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GENERAL SYSTEM OF $(A, \eta)-$MONOTONE VARIATIONAL INCLUSION PROBLEMS BASED ON A GENERALIZED HYBRID ALGORITHM

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Abstract: In this paper, a newsy stem of nonlinear set-valued variational inclusions involving (A, η)-monotone mappings in Hilbert spaces is introduced and examined. Using (A, η)-resolvent operator method, approximation solvability of this system based on a generalized iterative algorithm is investigated. The obtained results are general in nature.

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1. Introduction

The notion of (A, η)-monotonicity is introduced in [21], which generalizes A-monotonicity [18]-[20] as well as H-monotonicity [5], [6]. Based on the resolvent operator corresponding to (A, η)-monotonicity, the author examined the sensitivity analysis for quasivariational inclusion problems using (A, η)-resolvent operator technique. This class of systems seems to impact greatly the theory of general maximal monotone mappings in several modes of applications ranging from variational inequalities to variational inclusions. As a matter of fact, varia tional inclusion problems represent significant generalizations to classical variational inequalities, and do offer a wide range of applications to several other fields such as mechanics, physics, operations research, optimization and control theory, nonlinear programming, economics, and engineering sciences. In [10],

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Jin investigated the approximation solvability of a new system of nonlinear set-valued variational inclusions based on the convergence of (H, η)-resolvent operator technique applied in [7], while the convergence analysis for approximate solutions much depends on the existence of Cauchy sequences generated by a new iterative algorithm. Just recently the author [21] explored sensitivity analysis for strongly monotone variational inclusions using (A, η)-resolvent operator technique in a Hilbert space setting.

In this paper, we intend to consider, based on the generalized (A, η)-resolvent operator method, the existence and approximation of solutions for a general system of nonlinear set-valued variational inclusions involving relaxed cocoercive mappings in Hilbert spaces. The convergence analysis for a generalized iterative algorithm based on the generalized resolvent operator corresponding to (A, η)-monotonicity is discussed in detail. The obtained results generalize the results [7, 10] on (H, η)-monotone mappings, and others. For more details, we refer the reader to [1-21].

2. Preliminaries

Let X be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$. Let 2^X and $C(X)$ denote the family of all the nonempty subsets of X and the family of all closed subsets of X, respectively. Let us recall the following definitions and some known results.

Definition 2.1. Let $T, A : X \to X$ be single-valued mappings. The map T is said to be:

(i) Monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0 \qquad \text{for all } x, y \in X.$$

(ii) Strictly monotone, if T is monotone and

$$\langle Tx - Ty, x - y \rangle = 0$$

if and only if $x = y$.

(iii) r-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2 \qquad \text{for all } x, y \in X.$$

(iv) s-strongly monotone with respect to A, if there exists a constant $s > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y) \rangle \geq s\|x - y\|^2 \qquad \text{for all } x, y \in X\,.$$

(v) (c,μ)-relaxed cocoercive with respect to A, if there exist constants $c,\mu > 0$ such that

$$\langle T(x)-T(y), A(x)-A(y)\rangle \geq (-c)\|T(x)-T(y)\|^2+\mu\|x-y\|^2 \quad \text{for all } x,y\in X\,.$$

(vi) t-Lipschitz continuous, if there exists a constant $t>0$ such that

$$\|T(x)-T(y)\| \leq t\|x-y\| \qquad \text{for all } x,y\in X.$$

Definition 2.2. A single-valued mapping $\eta : X\times X \to X$ is said to be:
(i) Monotone, if

$$\langle x-y, \eta(x,y)\rangle \geq 0\,, \quad \text{for all } x,y\in X.$$

(ii) Strictly monotone, if

$$\langle x-y, \eta(x,y)\rangle \geq 0\,, \quad \text{for all } x,y\in X$$

and equality holds if and only if $x=y$.
(iii) δ-strongly monotone, if there exists a constant $\delta>0$ such that

$$\langle x-y, \eta(x,y)\rangle \geq \delta\|x-y\|^2\,, \quad \text{for all } x,y\in X\,.$$

(iv) τ-Lipschitz continuous, if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\|, \qquad \text{for all } x,y\in X.$$

Definition 2.3. Let $\eta : X\times X\to X$ and let $A,H : X\to X$ be single-valued mappings. A set-valued mapping $M : X\to 2^X$ is said to be:
(i) Monotone, if

$$\langle u-v, x-y\rangle \geq 0, \qquad \forall x,y\in X, \quad u\in Mx, v\in My.$$

(ii) η-monotone, if

$$\langle u-v, \eta(x,y)\rangle \geq 0, \qquad \forall x,y\in X, u\in Mx, v\in My.$$

(iii) Strictly η-monotone, if M is η-monotone and equality holds if and only if $x=y$.
(iv) r -strongly η-monotone, if there exists a constant $r>0$ such that

$$\langle u-v, \eta(x,y)\rangle \geq r\|x-y\|^2, \quad \forall x,y\in X, u\in Mx, v\in My.$$

(v) (m,η) -relaxed monotone, if there exists a constant $m>0$ such that

$$\langle u-v,\eta(x,y)\rangle \geq (-m)\|x-y\|^2, \qquad \forall x,y\in X, u\in Mx, v\in My.$$

(vi) Maximal monotone, if M is monotone and $(I+\lambda M)(X)=3$, for all $\lambda>0$, where I denotes the identity mapping on X.

