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OPTIMAL CONTROL IN THE CLASS OF SMOOTH AND BOUNDED FUNCTIONS

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Abstract: The paper is focused on the optimal control problem with boundary conditions. Unlike the traditional class of piecewise continuous functions, here the admissible controls are defined as continuously differentiable functions with inclusion or homogeneous inclusion constraints. Admissible perturbations of control functions are formed using the idea of simultaneous varying. Numerical solution algorithm, obtained as a result, is proved to be convergent to the necessary condition of optimality in the form of the maximum principle.

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1. Introduction

The mathematical theory of optimal processes emerged as an answer to the requirement of solving engineering problems, and the applicability of this theory depends naturally on algorithms for solving the problems of optimal control. In its general formulation, a problem of optimal control is aimed at determining the optimal value of an objective functional defined on the profiles of an ODE system subject to given initial conditions. The right-hand endpoint could be either left free or be subject to some constraints. This paper deals with the problem of optimal control where the ODE system is subject to given *boundary* conditions, which is substantially more complex than the problem with initial

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conditions and includes the latter as a special case. Aside from its importance for pure mathematics, the control problem with boundary conditions has numerous applications, for example, in the problem of choice of optimal compositions for the protections against nuclear radiation [3, p. 268], in the problem of optimization of manufacturing cycles [3, p. 263], in the problem of synthesis of stratified structures under the effect of various waves and temperature factors [4], etc.

The control problem with boundary conditions has been investigated by the author together with K. Mizukami. At first, there was obtained a necessary condition for optimality of the maximum principle type, which also provided the background for the development of solution techniques [8]. Then, this line of research was continued by the authors [10], [6] when the differential maximum principle has been justified and the idea of combined control variation has been introduced. In parallel, the theory of singular controls has been also proposed [9], [7]. Having analyzed various solved problems of the mentioned type it can be concluded that in many cases the extension of the class of admissible controls from continuous to piecewise continuous is stipulated by the desire to take into account the amplitude or inclusion constraints for control functions.

The objective of this paper is to develop the optimality condition and optimization technique for a control problem with boundary conditions whose class of admissible controls contains smooth (i.e., continuously differentiable) functions with inclusion or homogeneous inclusion constraints. The investigation technique remains the same as before. Namely, the increment of the objective functional together with conjugate BVP is being considered on a certain type of control variation, thus providing the admissibility of varied control under some adjustments of the parameters of variation. In contrast to the classic variation of Lagrange and the needle-shaped variation of Boltyanskii [2], it is proposed to use the idea of so-called "internal" or "interior" variation expressed as far back as by M.V. Ostrogradskii and presented in contemporary form, e.g., by L. Zabello for optimal control problems with delay in [11, 12] and for variational calculus in [13]. This idea consists in the simultaneous varying of independent variable and control function. Under such approach, the dominant term in the Taylor series expansion of the objective functional determines the necessary condition for optimality, and the formula itself serves as a basis for the development of optimization algorithm which converges to the necessary condition of optimality in the form of the maximum principle.

2. Statement of the Problem

Let a controllable process

$$\{\boldsymbol{u},\boldsymbol{x}\} = \{\boldsymbol{u}(t)\in\mathbb{R}^m;\ \ \boldsymbol{x}(t)\in\mathbb{R}^n,\ t\in T=[t_0,t_1]\}$$

be defined by the conditions

$$J(\boldsymbol{u}) = \varphi_0\left(\boldsymbol{x}(t_0),\boldsymbol{x}(t_1)\right) + \int_T F(\boldsymbol{x},\boldsymbol{u},t)\,dt \to \min, \tag{1}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t), \quad \boldsymbol{\varphi}\left(\boldsymbol{x}(t_0),\boldsymbol{x}(t_1)\right) = \mathbf{0}. \tag{2}$$

Here admissible controls $\boldsymbol{u}(t),\ t\in T$ are smooth vector-functions with fixed end-points $\boldsymbol{u}^0,\boldsymbol{u}^1\in\mathbb{R}^m$, that is,

$$\boldsymbol{u}(\cdot)\in\mathcal{U} = \left\{\boldsymbol{u}\in C_1^m(T):\ \boldsymbol{u}(t_0)=\boldsymbol{u}^0,\ \boldsymbol{u}(t_1)=\boldsymbol{u}^1\right\}. \tag{3}$$

In addition to (3) we impose two different types of direct constraints for the control variables in all $t\in T$. The first type is well-known as inclusion constraint and can be written formally as

$$\boldsymbol{u}(t)\in U_1\subset\mathbb{R}^m \ \text{ for } \ t\in T,\ U_1 - \text{compact set},\ \operatorname{int}U_1\neq\emptyset,\ \boldsymbol{u}\in\mathcal{U}\,. \tag{4}$$

In particular, the usual amplitude constraints of the form $\alpha_i\le u_i(t)\le\beta_i$ fit perfectly in the above description. The second type of direct constraint is known as homogeneous inclusion constraint and can be written formally as

$$\boldsymbol{u}(t)\in U_2 = U_1\cap\{\boldsymbol{u}(\cdot)\in\mathcal{U}:\ G_i(\boldsymbol{u},\dot{\boldsymbol{u}})\le 0,\ \ i=1,2,\ldots,l\,\} \ \text{ for } \ t\in T, \tag{5}$$

where $G_i:\mathbb{R}^{2m}\to\mathbb{R}$ are given scalar functions homogeneous with respect to $\dot{\boldsymbol{u}}$, that is,

$$G_i(\boldsymbol{u},\xi(t)\dot{\boldsymbol{u}}) = [\xi(t)]^p\, G_i(\boldsymbol{u},\dot{\boldsymbol{u}}), \quad p>0\,,$$

holds for any real function $\xi(t)>0$ and for all $t\in T$.

