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A MULTICRITERIA REWARD ALLOCATION MODEL: THE ORCHESTRATION OF THE SHAPLEY VALUES

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Abstract: This effort derived a mathematical model for the allocation of rewards in project-based or teamwork cases. The model, considering multicriteria decision environment, allocates a constant monetary reward to team members in an optimal frame such that the balance of equity and equality is established using an inequality index Gini coefficient and hence Lorenz curves as post analyses of the Shapley values of n-person game theory. The model generates a convex combination of the Shapley vectors, which is better than its parents in inequality base. The recipe offers optimal weights of the convex combination and cumulative frequencies of team members. A hypothetical illustrative example is solved and computed by *MS Excel's* add-in *Evolutionary Solver*, which uses genetic algorithms as a powerful spreadsheet tool. The model is a good example of the synergistic area consisting of social psychology, operations research, organizational behavior and human resources management.

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1. Introduction

A social psychological viewpoint, which has recently gained influence, is the notion that fairness is a basic human value that people want to see affirmed in social encounters, see [3]. In organizational aspect, distributive justice in reward allocation systems can be taken as one of the remarkable presence of fairness. The reward system is an important mediator that managers can use to channel employee motivation, which may also be achieved by ensuring that fairness exists in the workplace, see [8], in desired ways, see [13].

It is crucial that how employees percept the reward system. When a new compensation plan is made, employee reactions are an important part of the context, and the acceptance of human resource innovations by the employees is a necessary condition for their effective implementation and survival, see [1].

The most common allocation rules are equity, equality, need and seniority, which is the recent principle added [7], [6]. In different mediums, each of them may be employed by valid reasons within organizational climate. As an instance, Miller and Komorita [12] reported some variables of assessing the relative weight given to equity and equality in reward allocations in task-performing groups. However, especially in team-oriented cases that are mostly faced in project-based tasks, there is no one right answer to this issue, which emphasizes that effective allocation of aggregate pay, is a difficult decision, see [1]. As the foregoing considerations state, the basic allocation principles are equality and equity. Equality distributes relative proportion of rewards to each recipient regardless of their contribution to the organization, and equity that evolved from a social psychological theory called social comparison theory, [8], focuses on performance input of employees and this phrase include aspects like effort, talent and motivation, see [5]. Egalitarian rewards are intended to foster group cohesion, harmony via minimizing conflict, [18], and teamwork, while equitable distribution stimulates intrinsic satisfaction and maximization of individual performance, see [1].

There are many criteria upon which organizational rewards are distributed such as performance, effort, seniority, skills held, task difficulty, discretionary time [15], education experience [9] hierarchical level and gender [1]. Each of these indicators is separately important. Even it is evaluated that contribution to the organization refers to performance as the core factor, it is possible to constitute a synergy, a mix of appropriate criteria. Because such a mix may lead to an increase of inequality among team members gathered for the same goal and damage the team spirit, in order to de-emphasize equity rule, it will be more reasonable to approximate the allocation to equality meaning identical proportions as a fairness mechanism.

In the frame of reward allocation in work organizations, the literature is mostly composed of conceptual and empirical studies. For examples, Bettencourt et al [2] studied the cooperation and the reduction of intergroup bias within reward structure and social orientation. Barber et al [1] approached to group reward allocation in a conceptual aspect. There are some cross-cultural comparison studies. Fischer [5] is the one, who compared British and German organizations, when Puffer [14] compared Soviet and American managers in a dependency approach. Singh [18] determined the issue on group harmony and interpersonal fairness base. Brockner et al [3] underlined the procedural fairness and its effects on people's attitudes and behaviors. Zhang [21] investigated how the frequency of interaction affects decisions on reward allocation. Fischer and Smith [7] gave a well-designed review of the subject in a cross-cultural sense. However, the scarcity of the literature on mathematical reward allocation is salient.

In this study, a mathematical model is developed for the allocation of rewards in project-based or teamwork cases. The model, which considers different criteria, allocates a constant monetary reward to team members in an optimal frame such that the balance of equity and equality is established using an inequality index Gini coefficient and hence Lorenz curves as post analyses of the Shapley values of cooperative game theory. The orchestration of the Shapley vectors obtained with respect to different criteria is realized in a way that a convex combination of those vectors is created via optimal weights.

This paper is organized in the following way. Some preliminaries of n-person game theory and Gini coefficients are stated as a preparation for the proposed model. The optimization model is formulated and then a hypothetical illustrative example is given. Finally, some discussion and concluding remarks are addressed to the effort and further research.