(v) Maximal η-monotone, if M is η-monotone and $(I+\lambda M)(X)=X$, for all $\lambda>0$.

(vi) A-monotone, if M is (m)-relaxed monotone and $(A+\lambda M)(X)=X$, for all $\lambda>0$.

(vii) (A,η)-monotone, if M is (m,η)-relaxed monotone and $(A+\lambda M)(X)=X$, for all $\lambda>0$.

(viii) H-monotone, if M is monotone and $(H+\lambda M)(X)=X$, for all $\lambda>0$.

(ix) (H,η)-monotone, if M is η-monotone and $(H+\lambda M)(X)=X$, for all $\lambda>0$.

Lemma 2.1. *Let $\eta: X\times X\to X$ be a single-valued mapping, $A: X\to X$ be a strictly η-monotone mapping and $M: X\to 2^X$ be an (A,η) -monotone mapping. Then the mapping $(A+\lambda M)^{-1}$ is single-valued.*

By Lemma 2.1, we can define the resolvent operator $R^{A,\eta}_{M,\lambda}$ as follows.

Definition 2.4. Let $\eta: X\times X\to X$ be a single-valued mapping, $A: X\to X$ be a strictly η-monotone mapping and $M: X\to 2^X$ be an (H,η)-monotone mapping. The resolvent operator $R^{A,\eta}_{M,\lambda}: X\to X$ is defined by

$$R^{A,\eta}_{M,\lambda}(z)=(A+\lambda M)^{-1}(z) \qquad \text{for all } z\in X,$$

where $\lambda>0$ is a constant.

Lemma 2.2. (see [21]) *Let $\eta: X\times X\to X$ be a τ-Lipschtiz continuous mapping, $A: X\to X$ be an (r,η)-strongly monotone mapping and $M: X\to 2^X$ be an (A,η)-monotone mapping. Then the resolvent operator $R^{A,\eta}_{M,\lambda}: X\to X$ is $(\frac{\tau}{r-\lambda m})$-Lipschitz continuous, that is,*

$$\|R^{A,\eta}_{M,\lambda}(x)-R^{A,\eta}_{M,\lambda}(y)\|\leq \frac{\tau}{r-\lambda m}\|x-y\| \qquad \textit{for all } x,y\in X.$$

We define a Hausdorff pseudo-metric $D: 2^X\times 2^X\to(-\infty,+\infty)\cup\{+\infty\}$ by

$$D(\cdot,\cdot)=\max\{\sup_{u\in U}\inf_{v\in V}\|u-v\|, \sup_{u\in V}\inf_{v\in U}\|u-v\|\}$$

for any $U,V\in 2^X$. Note that if the domain of D is restricted to closed bounded subsets, then D is the Hausdorff metric.

Definition 2.5. A set-valued mapping $U : X \to 2^X$ is said to be D-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$D(U(u), U(v)) \leq \gamma \|u - v\|, \qquad \text{for all } u, v \in X.$$

3. System of Variational Inclusions

This section deals with the solvability of a generalsyste m of set-valued variational inclusions involving (A,η)-monotone mappings in Hilbert spaces. Let X_1 and X_2 be two real Hilbert sp aces, $K_1 \subset X_1$ and $K_2 \subset X_2$ be two nonempty, closed and convex sets. Let $F : X_1 \times X_2 \to X_1$, $G : X_1 \times X_2 \to X_2$, $A_i : X_i \to X_i$, and $\eta_i : X_i \times X_i \to X_i (i = 1, 2)$ be nonlinear mappings. Let $U : X_1 \to 2^{X_1}$ and $V : X_2 \to 2^{X_2}$ be set-valued mappings, $M_i : X_i \to 2^{X_i}$ be $(A_i,\ eta_i)$-monotone mappings $(i = 1, 2)$. The problem of determining an element $(a, b) \in X_1 \times X_2$, with $u \in U(a)$ and $v \in V(b)$ such that

$$\begin{cases} 0 \in F(a, v) + M_1(a) \\ 0 \in G(u, b) + M_2(b), \end{cases} \tag{3.1}$$

is called the nonlinear set-valued variational inclusion syste m problem.

Special Cases

Case (I) If $M_1(x) = \partial\varphi(x)$ and $M_2 = \partial\phi(y)$ for all $x \in X_1$ and $y \in X_2$, where $\varphi : X_1 \to R \cup \{+\infty\}$ and $\phi : X_2 \to R \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functionals, $\partial\varphi$ and $\partial\phi$ denote the subdifferential operators of φ and ϕ, respectively, then problem (3.1) reduces to the following problem: find $(a, b) \in X_1 \times X_2$, $u \in U(a)$, and $v \in V(v)$ such that

$$\begin{cases} \langle F(a, v), x - a\rangle + \varphi(x) - \varphi(a) \geq 0, & \forall x \in X_1, \\ \langle G(u, b), y - b\rangle + \phi(y) - \phi(b) \geq 0, & \forall y \in X_2, \end{cases} \tag{3.2}$$

is called a system of set-valued mixed variational inequalities.