Remark 1. The homogeneity of G_i with respect to $\dot{\boldsymbol{u}}$ will be beneficial for construction of admissible variation of interior type. It should be also noted that homogeneous constrains are not a rare exception. For instance,

$$G_i(\boldsymbol{u},\dot{\boldsymbol{u}}) = \sum_{k=1}^m \Phi_k(\boldsymbol{u}(t))\,[\dot{\boldsymbol{u}}_k(t)]^p \le 0,\ \ t\in T,\ \ p>0\,,$$

makes a general example of widely used homogeneous constraint. In particular, [1] provides various examples of homogeneous constraints in the form, e.g.,

$$G(\boldsymbol{u},\dot{\boldsymbol{u}}) = (u_1+u_2)\,\dot{u}_1^2 + (u_1-u_2)\,\dot{u}_2^2 \ge 0, \quad \boldsymbol{u}(t)\in U_1\subset\mathbb{R}^2.$$

Thus, this paper will study two types of optimal control problems with boundary conditions, the problem (1)–(3), (4) and the problem (1)–(3), (5). In both problems, vector-functions $\boldsymbol{f} = (f_1, \dots, f_n)$, $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ and scalar functions φ_0, F are continuous with respect to their arguments together with their partial derivatives with respect to $\boldsymbol{x}, \boldsymbol{u}$ and t. In addition, it is supposed that for any smooth and bounded admissible control $\boldsymbol{u}(t)$, $t \in T$, the boundary value problem (2) is solvable in the class of smooth functions $\boldsymbol{x}(\boldsymbol{u}, t)$, $t \in T$ (whether analytically or numerically). The latter assumption is quite reasonable since all admissible control functions are smooth.

Remark 2. A simple example of $\boldsymbol{\varphi}(\boldsymbol{x}(t_0), \boldsymbol{x}(t_1)) = \mathbf{0}$ could de given by separated two-point boundary conditions: $x_i(t_0) - a_i = 0$, $a_i \in \mathbb{R}$, $i = 1, \dots, k$, $k < n$ and $x_j(t_1) - b_j = 0$, $b_j \in \mathbb{R}$, $j = k+1, \dots, n$. That means that there is a group of state profiles x_j for which the initial conditions are not specified, and another group x_i for which the terminal conditions are not specified either. Thus, there are exactly n conditions in (2) which is just enough to define n arbitrary constants and to obtain the particular solution of ODE system for a specific control vector $\boldsymbol{u}(t)$. Under such approach, the boundary conditions are “linked” to the ODE system and we do not treat them as terminal state constraints. Thus, there are no additional state constraints and, therefore, no Lagrange multipliers will be used further on.

3. Increment Formula

For two admissible processes, the basic one $\{\boldsymbol{u}, \boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{u})\}$ and the varied one $\{\tilde{\boldsymbol{u}} = \boldsymbol{u} + \Delta \boldsymbol{u}, \tilde{\boldsymbol{x}} = \boldsymbol{x} + \Delta \boldsymbol{x} = \boldsymbol{x}(t, \tilde{\boldsymbol{u}})\}$, the formula for the increment of the functional (1) has been derived in [7] in the form of Taylor series expansion for more general case of measurable control functions $\boldsymbol{u} \in L_\infty$ with direct constraint $\boldsymbol{u}(t) \in U \subset \mathbb{R}^m, t \in T, U$ — compact set. The same work also provides an estimate of the state $\boldsymbol{x}$ caused by the control perturbation $\Delta \boldsymbol{u} = \tilde{\boldsymbol{u}} - \boldsymbol{u}$. Taking into account the smoothness of $\boldsymbol{u}$, the increment of (1) can be represented in the following form:

$$J(\tilde{\boldsymbol{u}}) - J(\boldsymbol{u}) = -\int_T \left\langle \frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{u}}, \Delta \boldsymbol{u}(t) \right\rangle dt + \int_T o\left(\|\Delta \boldsymbol{u}(t)\|\right) dt, \quad (6)$$

where

$$\frac{o(\alpha)}{\alpha} \to 0, \quad \alpha \to 0\,,$$

is a remainder term. Moreover, it should be noted that there exists some constant $\mathcal{K} > 0$ depending on $\boldsymbol{u}$ and $\boldsymbol{x}$ such that

$$| \, o \left(\|\Delta \boldsymbol{u}(t)\| \right) | \leq \mathcal{K} \, \|\Delta \boldsymbol{u}(t)\|^2, \quad t \in T. \tag{7}$$

