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A NOTE ON THERMOELASTIC PROBLEM OF TWO COLLINEAR GRIFFTH CRACKS IN AN ORTHOTROPIC MEDIUM

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Abstract: The problem of determining the thermal stresses and displacement fields in an infinite orthotropic plane containing a pair of equal collinear Griffith cracks when the shape of the cracks prescribed is considered. An integral transform technique, based upon displacement potential, is employed for the case of steady-state temperature field. Numerical results of the stresses for different forms of the prescribed displacement are presented in graphs.

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1. Introduction

It is unnecessary to mention that the presence of stress concentrations in struc-

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tural members is of great importance to a design engineer. Geometrical discontinuities such as holes, notches, fillets, grooves and load discontinuities can be controlled by the designer, where as inherent flaws such as cracks, segregations and voids, which are "metallurgical or fabrication discontinuities", are not very easy to control. In case of sharp cracks, the analysis become formidable. However, in recent years, there has been an increasing interest in analytical solutions of elasto-static and elasto-dynamic cracks problems in an anisotropic medium, particularly in orthotropic medium, due to their importance and usefulness from a technogical point of view. The problems of Griffith crack in an orthotropic medium have been considered by Dhaliwal [5], Satpathy and Parhi [8], Das and Behera [2] and Das and Patra [4]. Das and Patra [3] have solved the problem of a pair of equal collinear Griffith cracks in an orthotropic layer sandwiched between two identical orthotropic half planes. In this connection papers [7], [6], [11], [10] are of worth mentioning. In the present paper, a pair of equal collinear Griffith cracks has been considered in an orthotropic elastic plane, to study the effect of temperature on displacements and stresses. The displacements and stresses are determined in terms two potential functions which are harmonic in different planes. The problem has been reduced to a dual form of the integral equations, which have been solved analytically. Expressions for quantities of physical interest, e.g., stresses, crack energies for different cases are derived. Numerical results of crack energies are obtained for different forms of the displacement functions, which have been displayed graphically.

2. The Basic Equations

2.1. Temperature Field

The temperature distribution $T(x,y)$ in the orthotropic plane is assumed to satisfy the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} + k^2 \frac{\partial^2 T}{\partial y^2} = 0\,, \tag{1}$$

where $k^2 = \frac{k_y}{k_x}$ is the ratio of the thermal conductivity coefficients in the y and x directions respectively. A general solution for $T(x,y)$ is (c.f. Tauchert [1])

$$T(x,y) = \frac{1}{2\pi}\int_0^\infty A(p)e^{p[ix-(\frac{y}{k})]} + \bar{A}(p)e^{p[-ix-(\frac{y}{k})]}dp\,, \tag{2}$$

where $A(p)$ and $\bar{A}(p)$ are arbitrary functions of p. We shall assume that the prescribed temperature is

$$T(x,0) = h(x)\,, \tag{3}$$

and this temperature distribution may be written as a Fourier integral

$$T(x,0) = \frac{1}{2\pi}\int_0^{\infty}[\int_{-\infty}^{\infty}[A(p)e^{p[ix-(\frac{y}{k})]} + \bar{A}(p)e^{p[-ix-(\frac{y}{k})]}]d\xi]dp\,. \tag{4}$$

Comparing (4) with (2) evaluated at $y = 0$, we obtain

$$A(p) = \int_{-\infty}^{\infty} h(\xi)e^{-(\xi)p}d\xi, A(\bar{p}) = \int_{-\infty}^{\infty} h(\xi)e^{(\xi)p}d\xi\,. \tag{5}$$

Using (5), the temperature distribution $T(x,y)$ can be written in the form

$$T(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\frac{y}{k})h(\xi)d\xi}{(\frac{y}{k})^2 + (\xi - x)^2}\,. \tag{6}$$

Let us consider the prescribed temperature distribution $h(x)$ to be a point source on the y-axis, i.e.

$$h(x) = \delta(x), \tag{7}$$

where δ denotes the Dirac delta function. Therefore, the resultant distribution of temperature is given by

$$T(x,y) = \frac{1}{\pi}\frac{(\frac{y}{k})}{(\frac{y}{k})^2 + x^2}\,, \tag{8}$$

as obtained in [8].

