.

Let $\omega_0 = \varphi\cdot$ The basic wavelet packets $\omega_n,\ n = 0, 1, 2, \dots,$ associated with the scaling function φ are defined recursively by

$$\begin{cases} \omega_{2n}(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k\, \omega_n(2x - k)\,, \\ \omega_{2n+1}(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} g_k\, \omega_n(2x - k)\ . \end{cases} \tag{2.1}$$

It follows from the above definition that $\omega_1 = \psi$ is a mother wavelet and the set $\{\omega_n(x - k) : n = 0, 1, ..., k \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$.

Corresponding to some orthogonal scaling function $\varphi = \omega_0$, the family of wavelet packets $\{\omega_n\}$ defines a family of subspaces of $L^2(\mathbb{R})$ as follows:

$$U^n_j = \operatorname{span}\{2^j \omega_n(2^j x - k) :\ k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \dots\ . \tag{2.2}$$

Observe that

$$U^0_j = V_j \ \text{ and } \ U^1_j = W_j\,,$$

so that the orthogonal decomposition can be written as

$$U_{j+1}^0 = U_j^0 \oplus U_j^1. \tag{2.3}$$

A generalization of this result for other values of $n = 1, 2, 3, \ldots\ldots$can be written as

$$U_{j+1}^n = U_j^{2n} \oplus U_j^{2n+1}, \quad j \in \mathbb{Z}\,. \tag{2.4}$$

Now, we state a lemma which will be used in the proof of the preceding results.

Lemma 2.2. *For each $j = 1, 2, ...,$ decomposition trick (2.4) gives*

$$\begin{cases} W_j = U_j^1 = U_{j-1}^2 \oplus U_{j-1}^3\,, \\ W_j = U_{j-2}^4 \oplus U_{j-2}^5 \oplus U_{j-2}^6 \oplus U_{j-2}^7\,, \\ \vdots \\ W_j = U_{j-k}^{2^k} \oplus U_{j-k}^{2^k+1} \oplus \ldots\ldots \oplus U_{j-k}^{2^{k+1}-1}\,, \\ \vdots \\ W_j = U_0^{2^j} \oplus U_0^{2^j+1} \oplus \ldots\ldots \oplus U_0^{2^{j+1}-1}, \end{cases} \tag{2.4}$$

where U_j^n is defined in (2.2). Moreover, for each $j = 1, 2, \ldots\ ; k = 1, 2, \ldots, j$ and $m = 0, 1, 2, \ldots\ldots, 2^k - 1$, and the set $\left\{2^{\frac{j-k}{2}}\omega_p(2^{j-k}x - \ell) : \ell \in \mathbb{Z}\right\}$ is an orthonormal basis of U_{j-k}^p, where $p = 2^k + m$. However, all the elements of this basis have the general form

$$\omega_{j,n,k}(x) = 2^{j/2}\omega_n(2^j x - k)\,. \tag{2.6}$$

If a function $f \in L^2(\mathbb{R})$, then

$$f(x) \sim \sum_{j\in\mathbb{Z}} \sum_{n=2^p}^{2^{p+1}-1} \sum_{k\in\mathbb{Z}} C_{\ell,n,k}\, \omega_{\ell,n,k}(x)\,, \tag{2.7}$$

where $\ell = j - p,\ p = 0$ if $j < 0$ and $p = 0, 1, 2, \ldots, j$ if $j \geq 0$; will be a wavelet packet expansion of f and $C_{\ell,n,k}$ the wavelet packet coefficients defined as

$$C_{\ell,n,k} = \langle f, \omega_{\ell,n,k}\rangle. \tag{2.8}$$

3. Main Results

Throughout the section, we shall denote $\mathbb{S}$ and $\mathbb{S}^{'}$ for the Schwartz class and the space of tempered distributions, respectively. We also assume that ω_n will be the functions belonging to the Schwartz class $\mathbb{S}$ and its Fourier transforms be supported on $I \cup 2I$, with $I = [-2a, -a] \cup [a, 2a]$ for some $a > 0$. The Zak transform of any wavelet packet ω_n is defined as

$$Z(x, \xi) = \sum_{n=2^p}^{2^{p+1}-1} \sum_{k \in \mathbb{Z}} \omega_n(x-k)\, e^{i\xi k}.$$

Our goal in this note is to identify classes of functions f such that

$$S_m f(x) - \frac{1}{\sqrt{2\pi}} \int_{I_m} \hat{f}(\xi)\, e^{i\xi x}\, d\xi \to 0, \qquad m \to \infty\,, \tag{3.1}$$

uniformly on the real line, where

$$S_m f(x) = \sum_{n=2^p}^{2^{p+1}-1} \sum_{j=-m}^{m} \sum_{k \in \mathbb{Z}} \langle f, \omega_{\ell,n,k} \rangle\, \omega_{\ell,n,k}(x), \tag{3.2}$$

and $I_m = \left\{ t \in \mathbb{R} : 2^{-m} < \alpha^{-1}|t| < 2^m \right\}$, where α is some positive constant independent of f.