Here

$$H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t) = \langle \boldsymbol{\psi}(t), \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \rangle - F(\boldsymbol{x}, \boldsymbol{u}, t)$$

is the maximal Hamiltonian function; $\| \cdot \|$ is the vector's norm and $\langle \cdot \, , \cdot \rangle$ stands for inner product in the finite-dimensional Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. The adjoint vector-function $\boldsymbol{\psi}(t) \in \mathbb{R}^n$ is a solution profile of the boundary value problem

$$\dot{\boldsymbol{\psi}} = -\frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{x}}, \tag{8}$$

$$-\boldsymbol{B}_0 \boldsymbol{\psi}(t_0) + \boldsymbol{B}_1 \boldsymbol{\psi}(t_1) + \boldsymbol{B}_0 \frac{\partial \varphi_0}{\partial \boldsymbol{x}(t_0)} + \boldsymbol{B}_1 \frac{\partial \varphi_0}{\partial \boldsymbol{x}(t_1)} = \boldsymbol{0}, \tag{9}$$

where $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are some numerical $(n \times n)$ matrices which are chosen arbitrarily in order to satisfy the condition

$$\boldsymbol{B}_0 \left[\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}(t_0)} \right]' + \boldsymbol{B}_1 \left[\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}(t_1)} \right]' = \boldsymbol{0}. \tag{10}$$

Prime here denotes the transpose of the matrix. It was demonstrated in [7] that if the direct BVP (2) is solvable for some admissible process $\{\boldsymbol{u}, \boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{u})\}$ then the corresponding linear adjoint problem (8)–(10) is also solvable with respect to $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \boldsymbol{u})$.

4. Choice of Admissible Variation

In Section 2 it was proposed to consider two different sets of admissible controls. Now, it is worth to give the descriptions of admissible control variations suitable for both (4) and (5).

4.1. Inclusion Constraints

Let $\boldsymbol{u} \in U_1$ according to (4). Then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ can be chosen as

$$\boldsymbol{u}_\varepsilon(t) = \boldsymbol{u}(t + \varepsilon \delta(t)), \quad \varepsilon \in [0, 1], \tag{11}$$

where $\delta = \delta(t)$ is a smooth real function which satisfies

$$\delta(t_0) = \delta(t_1) = 0, \quad t_0 - t \leq \delta(t) \leq t_1 - t, \quad t \in T. \tag{12}$$

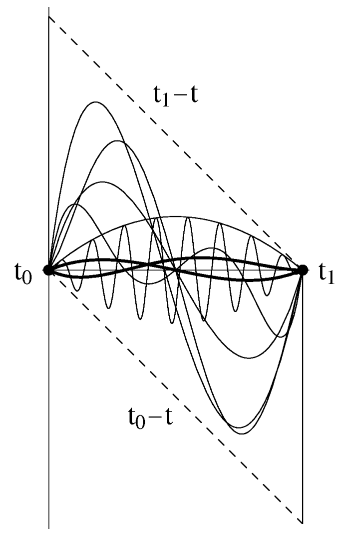

Figure 1: Infinite possibilities for choosing an appropriate $\delta(t)$

Here the end-point conditions for $\delta(t)$ are essential and destined for keeping the end-points of all admissible controls fixed, that is, $\boldsymbol{u}(t_0) = \boldsymbol{u}^0, \boldsymbol{u}(t_1) = \boldsymbol{u}^1$.

Proposition 3. *If the basic control $\boldsymbol{u} = \boldsymbol{u}(t)$ is admissible in the sense that $\boldsymbol{u} \in U_1$, i.e. satisfies (4), then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ defined by (11) is also admissible for all $\varepsilon \in [0,1]$ and for any smooth real function $\delta(t)$ satisfying (12).*

Apparently, by denoting

$$t_\varepsilon = t + \varepsilon\, \delta(t) \in T, \quad \varepsilon \in [0,1]\,,$$

it is concluded that $\boldsymbol{u}_\varepsilon = \boldsymbol{u}(t_\varepsilon) \in U$ due to the fact that $\boldsymbol{u}$ is admissible for all $t \in T$. It is also obvious that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u},\ t \in T$ when $\varepsilon \to 0$ since

$$\Delta_\varepsilon \boldsymbol{u}(t) = \boldsymbol{u}_\varepsilon(t) - \boldsymbol{u}(t) = \varepsilon\, \dot{\boldsymbol{u}}(t)\, \delta(t) + \hat{\boldsymbol{o}}(\varepsilon), \tag{13}$$

$$\hat{\boldsymbol{o}}(\varepsilon) = (o_1(\varepsilon), \ldots, o_m(\varepsilon)), \quad \lim_{\varepsilon \to 0} \frac{o_k(\varepsilon)}{\varepsilon} = 0,\ k = 1, \ldots, m.$$

According to the condition (12), real function $\delta(t)$ must have its graph within the parallelogram as it is shown in Figure 1.

4.2. Homogeneous Inclusion Constraints

Let $\boldsymbol{u} \in \mathcal{U}_2$ according to (5). In this case, an additional homogeneous constraints will considerably complicate the structure of admissible variations.