2.2. Thermal Stresses

The stresses induced in the plane by the temperature distribution (8) are related to the displacement components u and v by

$$\sigma_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} - \beta_x T\,, \tag{9}$$

$$\sigma_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} - \beta_y T\,, \tag{10}$$

$$\sigma_{xy} = A_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\,, \tag{11}$$

$$\sigma_{yz} = 0 = \sigma_{zx}\,, \tag{12}$$

where the A_{ij}'s are anistropic constants of the orthotropic material and β_x and β_y are the stress-temperature coefficients. The displacement equations of equilibrium are

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x \partial y} = \beta_x \frac{\partial T}{\partial x}\,, \tag{13}$$

$$A_{66}\frac{\partial^2 v}{\partial x^2} + A_{22}\frac{\partial^2 v}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 u}{\partial x \partial y} = \beta_y \frac{\partial T}{\partial y}\,. \tag{14}$$

Following the approach of Sharma [9], let us introduce displacement potentials $\psi(x, y)$ and $\phi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial x}\,, \tag{15}$$

and

$$v = \mu\frac{\partial \psi}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\,, \tag{16}$$

where λ and μ are constants and

$$\psi(x, y) = \frac{1}{2\pi}\int_0^\infty \{A(p)B(p)e^{p[ix-(\frac{y}{k})]} + \bar{A}(p)\bar{B}(p)e^{p[-ix-(\frac{y}{k})]}\}dp\,. \tag{17}$$

The displacement equations (13)-(14) are satisfied by (15)-(16), for nontrivial ϕ, if

$$\mu = \frac{\beta_y(A_{66} - k^2 A_{11}) + \beta_x(A_{12} + A_{66})k^2}{\beta_x(A_{22} - k^2 A_{66}) - \beta_y(A_{12} + A_{66})}\,, \tag{18}$$

$$p^2 B = p^2 \bar{B}$$

$$= \frac{\beta_x(A_{22} - k^2 A_{66}) - \beta_y(A_{12} + A_{66})}{(A_{22} - k^2 A_{66})(A_{66} - k^2 A_{11}) + k^2(A_{12} + A_{66})^2} = k_1(\text{say})\,, \tag{19}$$

and

$$\frac{A_{66} + \lambda(A_{12} + A_{66})}{A_{11}} = \frac{\lambda A_{22}}{\lambda A_{66} + A_{12} + A_{66}} = \gamma^2(\text{say})\,, \tag{20}$$

where k_1 and γ^2 are constants. Equations (20) yield two quadratic equations, one in λ and other in γ^2. λ_1, λ_2 and γ_1^2, γ_2^2 are the roots of the quadratic equations

$$\lambda^2 A_{66}(A_{12}+A_{66}) + \lambda[(A_{12}+A_{66})^2 + A_{66}^2 - A_{11}A_{22}] + A_{66}(A_{12}+A_{66}) = 0\,, \tag{21}$$

and

$$A_{11}A_{66}\gamma^4 + (A_{12}^2 + 2A_{12}A_{66})\gamma^2 - A_{22}A_{66}) = 0\,, \tag{22}$$

respectively. Hence associated with each root $\gamma_i^2 (i = 1, 2)$ of (22) there is a potential function ϕ satisfying the differential equation

$$\frac{\partial^2 \phi_i}{\partial x_i^2} + \frac{\partial^2 \phi_i}{\partial y_i^2} = 0 \quad (i = 1, 2)\,, \tag{23}$$

where

$$y_i = \frac{y}{\gamma_i} \,. \tag{24}$$

The displacements u and v are, therefore, expressed as

$$u = \frac{\partial}{\partial x}(\psi + \phi_1 + \phi_2), \quad v = \frac{\partial}{\partial y}(\mu\psi + \lambda_1\phi_1 + \lambda_2\phi_2) \,. \tag{25}$$

The corresponding thermal stresses are

$$\frac{\sigma_{xx}}{A_{66}} = \frac{1+\lambda_1}{\gamma_1^2}\frac{\partial^2\phi_1}{\partial x^2} + \frac{1+\lambda_2}{\gamma_2^2}\frac{\partial^2\phi_2}{\partial x^2} - \frac{1+\mu}{\gamma_1^2}\frac{\partial^2\psi}{\partial y_1^2} \,, \tag{26}$$

$$\frac{\sigma_{yy}}{A_{66}} = (1+\lambda_1)\frac{\partial^2\phi_1}{\partial y_1^2} + (1+\lambda_2)\frac{\partial^2\phi_2}{\partial y_2^2} - (1+\mu)\frac{\partial^2\psi}{\partial x^2} \,, \tag{27}$$

$$\frac{\sigma_{xy}}{A_{66}} = \frac{1+\lambda_1}{\gamma_1}\frac{\partial^2\phi_1}{\partial x\partial y_1} + \frac{1+\lambda_2}{\gamma_2}\frac{\partial^2\phi_2}{\partial x^2} - \frac{1+\mu}{\gamma_1^2}\frac{\partial^2\psi}{\partial x^2} \,. \tag{28}$$

3. Statement of the Problem

Let us consider a symmetric plane strain problem in an orthotropic elastic solid of infinite extent containing a pair of equal collinear Griffith cracks of finite length. Let the coordinate axes x, y and z coincide with axes of elastic symmetry of the material. It is assumed that the cracks are defined by the relations $a \le |x| \le b, y = \pm 0$. The corresponding plane strain problem can be easily deduced from the present analysis.