Along the scale of Sobolev spaces $\mathcal{H}^s, -\infty < s < \infty$, we shall prove that the smallest real number s for which (3.1) holds uniformly on the real line, is $s = 1/2$.

Theorem 3.1. *Let $f \in \mathbb{S}^{'}$ such that $\hat{f} \in L^1_{\ell oc}$. Given a real number x and a positive integer m,*

$$S_m f(x) = \frac{1}{\sqrt{2\pi}} \int_{I_m} \hat{f}(\xi)\, e^{i\xi x}\, d\xi + R_m f(x) + r_m f(x), \tag{3.3}$$

where $S_m f(x)$ has been defined in (3.2), $I_m = \bigcup_{s=-m+1}^{m} 2^s I$ and

$$r_m f(x) = \int_{2^{-m} I} \hat{f}(\xi)\, \overline{\hat{\omega}_n(2^m \xi)}\, Z(2^{-m}x, 2^m \xi)\, d\xi, \tag{3.4}$$

$$R_m f(x) = \int_{2^{m+1} I} \hat{f}(\xi)\, \overline{\hat{\omega}_n(2^{-m} \xi)}\, Z(2^m x, 2^{-m} \xi)\, d\xi. \tag{3.5}$$

The proof of Theorem 3.1 is broken down into two lemmas.

Lemma 3.2. *Let $f \in \mathbb{S}'$ such that $\hat{f} \in L^1_{\ell oc}$. Given a real number x and a positive integer m,*

$$S_m f(x) = \sum_{s=-m+1}^{m} \int_{2^s I} \hat{f}(\xi) H_s(2^{-s}\xi, x)\, d\xi + R_m f(x) + r_m f(x)\,,$$

where

$$H_s(\xi, x) = \overline{\hat{\omega}_n(\xi)}\, Z(2^s x, \xi) + \overline{\hat{\omega}_n(2\xi)}\, Z(2^{s-1}x, 2\xi).$$

Proof. Applying Parseval's identity, we can write

$$S_m f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) K_m(\xi, x)\, d\xi\,,$$

where

$$K_m(\xi, x) = \sum_{j=-m}^{m} \overline{\hat{\omega}_n(2^{-j}\xi)}\, Z(2^j x, 2^{-j}\xi).$$

Since the support of $K_m(.\,, x)$ is contained in $\bigcup_{u=-m}^{m+1} 2^u I$, we have

$$S_m f(x) = \sum_{j=-m}^{m+1} \int_{2^s I} \hat{f}(\xi) K_m(\xi, x)\, d\xi.$$

By isolating the terms in the summation corresponding to $s = -m$ and $s = m+1$, one can observe that $\hat{\omega}_n(2^{-m}\xi) = 0$ if $\xi \in 2^{-m}I$ and $-m+1 \le j \le m$. Hence,

$$K_m(\xi, x) = \overline{\hat{\omega}_n(2^m\xi)}\, Z(2^{-m}x, 2^m\xi), \qquad \xi \in 2^{-m}I. \tag{3.6}$$

Similarly, we obtain

$$K_m(\xi, x) = \overline{\hat{\omega}_n(2^{-m}\xi)}\, Z(2^m x, 2^{-m}\xi), \qquad \xi \in 2^{m+1}I. \tag{3.7}$$

Equations (3.6) and (3.7) give the desired forms of $r_m f(x)$ and $R_m f(x)$, respectively.

Now, if $-m+1 \le s \le m$, then $\hat{\omega}_n(2^{-j}\xi) = 0$ whenever $\xi \in 2^s I$ and either $s+1 \le j \le m$ or $-m \le j \le s-2$. Hence,

$$K_m(\xi, x) = \overline{\hat{\omega}_n(2^{1-s}\xi)}\, Z(2^{s-1}x, 2^{1-s}\xi) + \overline{\hat{\omega}_n(2^{-s}\xi)}\, Z(2^s x, 2^{-s}\xi)\,,$$

whenever $\xi \in 2^s I$ with $-m+1 \le s \le m$. This completes the proof of the lemma. □

Lemma 3.3. *Given $x \in \mathbb{R},\ \xi \in I,\ s \in \mathbb{Z}$,*

$$\frac{\exp(i2^s\xi x)}{\sqrt{2\pi}} = H_s(\xi, x)\,, \tag{3.8}$$

where $H_s(\xi, x)$ has been defined in the statement of Lemma 3.2.