Proposition 4. *If the basic control $\boldsymbol{u} = \boldsymbol{u}(t)$ is admissible in the sense that $\boldsymbol{u} \in \mathcal{U}_2$, i.e. satisfies (5), then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ defined by (11) is also admissible for all $\varepsilon \in [0,1]$ and for any smooth real function $\delta(t)$ which satisfies (12) and*

$$\left|\dot{\delta}(t)\right| \leq 1, \qquad t \in T. \tag{14}$$

In fact, if $\boldsymbol{u}(t) \in U_1$ and satisfies $G_i(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \leq 0,\ i = 1, \ldots, l$ then by virtue of Proposition 3 $\boldsymbol{u}_\varepsilon(t) \in U_1,\ t \in T,\ \varepsilon \in [0,1]$. Moreover, under homogeneity condition $G_i(\boldsymbol{u}, \xi(t)\dot{\boldsymbol{u}}) = [\xi(t)]^p\, G_i(\boldsymbol{u}, \dot{\boldsymbol{u}}),\ i = 1, \ldots, l$ with respect to $\dot{\boldsymbol{u}}$, it holds for $t_\varepsilon = t + \varepsilon\, \delta(t) \in T$ that

$$\begin{aligned} G_i\left(\boldsymbol{u}_\varepsilon(t), \dot{\boldsymbol{u}}_\varepsilon(t)\right) &= G_i\left(\boldsymbol{u}(t_\varepsilon), \left[1 + \varepsilon\dot{\delta}(t)\right] \frac{d\boldsymbol{u}}{dt_\varepsilon}\right) \\ &= \left[1 + \varepsilon\dot{\delta}(t)\right]^p G_i\left(\boldsymbol{u}(t_\varepsilon), \frac{d\boldsymbol{u}}{dt_\varepsilon}\right) \leq 0, \quad i = 1, \ldots, l. \end{aligned}$$

Since $1 + \varepsilon\dot{\delta}(t) \geq 0$ for all $\varepsilon \in [0,1]$ due to (14).

Thus, under the homogeneous constraints we can choose an appropriate $\delta(t)$ as any smooth function from the parallelogram in Figure 1 whose slope stays between -1 and 1 for all $t \in T$. The latter requirement narrows significantly our selection possibilities. However, we still have numerous potential possibilities to choose $\delta(t)$ as one of the curves shown in Figure 1 in bold lines.

5. Optimality Condition

The increment formula (6) serves as a basis to obtain the optimality condition (of the maximum principle type) for the problem (1)–(3) with control constraints (4)–(5). The replacement of $\tilde{\boldsymbol{u}}$ in (6) by an admissible perturbed control of the type (11) results in

$$J(\boldsymbol{u}_\varepsilon) - J(\boldsymbol{u}) = -\varepsilon \int_T W(\boldsymbol{u}, t)\, \delta(t)\, dt + o(\varepsilon), \quad \varepsilon \in [0,1], \tag{15}$$

for $\boldsymbol{u} \in U_1$ or $\boldsymbol{u} \in U_2$ and its perturbation $\boldsymbol{u}_\varepsilon$ given by (11), while $W(\boldsymbol{u}, t)$ stands for

$$W(\boldsymbol{u}, t) = \left\langle \frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{u}}, \dot{\boldsymbol{u}}(t) \right\rangle. \tag{16}$$

Formula (15) is valid for all $\varepsilon \in [0,1]$ and for all $\delta = \delta(t)$ that satisfy the conditions (12) or (12), (14) for control constraints (4) or (5), respectively. It is also obvious that for the remainder term $\hat{\boldsymbol{o}}(\varepsilon)$ of the expansion (11), (13) it is fulfilled that

$$\| \hat{\boldsymbol{o}}(\varepsilon) \| \leq \mathcal{K}\, \varepsilon^2.$$

The stationarity of the dominant term of (15) with respect to $\boldsymbol{u}$ and for all admissible $\delta(t)$ provides the necessary condition of optimality. Therefore, we can formulate the following theorem.

Theorem 5. *Suppose that $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ is optimal control for the problem (1)–(3) with direct control constraints of either type (4) or (5), and that $\boldsymbol{x}^*(t) = \boldsymbol{x}(t, \boldsymbol{u}^*)$, $\boldsymbol{\psi}^*(t) = \boldsymbol{\psi}(t, \boldsymbol{u}^*)$ are the correspondent profiles of the direct BVP (2) and the adjoint BVP (8)–(10). Then it should hold that*

$$W(\boldsymbol{u}^*, t) = 0, \qquad t \in T. \tag{17}$$

The proof of the Theorem 5 is almost immediate and arises out of the increment formula (15) considered for any admissible $\delta(t)$ with different signs.

Remark 6. This necessary condition of optimality in the class of smooth and bounded controls is defined directly by the problem entries and their derivatives (see formulae (16), (17)). However, if the class of admissible controls is extended from smooth to piecewise continuous functions (so that $\dot{\boldsymbol{u}}(t)$ is not defined for all $t \in T$), the optimality condition for bounded controls obtained in [8, 7] is to be written as

$$\left\langle \frac{\partial H(\boldsymbol{\psi}^*, \boldsymbol{x}^*, \boldsymbol{u}^*, t)}{\partial \boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}^* \right\rangle \leq 0\,, \quad \forall\, \boldsymbol{v} \in U\,,$$

and almost for all $t \in T$. The latter condition is obviously less useful for numerical calculations than (17) because of the presence of undefined parameter $\boldsymbol{v}$.

Remark 7. It is obvious that the necessary condition of optimality (17) holds trivially within all subsegments $T_* \subset T$, $\mathrm{mes} T_* > 0$, where $\dot{\boldsymbol{u}}^*(t) = \mathbf{0}$, i.e., $\boldsymbol{u}^*(t) = \mathrm{const}$. Therefore, the optimization technique presented in the following section will not tolerate any constant-wise controls even if they are admissible in the sense of direct constraints (4) or (5).