The two different types of boundary conditions (for symmetry) have been considered:

Case 1.

$$v(x,0) = -p_0 f(x), \quad a < x < b \,, \tag{29}$$

$$v(x,0) = 0, \quad 0 < x < b,\ b < x < \infty \,, \tag{30}$$

$$\sigma_{xy}(x,0), \quad 0 < x < \infty \,. \tag{31}$$

Case 2.

$$v(x,0) = -p_0 f(x)\sqrt{(x^2-a^2)(b^2-x^2)} \,, \tag{32}$$

along with the conditions (30) and (31), where p_0 is a constant.

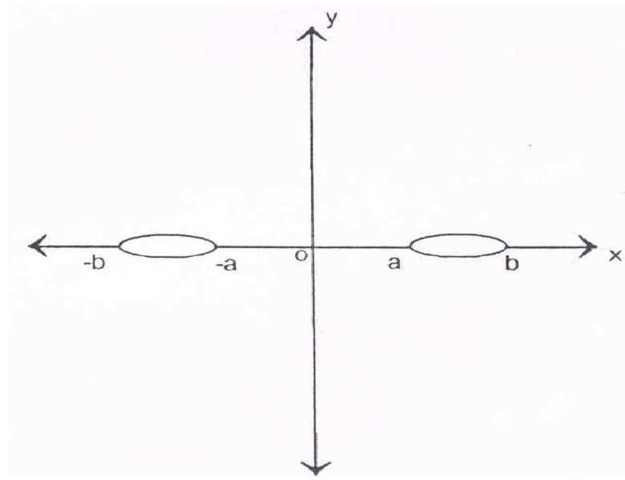

Figure 1: Geometry of the problem

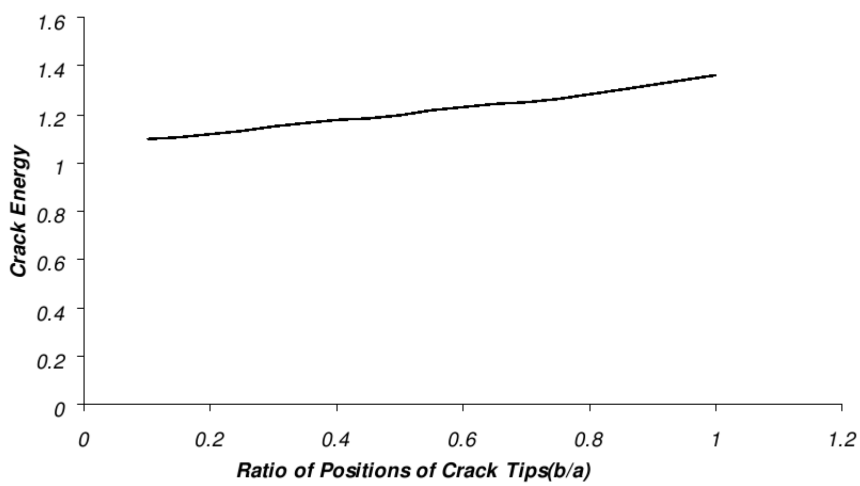

Figure 2: CrackEnergy(W_1)versus(b/a)

4. Solution of the Problem

Let us assume the appropriate integral solution of the equations (23) to be

$$\phi_1(x,y) = \int_0^\infty \alpha^{-2} C_1(\alpha) e^{-\alpha y_1} \cos(\alpha x) d\alpha, \tag{33}$$