2. The Background

In many competitive situations, there are more than two competitors. In this frame, any game with n players is an n-person game, which can be specified by the game's characteristic function v. Any solution concept for n-person games chooses some subset of the set of imputations (possibly empty) as the solution to the n-person game. Two main solution concepts are the core and the Shapley value, see [20].

The core has an important desirable property of efficiency, but it also has some undesirable ones. Most basically, there are games that have no core when many others have very large cores, and so the approach does not determine an outcome uniquely. Other concepts that do better in these respects have been constructed. The best known is the Shapley value. It is similarly grounded in a set of axioms or assumptions that a solution should satisfy and it is the unique solution that conforms to all these axioms [4]. Given any n-person game with the characteristic function v, there is a unique reward vector $x = (x_1, x_2, ..., x_n)$ satisfying the mentioned axioms. The reward of the i-th player is given by

$$x_i = \sum_{\forall S, i \notin S} p_n(S) \left[v(S \cup \{i\}) - v(S) \right], \text{ where } p_n(S) = \frac{|S|!\,(n - |S| - 1)!}{n!}.$$

In this expression, $|S|$ is the number of players in coalition S (see [20]-[11] for further discussion about the Shapley value).

Although the Shapley value in general gives more equitable solutions than the core does [20], neither guarantees any fairness of payoffs [4]. One of fairness concepts is proportionality, meaning that each of n people believes that his share is at least $1/n$ of the total. Another is envy-free, which means that no participant believes someone else got a better deal. Even though the two concepts are equivalent in two people case, envy-freeness is a stronger requirement in n-person case, since each may think he got $1/n$ while also thinking that someone else got more than he did [4]. Therefore, the allocation of payoffs especially rewards should be as possible as equal. Some real location methodologies, which accounts for equality may be needed as a post arrangement.

One of such instruments is the Gini coefficient, an inequality index [10]. The coefficient can be derived in terms of frequency distributions within m equal class intervals and interval width from the formula

$$G = \frac{\frac{2c}{n^2} \sum\limits_{k=1}^{m} F_k (n - F_k)}{\frac{2}{n} \sum\limits_{k=1}^{m} \overline{c_k} F_k},$$

where is $\overline{c_k}$ the mid-point of class k, F_k is the cumulative frequency for class k and n is the number of population [17]. The Gini coefficient is based on the Lorenz curve. It is a cumulative frequency curve plotted when subgroups are ordered hierarchically according to increasing frequency. The Gini coefficient, which is the most widely used summary measure of inequality, is the ratio of the area between the Lorenz curve and the diagonal (the line of perfect equality, i.e. $y = x$ [16]) to the area of the triangle beneath the diagonal [10]. This study stimulates the use of the Gini coefficients to allocate rewards in a less inequality way as a post analysis.

3. Developing the Model

The allocation of a constant monetary reward quantity to the employees can be modeled as an n-person game theoretic model, which may be better solved by the Shapley procedure. For a given criterion, the employees' contributions and efficiencies in a project may be measured via the Shapley vector by giving appropriate numerical values to the players' outputs. It is also possible to determine the payoffs to the employees with respect to different criteria such as performance, team hierarchy and seniority. In this case, it is needed to use different characteristic functions for different criteria. It is also a task that the obtained reward distributions should be combined into a synergistic distribution as a single measure (as usual in multicriteria optimization) so that the grand reward should be allocated more fairly. The mathematical model, which allocates the grand reward in an optimal frame, can be formulated as follows with the attention that the assumed sequences of the players should be kept fixed due to the axiomatic structure of the Shapley approach.

Let: $j = 1, 2, 3, ..., n$ be the number of players (people that will be rewarded), $n \geq 3$,

$i = 1, 2, 3, ..., m$ be the number of criteria used for reward allocation process,

s_{ij} be the Shapley value (reward) of j-th player for criterion i,

v_i be the characteristic function for criterion i,

S_i be the Shapley vector calculated for criterion i, i.e., $S_i^T = (s_{ij})_{1\times n}$,

w_i be the weight of convex combination for the Shapley vector,

S be the convex combination vector of the Shapley vectors,

$k = 1, 2, 3, ..., l$ be the number of equal class intervals for frequency distribution of S,

c be the equal class interval width,

$\overline{c_k}$ be the mid-point of class k,

F_k be the cumulative frequency for class interval k,

G_i be the Gini coefficient of the Shapley vector,

G be the overall Gini coefficient of the vector S.