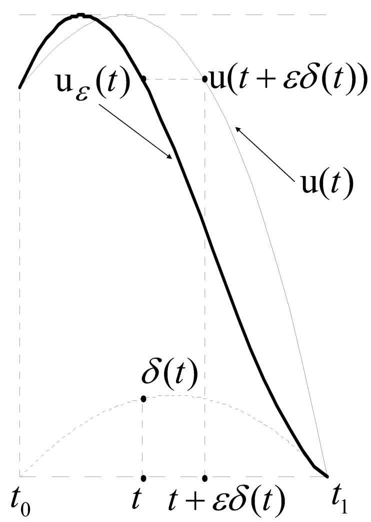

Figure 2: Geometrical interpretation of perturbation $u_\varepsilon(t) = u(t+\varepsilon\delta(t))$ by means of interior variation

6. Optimization Algorithms

Perturbations of the type $\Delta_\varepsilon \boldsymbol{u}(t) = \boldsymbol{u}_\varepsilon(t) - \boldsymbol{u}(t)$, where $\boldsymbol{u}_\varepsilon(t)$ is defined by (11) are often referred to as "internal" or "interior" variation due to the following feature. If a given control function $\boldsymbol{u}(t)$ is adjusted by means of such variation, the resulting function always remains within the limits of control domain for properly chosen $\delta(t)$. The mechanism of internal perturbation is illustrated in Figure 2 for amplitude constraints $\alpha_i \le u_i(t) \le \beta_i$.

Thus, in order to use these interior variations in practice, it will be useful to point out an appropriate way to choose the function $\delta(t)$.

Lemma 8. *Conditions (12) are fulfilled for*

$$\delta(t) = \delta_1(t) = \frac{(t-t_0)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, a(t), \qquad \mathcal{M} \ge \max_{t\in T} |a(t)|\,, \tag{18}$$

where $a = a(t),\ t \in T$ is some arbitrary smooth real function.

Proof. A mere glance reveals that $\delta(t_0) = \delta(t_1) = 0$. There is no difficulty to note that $\delta_1(t)$ has the same sign as $a(t)$. Therefore, if $a(t) \ge 0$ then $t_0 - t \le \delta_1(t)$, and if $a(t) \le 0$ then $\delta_1(t) \le t_1 - t$. It remains to show that:

(a) if $a(t) \ge 0$ then $\delta_1(t) \le t_1 - t$;

(b) if $a(t) \le 0$ then $t_0 - t \le \delta_1(t)$.

Item (a) holds due to the fact that

$$\frac{(t-t_0)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, a(t) \le \frac{(t_1-t_0)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, a(t) \le t_1 - t\,,$$

since $a(t)/\mathcal{M} \le 1$ and $t - t_0 \le t_1 - t_0$. On the other hand, item (b) holds due to the fact that

$$\frac{(t-t_0)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, a(t) = \frac{(t_0-t)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, |a(t)| \ge t_0 - t\,,$$

since $|a(t)|/\mathcal{M} \le 1,\ t_1 - t \le t_1 - t_0$ and $t_0 - t \le 0$. □

Lemma 9. *Conditions (12) and (14) are fulfilled for*

$$\delta(t) = \delta_2(t) = \frac{(t-t_0)(t_1-t)}{\mathcal{M}\cdot\mathcal{L}(t_1-t_0)}\, a(t) = \frac{\delta_1(t)}{\mathcal{L}}, \tag{19}$$

$$\mathcal{L} = \frac{1}{\mathcal{M}(t_1-t_0)} \cdot \max_{t\in T}\left[(t_1+t_0-2t)\, a(t) + (t-t_0)(t_1-t)\dot{a}(t)\right] \tag{20}$$

where $a = a(t),\ t \in T$ is some arbitrary smooth real function and $\delta_1(t),\ \mathcal{M}$ are defined by (18).

Proof. First, it is easy to check that (19) satisfies the conditions of Lemma 8. Moreover, it is noted that

$$\mathcal{L} = \max_{t\in T}\left|\dot{\delta}_1(t)\right|,$$

where $\delta_1(t)$ is defined by (18). Then, having calculated the derivative of $\delta_2(t)$, it is obtained that

$$\left|\dot{\delta}_2(t)\right| = \left|\frac{\dot{\delta}_1(t)}{\max_{t\in T}\left|\dot{\delta}_1(t)\right|}\right| \le 1.$$

This entirely proves all the statements of Lemma 9. □

It should be noted that the choice of $\delta_1(t),\ \delta_2(t)$ according to (18), (19)–(20) will establish the non-negativity of the dominant term in the expansion (15) for $a(t) = W(\boldsymbol{u}, t)$ since both $W(\boldsymbol{u}, t)$ and $\delta(t)$ will carry the same sign. This allows to introduce two nonnegative functionals $\mu_j(\boldsymbol{u}),\ \ j = 1, 2$. Each functional is determined by the corresponding control constraints (4) or (5) respectively:

$$\mu_1(\boldsymbol{u}) = \int_T W(\boldsymbol{u}, t)\, \delta_1(t)\, dt\,, \tag{21}$$

where $\delta_1(t)$ is defined by (18) for $a(t) = W(\boldsymbol{u}, t)$;

$$\mu_2(\boldsymbol{u}) = \int_T W(\boldsymbol{u}, t)\, \delta_2(t)\, dt\,, \tag{22}$$

where $\delta_2(t)$ is defined by (19) for $a(t) = W(\boldsymbol{u}, t)$. Then, for admissible perturbation (13), the following formula is correct for both types of control constraints (4) and (5):

$$J(\boldsymbol{u}_\varepsilon) - J(\boldsymbol{u}) = -\varepsilon\, \mu_j(\boldsymbol{u}) + o(\varepsilon), \qquad \mu_j(\boldsymbol{u}) \geq 0, \qquad j = 1, 2, \tag{23}$$

where by virtue of the estimate (7)

$$|\, o(\varepsilon)\,| \leq \mathcal{K}\, \varepsilon^2, \qquad \mathcal{K} = \text{const} > 0. \tag{24}$$