Case (II) If A and B are both identity mappings, then problem (3.2) reduces to the following problem: find $(a, b) \in X_1 \times X_2$ such that

$$\begin{cases} \langle F(a, b), x - a\rangle + \varphi(x) - \varphi(a) \geq 0, & \forall x\ inX_1, \\ \langle G(a, b), y - b\rangle + \phi(y) - \phi(b) \geq 0, & \forall y \in X_2, \end{cases} \tag{3.3}$$

is called system of nonlinear variational inequalities considered by Cho, Fang, Huang and Hwang [3]. Some special cases of problem (3.3) were studied by Kim and Kim [14].

Case (III) If $M_1(x) = \partial\delta_{K_1}(x)$ and $M_2(y) = \partial\delta_{K_2}(y)$, for all $x \in K_1$ and $y \in K_2$, where $K_1 \subset X_1$ and $K_2 \subset X_2$ are two nonempty, closed, and convex subsets, and δ_{K_1} and δ_{K_2} denote the indicator functions of K_1 and K_2, respectively, then problem (3.2) reduces to the following system of variational inequalities: find $(a, b) \in K_1 \times K_2$ such that

$$\begin{cases} \langle F(a,b), x-a\rangle \geq 0, & \forall x \in K_1, \\ \langle G(a,b), y-b\rangle \geq 0, & \forall y \in K_2, \end{cases} \tag{3.4}$$

is the problem in [20] with both F and G single-valued.

Case (IV) If $X_1 = X_2 = X, K_1 = K_2 = K, F(x,y) = \rho T(y) + x - y$, and $G(x,y) = \gamma T(x) + y - x$, for all $x, y \in X$, where $T : K \to X$ is a nonlinear mapping, $\rho > 0$ and $\gamma > 0$ are two constants, then problem (3.4) reduces to the following system of variational inequalities: find $(a, b) \in K \times K$ such that

$$\begin{cases} \langle \rho T(b) + a - b, x-a\rangle \geq 0, & \forall x \in K, \\ \langle \gamma T(a) + b - a, x-b\rangle \geq 0, & \forall x \in K. \end{cases} \tag{3.5}$$

Case (V) If A and B are both identity mappings, then problem (3.1) reduces to the following problem [7]: find $(a, b) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in F(a,b) + M_1(a) \\ 0 \in G(a,b) + M_2(b). \end{cases} \tag{3.6}$$

4. Convergence Analysis and Algorithms

In this section, using the resolvent operator method associated with (A, η)-monotone mappings, a new iterative algorithm for solving problem (2.1) is examined. The convergence of the iterative sequence generated by the algorithm is discussed.

Theorem 4.1. *For given $(a, b) \in X_1 \times X_2, u \in U(a), v \in V(b)$, (a, b, u, v) is a solution of problem (3.1) if and only if (a, b, u, v) satisfies the relation*

$$\begin{cases} a = R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(a) - \rho_1 F(a,v)], \\ b = R^{A_2,\eta_2}_{M_2,\rho_2}[A_2(b) - \rho_2 G(u,b)], \end{cases} \tag{4.1}$$

where $\rho_i > 0$ are two constants for $i = 3, 2$.

Proof. This directly follows from Definition 2.4. □

The relation (4.1) and Nadler [22] allows us to suggest the following iterative algorithms.

Algorithm 4.1. *Step 1.* Choose $(a_0, b_0) \in X_1 \times X_2$ and choose $u_0 \in U(a_0)$ and $v_0 \in V(b_0)$.

Step 2. Let

$$\begin{cases} a_{n+1} = (1 - \lambda_n - \delta_n)a_n + \lambda_n R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(a_n) - \rho_1 F(a_n, v_n)], \\ b_{n+1} = (1 - \lambda_n - \delta_n)b_n + \lambda_n R^{A_2,\eta_2}_{M_2,\rho_2}[A_2(b_n) - \rho_2 G(u_n, b_n)], \end{cases} \tag{4.2}$$

where λ_n and δ_n are nonnegative constants such that $0 < \lambda_n + \delta_n \leq 1$, and $\limsup_{n\geq 0} \lambda_n < 1$.

Step 3. Choose $u_{n+1} \in U(a_{n+1})$ and $v_{n+1} \in V(b_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \leq (1 + (1 + n)^{-1})D_1(U(a_{n+1}), U(a_n)), \\ \|v_{n+1} - v_n\| \leq (1 + (1 + n)^{-1})D_2(V(b_{n+1}), V(b_n)), \end{cases} \tag{4.3}$$

where $D_i(\cdot, \cdot)$ is the Hausdorff pseudo-metric on 2^{X_i} for $i = 1, 2$.