First and foremost, according to selected criteria, the Shapley vectors S_i should be calculated in a better way that they may be designed in normalized form as probability distributions. The Shapley value, which is reward, of j-th player within criterion i, s_{ij} should be obtained by

$$s_{ij} = \sum_{\forall S, j \notin S} p_n(L)\left[v_i(L \cup \{j\}) - v_i(L)\right],$$

where $p_n(L) = \frac{|L|!(n-|L|-1)!}{n!}$, $|L|$ is the number of players in coalition L.

Once the reward distribution vectors with respect to the given criteria have been obtained, the Gini coefficients G_i, inequality measure, for reward allocation can be determined optionally. The main effort is to hold an optimal mix of these vectors yielding a more equitable distribution in cooperative game theoretic frame. Therefore, the objective function, to be minimized, should be the overall Gini coefficient. The overall Gini coefficient is of the convex combination of the Shapley vectors obtained for different criteria. In other words, the problem is to find optimal weights and hence frequency distribution of the convex combination vector S. Therefore, by labeling w_i's as decision variables, the objective function can be formulated by means of the frequency distribution as

$$\text{minimize } G = \frac{\frac{2c}{n^2}\sum_{k=1}^{l} F_k(n - F_k)}{\frac{2}{n}\sum_{k=1}^{l} \overline{c_k} F_k}.$$

Naturally, there are some constraints to be satisfied. The overall probability distribution S is a convex combination of the Shapley vectors obtained for the criteria,

$$S = w_1 S_1 + w_2 S_2 + ... + w_m S_m = \begin{pmatrix} w_1 s_{11} + w_2 s_{21} + ... + w_m S_{m1} \\ w_1 s_{12} + w_2 s_{22} + ... + w_m S_{m2} \\ \\ w_1 s_{1n} + w_2 s_{2n} + ... + w_m S_{mn} \end{pmatrix}.$$

The resultant vector is both a probability distribution and a convex combination and therefore,

$$w_1 + w_2 + ... + w_m = 1 \quad \text{and}$$

$$w_1, w_2, ..., w_m \geq 0,\ \forall j = 1, 2, 3, ..., n,\ \forall i = 1, 2, 3, ..., m,\ \forall k = 1, 2, 3, ..., l.$$

The model finds optimal, in theory, convex combination weights and hence frequency distribution for resultant probability distribution of multicriteria reward allocation policy. The optimization problem is nonsmooth and thus might be difficult to be solved by any calculus-based solver (see [20], [19]). Alternatively, it is a better way to employ heuristic methods such as genetic algorithms or softwares based on such heuristics (e.g., *Evolutionary Solver* as *MS Excel* add-in [19], [22]) to attain at least good or near optimal solutions. The *Evolutionary Solver* that uses genetic algorithms searches the entire feasible region and is much less likely to get stuck at local optima, see [19].

4. Numerical Application

A hypothetical illustrative example may be developed in the following way. Assume that a team of $n = 10$ employees of a firm has successfully finished a project. The team is made up of different hierarchies. The management gave a reference number to each employee ranging from 1 to 10 and calculated the Shapley vectors of the team members according to three criteria ($m = 3$); performance, team hierarchy and seniority. Suppose that these vectors are as follows respectively;

$$S_1^T = (0.10, 0.06, 0.03, 0.40, 0.03, 0.05, 0.08, 0.11, 0.10, 0.04) \ ,$$
$$S_2^T = (0.20, 0.04, 0.06, 0.05, 0.47, 0.01, 0.02, 0.06, 0.06, 0.03) \ ,$$
$$S_3^T = (0.05, 0.07, 0.09, 0.60, 0.05, 0.01, 0.05, 0.01, 0.02, 0.05) \ .$$

As observing the vectors, the 4-th player gets 40% of the reward for the first criterion and 60% for the third criterion when the 5-th player gains 47% of the reward for the second criterion. The Gini coefficients associated with the distributions are $G_1 = 0.32$, $G_2 = 0.37$, $G_3 = 0.39$. Also, the Gini coefficient for the average distribution (equally weighted) of the Shapley vectors is calculated as 0.33.

The mission is to create a synergistic reward distribution. For this purpose, we take $l = 10$ equal class intervals with $c = 0.10$. The formulation and solution of the mathematical programming model is implemented via *MS Excel* as a powerful spreadsheet tool with an add-in *Evolutionary Solver*, which uses genetic algorithms.