Theorem 10. *Let $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ be an optimal control in the problem (1)–(3) with one type of control constraints (4) or (5), i.e., $\boldsymbol{u} \in U_j,\ j = 1, 2$. Then, for corresponding j, it holds that*

$$\mu_j(\boldsymbol{u}^*) \;=\; 0, \qquad\qquad j = 1, 2. \tag{25}$$

The result of Theorem 10 is proved by the increment formula (23).

It is known that in order to use the successive approximation technique based on the maximum principle [8, 10, 6] the direct constraints $\boldsymbol{u}(t) \in U_j$, $j = 1, 2$ should be rather simple because this technique relies on the supposition of explicit solvability of the maximum condition for Hamiltonian function:

$$\hat{\boldsymbol{u}}(t) = \arg \max_{\boldsymbol{v} \in U_j} H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{v}, t) \quad \text{almost for all} \quad t \in T.$$

In other words, the classic method of successive approximations requires to solve problems of nonlinear programming with respect to U_j and for all $t \in T$ at every iteration. This part of the technique can be excluded when admissible controls are of the form (4) or (5) and the variation is chosen according to (11), (12) and (14). In fact, it is sufficient to find only one basic admissible control, since the varied control $\boldsymbol{u}_\varepsilon$ will always remain within the class of admissible controls.

The algorithms for numerical solution of the problem (1)–(3) have the same structure for both types of control constraints (4) and (5). Both algorithms are designed so as to be convergent to the necessary conditions for optimality (25). This implies that a numerical solution $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ will be a *Pontryagin extremal*, that is, an admissible control satisfying the necessary condition of optimality of the form (17) or (25).

Let an admissible control $\boldsymbol{u}^k \in \mathcal{U}_j,\ \ j \in \{1,2\},\ \ k = 0$ be given. It should be emphasized that $\boldsymbol{u}^k = \boldsymbol{u}^k(t)$ must not contain constant sections, i.e., $\boldsymbol{u}^k(t) \neq \text{const},\ \ t \in T_k,\ \ \text{mes}\, T_k > 0$ (see Remark 7). Then one should integrate numerically the direct BVP (2) and adjoint BVP (8)–(10) and store their profiles $\boldsymbol{x}^k = \boldsymbol{x}(t,\boldsymbol{u}^k),\ \ \boldsymbol{\psi}^k = \boldsymbol{\psi}(t,\boldsymbol{u}^k)$. After that, for $j \in \{1,2\}$ subject to (4) or (5), the corresponding $\mu_j(\boldsymbol{u}^k) \geq 0$ must be calculated using the formula (21) or (22) respectively. If $\mu_j(\boldsymbol{u}^k) = 0$, then by virtue of Theorem 5, the control function $\boldsymbol{u}^k = \boldsymbol{u}^k(t)$ is a possible solution to the problem (1)–(3) with additional direct constraint (4) or (5); therefore, the algorithm is depleted then.

Otherwise, it is concluded that $\boldsymbol{u}^k(t)$ is not a Pontryagin extremal and we have

$$\mu_j(\boldsymbol{u}^k) > 0, \qquad j \in \{1,2\}. \tag{26}$$

The next stage of the solution process is dedicated to construction of perturbed control function $\boldsymbol{u}^k_\varepsilon(t)$ by means of interior variation. First step in this direction is to define a smooth real function $\delta_1(t)$ or $\delta_2(t)$ according to (18) or (19)–(20), respectively, for $a(t) = W(\boldsymbol{u}^k,t)$ and then to construct one-parameter family of admissible controls $\boldsymbol{u}^k_\varepsilon = \boldsymbol{u}^k_\varepsilon(t),\ \ t \in T,\ \ \varepsilon \in [0,1]$ using the formula (11): $\boldsymbol{u}^k_\varepsilon \in \mathcal{U}_j,\ \ j \in \{1,2\}$. Second step is to solve the problem of one-parameter minimization

$$\varepsilon_k \ = \ \arg\min_{\varepsilon\in[0,1]} J\left(\boldsymbol{u}^k_\varepsilon\right) \tag{27}$$

and then to determine the successive approximation as

$$\boldsymbol{u}^{k+1}(t) \ = \ \boldsymbol{u}^k_{\varepsilon_k}(t), \qquad k = 0,1,2,\ldots. \tag{28}$$

After that, the whole process must be repeated all over again until the necessary condition of optimality (25) is satisfied within the limits of specified precision. The convergence of this iterative process is justified by the following theorem.