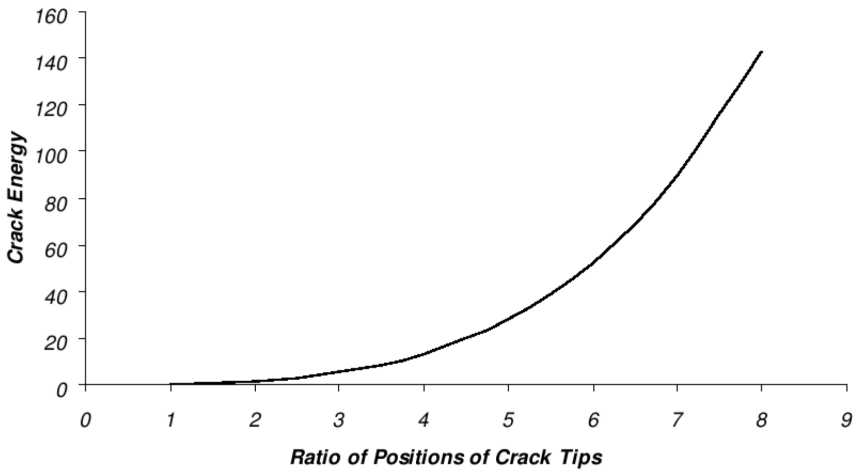

Figure 3: CrackEnergy (W_2)versus(b/a)

and

$$\phi_2(x,y) = \int_0^\infty \alpha^{-2} C_2(\alpha) e^{-\alpha y_2} \cos(\alpha x) d\alpha, \tag{34}$$

where $C_1(\alpha)$ and $C_2(\alpha)$ are arbitrary functions of α. Boundary condition (31) will be satisfied, if we take

$$C_1(\alpha) = -\frac{\gamma_1}{(1+\lambda_1)}\left[\frac{(1+\lambda_2)}{\gamma_2} C_2(\alpha) + \frac{(1+\mu)}{\pi}\frac{k_1}{k}\right]. \tag{35}$$

Boundary conditions (30) and (29) with the help of (35) give respectively

$$\int_0^\infty \alpha^{-1} D_1(\alpha) \cos(\alpha x) d\alpha = 0, \quad 0 < x < b,\ b < x < \infty\,, \tag{36}$$

and

$$\int_0^\infty \alpha^{-1} D_1(\alpha) \cos(\alpha x) d\alpha = p_0 f(x), \quad a < x < b, \tag{37}$$

where $D_2(\alpha)$is related to $C_2(\alpha)$by

$$D_2(\alpha) = C_2(\alpha) + \frac{\gamma_2(\lambda_1 - \mu)k_1}{(\lambda_1 - \lambda_2)k\pi}\,. \tag{38}$$

From the equations (36) and (32), we get

$$D_2(\alpha) = -\frac{2\alpha}{\pi}\int_a^b p_0 f(x)\cos(\alpha x) dx\,, \tag{39}$$

$$C_2(\alpha) = A_2(\alpha)\int_a^b p_0 f(x)\cos(\alpha x) dx + B_2\,, \tag{40}$$

where

$$A_2 = -\frac{2p_0}{\pi},\ B_2 = -\frac{\gamma_2(\lambda_1-\mu)k_1}{(\lambda_1-\lambda_2 k\pi)}\,. \tag{41}$$

From the equation (35) and with the help of the equation (40), we obtain

$$C_1(\alpha) = A_1\alpha\int_a^b f(x)\cos(\alpha x)dx + B_1\,, \tag{42}$$

where

$$A_1 = \frac{(1+\lambda_2)}{(1+\lambda_1)}.\frac{\gamma_1}{\gamma_2}.\frac{2p_0}{\pi},\ B_1 = \frac{\gamma_1}{1+\lambda_1}.\frac{k_1}{k\pi}.[\frac{(1+\lambda_2)(\lambda_1-\mu)}{(\lambda_1-\lambda_2)} - (1+\mu)]\,. \tag{43}$$

Therefore,

$$\phi_1(x,y) = \int_0^\infty \alpha^{-2}\{A_1\alpha\int_a^b f(x')\cos(\alpha x')dx' + B_1\}e^{-\alpha y_1}\cos(\alpha x)d\alpha\,, \tag{44}$$

and

$$\phi_2(x,y) = \int_0^\infty \alpha^{-2}\{A_2\alpha\int_a^b f(x')\cos(\alpha x')d(x') + B_2\}e^{-\alpha y_i}\cos(\alpha x)d\alpha\,. \tag{45}$$