Step 4. If $a_{n+1}, b_{n+1}, u_{n+1}$ and v_{n+1} satisfy (4.2) to a sufficient degree of accuracy, stop; otherwise, set $n := n + 1$ and return to Step 2.

Algorithm 4.2. *Step 1.* Choose $(a_0, b_0) \in X_1 \times X_2$ and choose $u_0 \in U(a_0)$ and $v_0 \in V(b_0)$.

Step 2. Let

$$\begin{cases} a_{n+1} = (1 - \lambda - \delta)a_n + \lambda R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(a_n) - \rho_1 F(a_n, v_n)], \\ b_{n+1} = (1 - \lambda - elta)b_n + \lambda R^{A_2,\eta_2}_{M_2,\rho_2}[A_2(b_n) - \rho_2 G(u_n, b_n)], \end{cases} \tag{4.4}$$

where λ and δ are nonnegative constants such that $0 < \lambda + \delta \leq 1$ is a constant.

Step 3. Choose $u_{n+1} \in U(a_{n+1})$ and $v_{n+1} \in V(b_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \leq (1 + (1 + n)^{-1})D_1(U(a_{n+1}), U(a_n)), \\ \|v_{n+1} - v_n\| \leq (1 + (1 + n)^{-1})D_2(V(b_{n+1}), V(b_n)), \end{cases} \tag{4.5}$$

where $D_i(\cdot, \cdot)$ is the Hausdorff pseudo-metric on 2^{X_i} for $i = 1, 2$.

Step 4. If $a_{n+1}, b_{n+1}, u_{n+1}$ and v_{n+1} satisfy (4.4) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 2.

Algorithm 4.3. *Step 1.* Choose $(a_0, b_0) \in X_1 \times X_2$ and choose $u_0 \in U(a_0)$ and $v_0 \in V(b_0)$.

Step 2. Let

$$\begin{cases} a_{n+1} = (1 - \lambda)a_n + \lambda R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(a_n) - \rho_1 F(a_n, v_n)], \\ b_{n+1} = (1 - \lambda)b_n + \lambda R^{A_2,\eta_2}_{M_2,\rho_2}[A_2(b_n) - \rho_2 G(u_n, b_n)], \end{cases} \tag{4.6}$$

where λ is nonnegative constant such that $0 < \lambda \leq 1$ is a constant.

Step 3. Choose $u_{n+1} \in U(a_{n+1})$ and $v_{n+1} \in V(b_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \leq (1 + (1+n)^{-1})D_1(U(a_{n+1}), U(a_n)), \\ \|v_{n+1} - v_n\| \leq (1 + (1+n)^{-1})D_2(V(b_{n+1}), V(b_n)), \end{cases} \tag{4.7}$$

where $D_i(\cdot, \cdot)$ is the Hausdorff pseudo-metric on 2^{X_i} for $i = 1, 2$.

Step 4. If $a_{n+1}, b_{n+1}, u_{n+1}$ and v_{n+1} satisfy (4.6) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 2.

Theorem 4.2. *Let $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous mappings, $A_i : X_i \to X_i$ (r_i, η)-strongly monotone and β_i-Lipschitz continuous mappings, $M_i : X_i \to 2_i^X$ be (A_i, η_i)-monotone mappings for $i = 1, 2$. Let $U : X_1 \to C(X_1)$ be D_1-γ_1-Lipschitz continuous and $V : X_2 \to C(X_2)$ be D_2-γ_2-Lipschitz continuous. Let $F : X_1 \times X_2 \to X_1$ be a nonlinear mapping such that for any given $(a, b) \in X_1 \times X_2$, $F(\cdot, b)$ is (c_1, μ_1)-relaxed cocoercive with respect to A_1 and α_1-Lipschitz continuous and $F(a, \cdot)$ is ζ_1-Lipschitz continuous. Let $G : X_1 \times X_2 \to X_2$ be another nonlinear mapping such that for any given $(x, y) \in X_1 \times X_2$, $G(x, \cdot)$ is (c_2, μ_2)-relaxed cocoercive with respect to A_2 and α_2-Lipschitz continuous and $G(\cdot, y)$ is ζ_2-Lipschitz continuous. If there exist constants $\rho_i > 0$ for $i = 1, 2$ such that*

$$\begin{cases} \tau_1 r_2 \sqrt{\beta_1^2 - 2\rho_1\mu_1 + \ rho_1^2\alpha_1^2 + 2\rho_1 c_1\alpha_1^2} + \tau_2 r_1\zeta_2\gamma_1 < r_1 r_2, \\ \tau_2 r_1 \sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2 + 2\rho_2 c_2\alpha_2^2} + \tau_1 r_2\zeta_1\gamma_2 < r_1 r_2. \end{cases} \tag{4.8}$$

Then problem (3.1) admits a solution (a, b, u, v) and iterative sequences $\{a_n\}, \{b_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to a, b, u and v, respectively, where $\{a_n\}, \{b_n\}, \{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 4.1.