Theorem 11. *Suppose that $J(\boldsymbol{u})$ in the problem (1)–(3) is bounded from below for both types of control constraints (4) and (5). Then the sequence of admissible controls $\{\boldsymbol{u}^k\}$ generated by the algorithm (26)–(28) is a strictly relaxational one, i.e., $J\left(\boldsymbol{u}^{k+1}\right) < J\left(\boldsymbol{u}^k\right),\ k = 0,1,2,\ldots$ and convergent to the necessary condition of optimality (25) in the sense that*

$$\lim_{k\to\infty} \mu_j\left(\boldsymbol{u}^k\right) = 0, \qquad j \in \{1,2\}. \tag{29}$$

Proof. To begin with, the increment formula (23) should be examined for $\boldsymbol{u} = \boldsymbol{u}^k,\ \boldsymbol{u}_\varepsilon = \boldsymbol{u}^k_\varepsilon$ taking into account the estimate (24):

$$J\left(\boldsymbol{u}^k_\varepsilon\right) - J\left(\boldsymbol{u}^k\right) \leq -\varepsilon\, \mu_j(\boldsymbol{u}^k) + \mathcal{K}\varepsilon^2.$$

By virtue of the inequality (26), the strict relaxation for small $\varepsilon > 0$ becomes obvious. Hence, taking into consideration the minimization problem (27)

$$J(\boldsymbol{u}^{k+1}) - J(\boldsymbol{u}^k) \leq -\varepsilon\, \mu_j(\boldsymbol{u}^k) + \mathcal{K}\varepsilon^2, \tag{30}$$

$$\varepsilon \in [0,1], \quad j \in \{1,2\}.$$

Inequality (30) can be transformed into

$$0 \leq \varepsilon\, \mu_j(\boldsymbol{u}^k) \leq J(\boldsymbol{u}^k) - J(\boldsymbol{u}^{k+1}) + \mathcal{K}\varepsilon^2. \tag{31}$$

Due to the relaxation and to the boundedness of $J(\boldsymbol{u})$ from below

$$0 \leq J(\boldsymbol{u}^k) - J(\boldsymbol{u}^{k+1}) \to 0, \qquad k \to \infty.$$

Then, passing to the limit in (31)

$$0 \leq \varepsilon \left[\lim_{k\to\infty} \mu_j(\boldsymbol{u}^k) \right] \leq \mathcal{K}\varepsilon^2,$$

$$\varepsilon \in [0,1], \quad j \in \{1,2\}.$$

The latter is valid only if (29) holds. □

Remark 12. Apparently, due to (16), function $W(\boldsymbol{u}^k, t)$ may have rather complicated structure in terms of t since it will always include $\boldsymbol{x}(t)$ and $\boldsymbol{\psi}(t)$. Quite often, however, the solution profiles $\boldsymbol{x}(t)$ and $\boldsymbol{\psi}(t)$ may only be recovered by means of numerical integration. In that case, the explicit forms of (16) and (21), (22) may seem useless. On the other hand, by defining

$$\delta_1(t) = \frac{(t-t_0)(t_1-t)}{\mathcal{M}(t_1-t_0)}\, W(\boldsymbol{u}^k, t) \quad \text{and} \quad \delta_2(t) = \frac{\delta_1(t)}{\mathcal{L}} \tag{32}$$

we just wanted to guarantee that both $\delta_j(t), j = 1,2$ and $W(\boldsymbol{u}^k, t)$ carry the same sign and thus to assure the non-negativity of the dominant term of (15). Therefore, the *exact* form of $W(\boldsymbol{u}^k, t)$ is not actually required and, for computational purposes, we can approximate $W(\boldsymbol{u}^k, t)$ in (32) with a minimum-degree interpolating polynomial $P_k(t)$ carrying the same sign as $W(\boldsymbol{u}^k, t)$ everywhere on T. In other words, $P_k(t)$ must interpolate all zeros of $W(\boldsymbol{u}^k, t)$ and carry the same sign as $W(\boldsymbol{u}^k, t)$ for all $t \in T$.

The following example illustrates the application of the solution procedure described above.

Example 13. Consider a simplified version linear-convex version of the problem (1)–(3):

$$\begin{cases} \dot{x}_1 &=& x_2, & x_1(0) &=& 1, \\ \dot{x}_2 &=& x_1 + u(t), & x_2(1) &=& 0, \end{cases} \quad t \in T = [0,1],$$

$$J(u) = [3x_2(0) + 2.16]^2 + [10x_1(1) - 5.8]^2 \to \min,$$

$$u(t) \in \mathcal{U} = \{u(\cdot) \in C_q(T) : |u| \leq 1,\ u(0) = -1, u(1) = 1\},$$

with scalar control function $\boldsymbol{u}(t) = u(t)$ and $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2,\ t \in T$. In this particular example, the optimal control is $u^*(t) = (2t-1)^3$, with the correspondent states $x_1(1, u^*) = 0.58,\ x_2(0, u^*) = -0.72$ that guarantee that $J(u^*) = 0,\ W(u^*, t) = 0$. It should be noted that the optimal control $u^*(t) = (2t-1)^3$ is also singular in the sense of the maximum principle if the end-points of admissible controls are left free, see [9].