The run of the *Evolutionary Solver* yields the solution in 40 s on a PC with *Pentium(R)* 4, CPU 3.00GHz and 512MB RAM. The objective function value, the overall Gini coefficient, is 0.28. This value, which is a good solution, is lower than all the Shapley vectors have. The good (close to optimal) weights of the convex combination are $w_1 = 0.635$, $w_2 = 0.332$ and $w_3 = 0.033$. Also, the good cumulative frequency distribution is $F_1 = 7$, $F_2 = 9$, and $F_3 = F_4 = \ldots = F_{10} = 10$. Consequently, the good multicriteria reward allocation policy is obtained as

$$S^T = (0.13, 0.05, 0.04, 0.29, 0.18, 0.04, 0.06, 0.09, 0.08, 0.04\ ) \,.$$

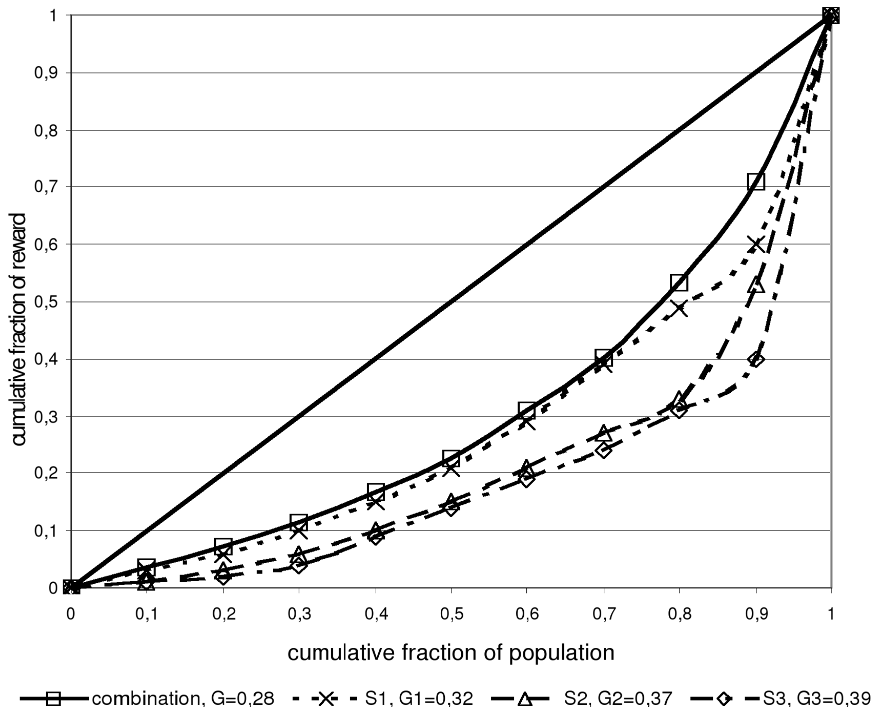

Figure 1: Lorenz curves and the Gini coefficients

As seen, this solution smoothed the extreme values of 4-th and 5-th employees in a reasonable way and offered a more equitable reward distribution as a fairness mechanism. A graphical comparison in Lorenz curves and Gini coefficients is drawn in Figure 1.

In addition, it is possible to give some statistics related to genetic algorithm solution process of the problem using 'population report' that *MS Excel* offers. Mean values of the weights are $0.628, 0.340$ and 0.032 and associated standard deviations are 0.066, 0.070 and 0.003.

5. Discussion and Concluding Remarks

This effort derived a mathematical reward allocation model, which optimally distributes a constant monetary value to team members. The proposed model takes individual-based parameters and smoothes the extreme ones, in order to establish a distribution balance within recipients, in multicriteria cases. The instruments are Gini coefficients and hence Lorenz curves and the Shapley value of n-person game theory.

One important point in this paper is that, it is used both the Shapley values to reveal the individual contributions and the Gini coefficient for social psychological care within group as a fairness mechanism. In addition, the recipe is a general procedure that considers pre-determined number of criteria appropriate to its psychological philosophy. Furthermore, it is shown that the use of *MS Excel* and the *Evolutionary Solver*, which uses genetic algorithms, easies the complex allocation process.

Although the road map proposes optimal weights of convex combination of the criteria, it is also possible to specify them in a desired way. The Shapley value may also be used to obtain the weights with respect to criteria power. Another approach is the arbitrarily chosen weights and even the probability vectors by the manager. In addition, some other multicriteria decision making techniques such as scoring and the analytical hierarchy process are welcome. The latter may facilitate the subordinates to participate the reward management procedure.

The study itself is also a good orchestration example of social psychology, operations research, organizational behavior and human resources management. As another future research, some similar approaches can successfully be adopted to the synergistic area by using different instruments of these disciplines.

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