Proof. Applying Algorithm 4.1 and Lemma 2.2, we have

$$\begin{aligned} \|a_{n+1} - a_n\| &= \|(1 - \lambda_n - \delta_n)a_n + \lambda - nR_{M_1,\rho_1}^{A_1,\eta_1}(A_1(a_n) - \rho_1 F(a_n, v_n)) \\ &\quad - [(1 - \lambda_n - \delta_n)a_{n-1} + \lambda_n R_{M_1,\rho_1}^{A_1,\eta_1}(A_1(a_{n-1}) - \rho_1 F(a_{n-1}, v_{n-1}))]\| \\ &\leq (1 - \lambda_n - \delta_n)\|a_n - a_{n-1}\| + \lambda_n \|R_{M_1,\rho_1}^{A_1,\eta_1}(A_1(a_n) - \rho_1 F(a_n, v_n)) \\ &\quad - R_{M_1,\rho_1}^{A_1,\eta_1}(A_1(a_{n-1}) - \rho_1 F(a_{n-1}, v_{n-1}))\| \\ &\leq (1 - \lambda_n - \delta_n)\|a_n - a_{n-1}\| \\ &\quad + \lambda_n \frac{\tau_1}{r_1 - \rho_1 m_1} \|A_1(a_n) - A_1(a_{n-1}) - \ rho_1[F(a_n, v_n) - F(a_{n-1}, v_{n-1})]\| \\ &\leq (1 - \lambda_n)\|a_n - a_{n-1}\| \\ &\quad + \lambda_n \frac{\tau_1}{r_1 - \ rho_1 m_1} (\|A_1(a_n) - a_1(a_{n-1}) - \rho_1[F(a_n, v_n) - F(a_{n-1}, v_n)]\| \end{aligned}$$

$$+\|F(a_{n-1},v_n)-F(a_{n-1},v_{n-1})\|). \quad (4.9)$$

Similarly, we have

$$\begin{aligned}\|b_{n+1}-b_n\| &\le (1-\lambda_n)\|b_n-b_{n-1}\| \\ &+\lambda_n\frac{\tau_2}{r_2-\rho_2 m_2}(\|A_2(b_n)-A_2(b_{n-1})-\rho_2[G(u_n,b_n)-G(u_n,b_{n-1})]\| \\ &+\|G(u_n,b_{n-1})-G(u_{n-1},b_{n-1})\|). \quad (4.10)\end{aligned}$$

Since A_i are β_i-Lipschitz continuous for $i=1,2,$ $F(\cdot,b)$ is (c_1,μ_1)-relaxed cocoercive with respect to A_1 and α_1-Lipschitz continuous, $G(x,\cdot)$ is (c_2,μ_2)-relaxed cocoercive with respect to A_2 and α_2-Lipschitz continuous, we obtain

$$\begin{aligned}&\|A_1(a_n)-A_1(a_{n-1})-\rho_1[F(a_n,v_n)-F(a_{n-1},v_n)]\|^2 \\ &=\|A_1(a_n)-A_1(a_{n-1})\|^2-2\rho_1\langle F(a_n,v_n)-F(a_{n-1},v_n),A_1(a_n)-A_1(a_{n-1})\rangle \\ &\quad+\rho_1^2\|F(a_n,v_n)-F(a_{n-1},v_n)\|^2 \\ &\quad\le(\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2+2\rho_1 c_1\alpha_1^2)\|a_n-a_{n-1}\|^2 \quad (4.11)\end{aligned}$$

and

$$\begin{aligned}&\|A_2(b_n)-A_2(b_{n-1})-\rho_2[G(u_n,b_n)-G(u_n,b_{n-1})]\|^2 \\ &=\|A_2(b_n)-A_2(b_{n-1})\|^2-2\rho_2\langle G(u_n,b_n)-G(u_n,b_{n-1}),A_2(b_n)-A_2(b_{n-1})\rangle \\ &\quad+\rho_2^2\|G(u_n,b_n)-G(u_n,b_{n-1})\|^2 \\ &\quad\le(\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2+2\rho_2 c_2\alpha_2^2)\|b_n-b_{n-1}\|^2. \quad (4.12)\end{aligned}$$

Furthermore, from the assumptions, we have

$$\begin{aligned}&\|F(a_{n-1},v_n)-F(a_{n-1},v_{n-1})\| \\ &\quad\le\zeta_1\|v_n-v_{n-1}\|\le\zeta_1\gamma_2(1+n^{-1})\|b_n-b_{n-1}\|, \quad (4.13)\end{aligned}$$

$$\begin{aligned}&\|G(u_n,b_{n-1})-G(u_{n-1},b_{n-1})\| \\ &\quad\le\zeta_2\|u_n-u_{n-1}\|\le\zeta_2\gamma_1(1+n^{-1})\|a_n-a_{n-1}\|. \quad (4.14)\end{aligned}$$