In order to perform an iteration of the optimization algorithm it is convenient to define analytically several useful quantities. First, in this example

$$H(\boldsymbol{\psi}, \boldsymbol{x}, u, t) = \psi_1 x_2 + \psi_2 x_1 + \psi_2 u(t).$$

There is no difficulty to determine the conjugate BVP according to (8)–(10)

$$\begin{cases} \dot{\psi}_1 &=& -\psi_2, & \psi_1(1) &=& -20\left[10x_1(1) - 5.8\right], \\ \dot{\psi}_2 &=& -\psi_1, & \psi_2(0) &=& 6\left[3x_2(0) + 2.16\right], \end{cases}$$

whose boundary conditions depend on the missing end-points of the state system. These end-points can be obtained using the matrix representation of the solution of linear BVP [8, 6]:

$$\begin{aligned} x_1(1, u) &= 0.65 - 0.32\left[\int_0^1 e^t u(t)\,dt - \int_0^1 e^{-t} u(t)\,dt\right], \\ x_2(0, u) &= -0.76 - 0.12\int_0^1 e^t u(t)\,dt - 0.88\int_0^1 e^{-t} u(t)\,dt. \end{aligned}$$

Thus, the deviation function (16) turns into

$$W(u, t) = \psi_2(t)\dot{u}(t),$$

where

$$\psi_2(t, u) = 0.32\,\psi_2(0)\left[0.37e^t + 2.72e^{-t}\right] - 0.32\,\psi_1(1)\left[e^t - e^{-t}\right].$$

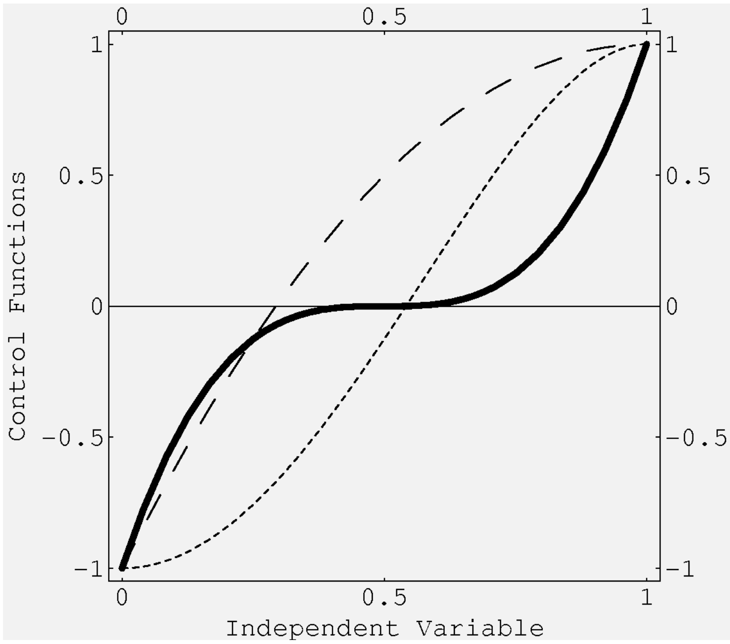

Figure 3: Smooth control functions, Example 1

Let the initial approximation be given by $u^0(t) = -2t^2 + 4t - 1$. Under this control $x_1(1, u^0) = 0.42$, $x_2(0, u^0) = -0.96$ and $J(u^0) = 3.08$. A smooth function $\delta = \delta^0(t)$ can be defined by (18) for $a(t) = W(u^0, t)$. However, we can observe that $W(u^0(t), t) < 0$ for all $t \in [0, 1)$ while $W(u^0(1), 1) = 0$. Therefore, for manual illustrative computations the function δ^0 can be simplified as follows:

$$\delta^0(t) = t^2 - t \leq 0, \qquad t \in T = [0, 1].$$

Then by virtue of (11)

$$u^0_\varepsilon(t) = -2\left[t + \varepsilon\, \delta^0(t)\right]^2 + 4\left[t + \varepsilon\, \delta^0(t)\right] - 1, \quad \varepsilon \in [0, 1]$$

and for $\varepsilon = 1$ it is obtained that $u^1(t) = -2t^4 + 4t^2 - 1$, $x_1(1, u^1) = 0.61$, $x_2(0, u^1) = -0.84$, and $J(u^1) = 0.22 < J(u^0) = 3.08$. Therefore, for the first illustrative iteration the relaxation becomes evident. Then we can continue the iterative process numerically starting from $u^1(t)$.

Figure 3 shows the control functions. Here $u^0(t)$ is given by dashed line, dotted line stands for $u^1(t)$, and $u^*(t)$ is drawn using thick solid line. On the sixth iteration of the computer implementation, $u^6(t)$ almost coincides with

$u^*(t)$ in the sense that

$$\int_0^1 \left|u^*(t) - u^6(t)\right| \, dt \leq 0.01 \quad \text{and} \quad J(u^6) \approx 4 \cdot 10^{-5}.$$

7. Conclusion

The application of general methods for solving real world problems requires a lot of computational effort and is rarely successful. Algorithmic support to non-linear optimal control problems is provided by numerical methods of the gradient type, linearization methods, and methods based on the maximum principle (see, e.g. the survey paper [5]). The presence of control constraints in many cases leads to the application of numerical techniques based on the maximum principle, where the control trajectories are adjusted by mean of needle-shaped or combined variation. The majority of such techniques rest on a compulsory assumption of explicit solvability of the maximum condition with respect to the maximal Hamiltonian function. Such an assumption becomes completely excessive if the control trajectories are adjusted by means of interior variations.

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