Proof. Fix a real number x_0, an integer k, and define $f \in L^2(\mathbb{R})$ such that

$$\hat{f}(\xi) = \begin{cases} \overline{H_k(2^{-k}\xi, x_0)} - \dfrac{\exp(-i\xi x_0)}{\sqrt{2\pi}}, & \text{if } \xi \in 2^k I, \\ 0, & \text{otherwise.} \end{cases}$$

Now, if $-m < k < m+1$, then by Lemma 3.2, we have

$$S_m f(x) = 2^k \int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) H_k(\xi, x)\, d\xi.$$

Obviously, $S_m f$ is independent of m. Since $S_m f$ converges to f in $L^2(\mathbb{R})$, then for almost every x,

$$\begin{aligned} &\int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) H_k(\xi, x)\, d\xi \\ &\qquad = \frac{1}{\sqrt{2\pi}} \int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) \exp(i2^k\xi x)\, d\xi. \end{aligned}$$

Since both the sides of equation are continuous functions of x, so equality must hold everywhere. For $x = x_0$, the equality can be written as

$$\int_I \left| H_k(\xi, x_0) - \frac{\exp(i2^k\xi x_0)}{\sqrt{2\pi}} \right| d\xi = 0. \qquad \square$$

Theorem 3.4. *Suppose that $f \in \mathcal{H}^{1/2}$, then*

$$\sup_{x\in\mathbb{R}} \left| S_m f(x) - \frac{1}{\sqrt{2\pi}} \int_{I_m} \hat{f}(\xi)\, e^{i\xi x}\, d\xi \right| \to 0, \text{ as } m \to \infty. \tag{3.9}$$

This result is sharp as the following example shows. Recall that $I_m = \bigcup_{s=-m+1}^{m} 2^s I$, the support of $\hat{\omega}_n$ is contained in $I \cup 2I$ and $I = [-2a, -a] \cup [a, 2a]$, where $a > 0$.

Fix a real number x_0 and define $f \in L^2(\mathbb{R})$ such that $\hat{f}(\xi) = 0$ if $|\xi| \le 2a$ and

$$\hat{f}(\xi) = 2^{-m-1}\hat{\omega}_n(2^{-m}\xi)\, \overline{Z(2^m x_0, 2^{-m}\xi)} \quad \text{if } \xi \in 2^{m+1} I,$$

for $m = 0, 1, 2, ...$ The boundedness of Z shows that for each $s < 1/2, f \in \mathcal{H}^s$. Indeed,

$$\int_{-\infty}^{\infty} |u|^{2s} |\hat{f}(\xi)|^2 \, d\xi \leq C \sum_{m=0}^{\infty} 2^{m(2s-1)} \int_{2I} |\xi|^{2s} \Big| \hat{\omega}_n(\xi) \Big|^2 \, d\xi \,,$$

where $4C = \sup \big\{ |Z(x, \xi)|^2 : x, \xi \in \mathbb{R} \big\}$.

However, (3.9) does not hold. If it were true that

$$S_m f(x_0) - \frac{1}{\sqrt{2\pi}} \int_{I_m} \hat{f}(\xi) \, e^{i\xi x_0} \, d\xi \to 0,$$

then by Theorem 3.1, $r_m f(x_0) + R_m f(x_0) \to 0$. Observe that

$$R_m f(x_0) = \frac{1}{2} \int_{2I} \Big| \hat{\omega}_n(\xi) \, Z(2^m x_0, \xi) \Big|^2 d\xi \,,$$

while $r_m f(x_0) = 0$ for each positive integer m.

This would imply that for some sequence of positive integers $\{n_k\}$,

$$\lim_{k \to \infty} \Big| \hat{\omega}_n(\xi) \, Z(2^{n_k} x_0, \xi) \Big| = 0 \,,$$

for almost every ξ in $2I$. By Lemma 3.2, we obtain

$$\lim_{k \to \infty} \Big| \hat{\omega}_n(\xi) \, Z(2 \,.\, 2^{n_k} x_0, \xi) \Big| = \frac{1}{\sqrt{2\pi}} \,,$$

for almost every ξ in I. This is impossible since Z is bounded while $\hat{\omega}_n$ is continuous with $\hat{\omega}_n(a) = 0$.

Proof. We only have to show that both $R_m f$ and $r_m f$ tend to zero uniformly on $\mathbb{R}$. Using Hölder's inequality and the boundedness of Z, we obtain for all real numbers x,

$$|R_m f(x)|^2 \leq b_m \int_{2^{m+1} I} |\xi| \,. |\hat{f}(\xi)|^2 \, d\xi \,, \quad \text{where} \quad b_m \leq C \int_{2I} \frac{d\xi}{|\xi|},$$

and the constant C depends only on the wavelet packets ω_n. Hence if f belongs to $\mathcal{H}^{1/2}$,

$$\sup_{x \in \mathbb{R}} |R_m f(x)| \to 0,$$

as m tends to ∞. Similarly, we have $\sup_{x \in \mathbb{R}} |r_m f(x)| \to 0$. □

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