It follows from (4.9)-(4.14) that

$$\begin{cases}\|a_{n+1}-a_n\|\le(1-\lambda_n+\lambda_n\frac{\tau_1}{r_1-\rho_1 m_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2+2\rho_1 c_1\alpha_1^2}) \\ \times\|a_n-a_{n-1}\|+\lambda\frac{\tau_1}{r_1-\rho_1 m_1}\zeta_1\gamma_2(1+n^{-1})\|b_n-b_{n-1}\|, \\ \|b_{n+1}-b_n\|\le(1-\lambda_n+\lambda_n\frac{\tau_2}{r_2-\rho_2 m_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2+2\rho_2 c_2\alpha_2^2}) \\ \times\|b_n-b_{n-1}\|+\lambda\frac{\tau_2}{r_2-\rho_2 m_2}\zeta_2\gamma_1(1+n^{-1})\|a_n-a_{n-1}\|.\end{cases} \quad (4.15)$$

Now (4.15) implies that

$$\begin{aligned}
& \|a_{n+1}-a_n\|+\|b_{n+1}-b_n\| \\
\leq\ & (1-\lambda_n+\lambda_n\frac{\tau_1}{r_1-\rho_1 m_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2+2\rho_1 c_1\alpha_1^2} \\
+\ & \lambda_n\frac{\tau_2}{r_2-\rho_2 m_2}\zeta_2\gamma_1(1+n^{-1}))\|a_n-a_{n-1}\| \\
+\ & (1-\lambda_n+\lambda_n\frac{\tau_2}{r_2-\rho_2 m_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2+2\ rho_2 c_2\alpha_2^2} \\
+\ & \lambda_n\frac{\tau_1}{r_1-\rho_1 m_1}\zeta_1\gamma_2(1+n^{-1}))\|b_n-b_{n-1}\| \\
\leq\ & (1-\lambda_n(1-\theta_n))(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|) \\
\leq\ & (1-\triangle(1-\theta_n))(\|a_n-a_{n-1}\|+\|b_n-b_{n-1}\|)\,,
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
& \theta_n=\max\{\frac{\tau_1}{r_1-\rho_1 m_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2+2\rho_1 c_1\alpha_1^2} \\
+\ & \frac{\tau_2}{r_2-\rho_2 m_2}\zeta_2\gamma_1(1+n^{-1}) \\
\times\ & \frac{\tau_2}{r_2-\rho_2 m_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2+2\rho_2 c_2\alpha_2^2} \\
+\ & \frac{\tau_1}{r_1-\rho_1 m_1}\zeta_1\gamma_2(1+n^{-1})\},
\end{aligned}$$

and $\limsup_{k\geq 0}\lambda_n<1$. If we set

$$\begin{aligned}
& \theta=\max\{\frac{\tau_1}{r_1}\sqrt{\beta_1^2-2\rho_1\mu_1+\rho_1^2\alpha_1^2+2\rho_1 c_1\alpha_1^2}+\frac{\tau_2}{r_2}\zeta_2\gamma_1 \\
\times\ & \frac{\tau_2}{r_2}\sqrt{\beta_2^2-2\rho_2\mu_2+\rho_2^2\alpha_2^2+2\rho_2 c_2\alpha_2^2}+\frac{\tau_1}{r_1}\zeta_1\gamma_2\},
\end{aligned}$$

we have that $\theta_n\to\theta$ as $n\to\infty$. It follows from condition (4.8) that $0<\theta<1$. Therefore, by (4.16) and $0<\lambda_n+\delta_n\leq 1$ implies that $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences and so there exist $a\in X_1$ and $b\in X_2$ such that $a_n\to a$ and $b_n\to b$ as $n\to\infty$.

Next we show that $u_n\to u\in U(u)$ and $v_n\to v\in V(b)$ as $n\to\infty$. It follows from (4.13) and (4.14) that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences. Therefore, there exist $u\in X_1$ and $v\in X_2$ such that $u_n\to u$ and $v_n\to v$ as $n\to\infty$. Furthermore,

$$d(u,U(u))=\inf\{\|u-t\|:t\in U(a)\}\leq\|u-u_n\|+d(u_n,U(a))$$

$$\leq \|u-u_n\| + D_1(U(a_n),U(a)) \leq \|u-u_n\| + \zeta_1\|a_n-a\| \to 0.$$

Since $U(a)$ is closed, we have $u \in U(a)$. Similarly, we can show that $v \in V(b)$.

By continuity, a, b, u and v satisfy the following relation

$$\begin{cases} a = R^{A_1,\eta_1}_{M_1,\rho_1}[A_1(a) - \ rho_1 F(a,v)], \\ b = R^{A_2,\eta_2}_{M_2,\rho_2}[A_2(b) - \rho_2 G(u,b)], \end{cases}$$

by Theorem 4.1, we know that (a,b,u,v) is a solution of problem (3.1). This completes the proof. □