From (2)-(11), we obtain

$$\sigma_{xx}/A_{66} = -\sum_{i=1}^{2}\frac{1+\lambda_i}{\gamma_i^2}\int_0^\infty [A_i\alpha\int_a^b f(x')\cos(\alpha x')dx' + B_i]e^{-\alpha y_i}\cos(\alpha x)d\alpha - \frac{1+\mu}{\pi}.\frac{k_1}{k^3}.\frac{y}{x^2+(y/k)^2}\,, \tag{46}$$

$$\sigma_{yy}/A_{66} = \sum_{i=1}^{2}(1+\lambda_i)\int_0^\infty [A_i\alpha\int_a^b f(x')\cos(\alpha x')dx' + B_i]e^{-\alpha y_i}\cos(\alpha x)d\alpha + \frac{1+\mu}{\pi}.\frac{k_1}{k}.\frac{y}{x^2+(y/k)^2}\,, \tag{47}$$

$$\sigma_{xy}/A_{66} = \sum_{i=1}^{2}\frac{1+\lambda_i}{\gamma_i}\int_0^\infty [A_1\alpha\int_a^b f(x')\cos(\alpha x')dx' + B_i]e^{-\alpha y_i}\sin(\alpha x)d\alpha + \frac{1+\mu}{\pi}.\frac{k_1}{k^3}.\frac{y}{x^2+(y/k)^2}\,. \tag{48}$$

Now, if we take $f(x) = 1$, we get

$$\sigma_{xx}/A_{66} = -\sum_{i=1}^{2}\frac{1+\lambda_i}{\gamma_i^2}[\frac{p_0 A_i}{2}.\{\frac{b+x}{(b+x)^2+y_i^2} + \frac{b-x}{(b-x)^2+y_i^2} - \frac{a+x}{(a-x)^2+y_i^2} - \frac{a-x}{(a-x)^2+y_i^2}\} + B_i\frac{y_i}{x^2+y_i^2}] + \frac{k_1(1+\mu)}{k^3\pi}.\frac{y}{x^2+(\frac{y}{k})^2}\,, \tag{49}$$

$$\sigma_{yy}/A_{66} = \sum_{i=1}^{2}(1+\lambda_i)[\frac{p_0 A_i}{2}.\{\frac{b+x}{(b+x)^2+y_i^2} + \frac{b-x}{(b-x)^2+y_i^2} - \frac{a+x}{(a+x)^2+y_i^2} - \frac{a-x}{(a-x)^2+y_i^2}\} + B_i\frac{y_i}{x^2+y_i^2}] + \frac{k_1(1+\mu)}{k\pi}.\frac{y}{x^2+(\frac{y}{k})^2}\,, \quad (50)$$

$$\sigma_{xy}/A_{66} = \sum_{i=1}^{2}\frac{1+\lambda_i}{\gamma_i}[\frac{p_0 A_i}{2}.\{\frac{y_i}{(b+x)^2+y_i^2} + \frac{y_i}{(b-x)^2+y_i^2} - \frac{y_i}{(a-x)^2+y_i^2} - \frac{y_i}{(a-x)^2+y_i^2}\} + B_i\frac{y_i}{x^2+y_i^2}] + \frac{k_1(1+\mu)}{k\pi}.\frac{y}{x^2+(\frac{y}{k})^2}\,. \quad (51)$$

The crack energy for Case 1 of the problem under discussion is

$$W_1 = 2p_0 A_{66}\int_a^b \sum_{i=1}^{2}(1+\lambda_i[\frac{p_0 A_i}{2}\{\frac{b}{b+x} + \frac{b}{b-x} + \frac{a}{a+x} - \frac{a}{a-x}\}]dx\,. \quad (52)$$

Case 2. If we consider the boundary condition (32) along with the conditions (30) and (31), the stresses can be obtained as

$$W_2 = -2p_0 A_{66}[(1+\lambda_1)A_1 + (1+\lambda_2)A_2]\int_a^b f(x)\sqrt{(x^2-a^2)(b^2-x^2)} \times [\int_a^b f(x')\sqrt{(x'^2-a^2)(b^2-x'^2)}\frac{(x^2+x'^2)}{(x^2-x'^2)}dx']dx\,. \quad (53)$$

5. Numerical Results and Discussion

Numerical calculations have been carried out to determine the physical quantities W_1 and W_2. These results have been presented through graphs. The material constants for this purpose are taken to be $A_{11} = 30.3 \times 10^6$psi, $A_{12} = 3.78 \times 10^6$psi, $A_{66} = 1.13 \times 10^6$psi, $A_{22} = 4.04 \times 10^6$psi, $k_1 = 1.6$, $\beta_x = 7.04 \times 10^6 Nm^2 deg^{-1}$ and $\beta_y/\beta_x = 0.5$.

Figure 2 and Figure 3 show the graphs for the crack energy, which have been drawn for different positions of the crack tips. It is observed from the Figure 2 that the crack energy is almost linear in nature for Case 1. But for Case 2 the crack energy is parabolic in nature.

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