Theorem 4.3. *Let $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous mappings, $A_i : X_i \to X_i$ (r_i,η)-strongly monotone and β_i-Lipschitz continuous mappings, $M_i : X_i \to 2^X_i$ be (A_i,η_i)-monotone mappings for $i = 1,2$. Let $U : X_1 \to C(X_1)$ be D_1-γ_1-Lipschitz continuous and $V : X_2 \to C(X_2)$ be D_2-γ_2-Lipschitz continuous. Let $F : X_1 \times X_2 \to X_1$ be a nonlinear mapping such that for any given $(a,b) \in X_1 \times X_2$, $F(\cdot,b)$ is (c_1,μ_1)-relaxed cocoercive with respect to A_1 and α_1-Lipschitz continuous and $F(a,\cdot)$ is ζ_1-Lipschitz continuous. Let $G : X_1 \times X_2 \to X_2$ be another nonlinear mapping such that for any given $(x,y) \in X_1 \times X_2$, $G(x,\cdot)$ is (c_2,μ_2)-relaxed cocoercive with respect to A_2 and α_2-Lipschitz continuous and $G(\cdot,y)$ is ζ_2-Lipschitz continuous. If there exist constants $\rho_i > 0$ for $i = 1,2$ such that*

$$\begin{cases} \tau_1 r_2\sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2 + 2\rho_1 c_1\alpha_1^2} + \tau_2 r_1\zeta_2\gamma_1 < r_1 r_2, \\ \tau_2 r_1\sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2 + 2\rho_2 c_2\alpha_2^2} + \tau_1 r_2\zeta_1\gamma_2 < r_1 r_2. \end{cases} \tag{4.17}$$

Then problem (3.1) admits a solution (a,b,u,v) and iterative sequences $\{a_n\}, \{b_n\}, \{u_n\}$ and $\{v_n\}$ converge strongly to a, b, u and v, respectively, where $\{a_n\}, \{b_n\}, \{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 4.2.

Proof. The proof is similar to that of Theorem 4.2.

Theorem 4.4. *Let $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous mappings, $A_i : X_i \to X_i$ (r_i,η)-strongly monotone and β_i-Lipschitz continuous mappings, $M_i : X_i \to 2^X_i$ be (A_i,η_i)-monotone mappings for $i = 1,2$. Let $U : X_1 \to C(X_1)$ be D_1-γ_1-Lipschitz continuous and $V : X_2 \to C(X_2)$ be D_2-γ_2-Lipschitz continuous. Let $F : X_1 \times X_2 \to X_1$ be a nonlinear mapping such that for any given $(a,b) \in X_1 \times X_2$, $F(\cdot,b)$ is (c_1,μ_1)-relaxed cocoercive with respect to A_1 and α_1-Lipschitz continuous and $F(a,\cdot)$ is ζ_1-Lipschitz continuous. Let $G : X_1 \times X_2 \to X_2$ be another nonlinear mapping such that for any given $(x,y) \in X_1 \times X_2$, $G(x,\cdot)$ is (c_2,μ_2)-relaxed cocoercive with respect to A_2 and α_2-Lipschitz continuous and $G(\cdot,y)$ is ζ_2-Lipschitz continuous. If there exist constants $\rho_i > 0$ for $i = 1,2$ such that*

$$\begin{cases} \tau_1 r_2\sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2 + 2\rho_1 c_1\alpha_1^2} + \tau_2 r_1\zeta_2\gamma_1 < r_1 r_2, \\ \tau_2 r_1\sqrt{\beta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2 + 2\rho_2 c_2\alpha_2^2} + \tau_1 r_2\zeta_1\gamma_2 < r_1 r_2. \end{cases} \tag{4.18}$$

Then problem (3.1) admits a solution (a, b, u, v) *and iterative sequences* $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *converge strongly to* a, b, u *and* v*, respectively, where* $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *are the sequences generated by Algorithm 4.3.*

Proof. The proof is quite similar to that of Theorem 4.3. □

Corollary 4.5. *Let* $\eta_i : X_i \times X_i \to X_i$ *be* τ_i*-Lipschitz continuous mappings,* $A_i : X_i \to X_i$ (r_i, η)*-strongly monotone and* β_i*-Lipschitz continuous mappings,* $M_i : X_i \to 2_i^X$ *be* (A_i, η_i)*-monotone mappings for* $i = 1, 2$*. Let* $U : X_1 \to C(X_1)$ *be* D_1*-*γ_1*-Lipschitz continuous and* $V : X_2 \to C(X_2)$ *be* D_2*-*γ_2*-Lipschitz continuous. Let* $F : X_1 \times X_2 \to X_1$ *be a nonlinear mapping such that for any given* $(a, b) \in X_1 \times X_2$*,* $F(\cdot, b)$ *is* μ_1*-strongly monotone with respect to* A_1 *and* α_1*-Lipschitz continuous and* $F(a, \cdot)$ *is* ζ_1*-Lipschitz continuous. Let* $G : X_1 \times X_2 \to X_2$ *be another nonlinear mapping such that for any given* $(x, y) \in X_1 \times X_2$*,* $G(x, \cdot)$ *is* μ_2*-strongly monotone with respect to* A_2 *and* α_2*-Lipschitz continuous and* $G(\cdot, y)$ *is* ζ_2*-Lipschitz continuous. If there exist constants* $\rho_i > 0$ *for* $i = 1, 2$ *such that*

$$\begin{cases} \tau_1 r_2 \sqrt{\beta_1^2 - 2\rho_1\mu_1 + \rho_1^2\alpha_1^2} + \tau_2 r_1 \zeta_2 \gamma_1 < r_1 r_2, \\ \tau_2 r_1 \sqrt{\underline{e}ta_2^2 - 2\rho_2\mu_2 + \rho_2^2\alpha_2^2} + \tau_1 r_2 \zeta_1 \gamma_2 < r_1 r_2. \end{cases} \tag{4.19}$$

Then problem (3.1) admits a solution (a, b, u, v) *and iterative sequences* $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *converge strongly to* a, b, u *and* v*, respectively, where* $\{a_n\}, \{b_n\}, \{u_n\}$ *and* $\{v_n\}$ *are the sequences generated by Algorithm 4.1.*

References

[1] R.P. Agarwa l, N.J. Huang, Y.J. Cho, Generalized nonlinear mixed implicit Aquasi-variational inclusions with set-valued mappings, *Journal of Inequalities and Applications*, **7**, No. 6 (2002), 807-828.

[2] S.S. Chang, Y.J. Cho, H.Y. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Science Publ., New York (2002).

[3] Y.J. Cho, Y.P Fang, N.J. Huang, H.J. Hwang, Algorithms for systems of nonlinear variational inequalities, *Journal of Korean Mathematical Society*, **41** (2004), 489-499.

[4] X.P. Ding, C.L. Luo, Perturbed proximal point algorithms for generalized quasi-variational-like inclusions, *Journal of Computational and Applied Mathema tics*, **210** (2000), 153-165.

[5] Y.P. Fang, N.J. Huang, H-monotone operator and resolvent operator technique for variational inclusions, *Applied Mathematics and Computation*, **14**, No. 5 (2003), 795-803.

[6] Y.P. Fang, N.J. Huang, H-monotone operators and system of variational inclusions, *Communications on Applied Nonlinear Analysis*, **11**, No. 1 (2004), 93-101.

[7] Y.P. Fang, N.J. Huang, Thompson, A new system of variational inclusions with (H, η)-monotone operators in Hilbert spaces, *Computers and Mathematics with Applications*, **49**, No. 36 (2005), 365-374.

[8] A. Hassouni, A. Moudafi, A perturbed algorithms for variational inequalities, *Journal of Mathematical Analysis and Applications*, **185** (1994), 706-712.

[9] N.J. Huang, Y.P. Fang, A new class of general variational inclusions involving maximal η-monotone mappings, *Publ. Math. Debrecen*, **62** (2003), 83-98.

[10] M.M. Jin, Iterative algorithm for a new system of nonlinear set-valued variational inclusions involving (H, η)-monotone mappings, *Journal of Inequalities in Pure and Applied Mathematics,* To Appear.

[11] M.M. Jin, Q.K. Liu, Nonlinear quasi-variational inclusions involving generalized m-accretive mappings, *Nonlinear Functional Analysis and Applications*, **9**, No. 3 (2004), 485-494.

[12] M.M. Jin, Generalized nonlinear implicit quasi-variational inclusions with relaxed monotone mappings, *Advances in Nonlinear Variational Inequalities*, **7**, No. 2 (2004), 173-181.

[13] K.R. Kazmi, Mann and Ishikawa type perturbed iterative algorithms for generalized quasivariational inclusions, *Journal of Mathematical Analysis and Applications*, **209** (1997), 572-584.

[14] J.K. Kim, D.S. Kim, A new system of generalized nonlinear mixed variational inequalities in Hilbert spaces, *Journal of Convex Analysis*, **11** (2004), 117-124.

[15] H.Y. Lan, J.H. Kim, Y.J. Cho, On a new system of nonlinear A-monotone multivalued variational inclusions, *Journal of Mathematical Analysis and Applications*, **327** (2007), 481-493.

[16] S.B. Nadler, Multi-valued contraction mappings, *Pacific Journal of Mathematics*, **30** (1969), 475-488.

[17] R.U. Verma, Generalized system for relaxed coercive variational inequalities and projection methods, *Journal of Optimization Theory and Applications*, **121** (2004), 203-210.

[18] R.U. Verma, A-monotonicity and applications to nonlinear variational inclusions, *Journal of Applied Mathematics and Stochastic Analysis*, **17**, No. 2 (2004), 193-195.

[19] R.U. Verma, Approximation-solvability of a class of A-monotone variational inclusion problems, *Journal of the Korean Society of Industrial and Applied Mathematics*, **8**, No. 1 (2004), 55-66.

[20] R.U. Verma, A-monotonicity and its role in nonlinear variational inclusions, *Journal of Optimization Theory and Applications*, **129**, No. 3 (2006), 457-467.

[21] R.U. Verma, Sensitivity analysis for generalized strongly monotone variational inclusions based on (A, η)-resolvent operator technique, *Applied Mathematics Letters*, **19** (2006), 1409-1413.

[22] George X.Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis,* Marcel Dekker, New York (1999).