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MINIMAL SUBMANIFOLDS OF CONSTANT SECTIONAL CURVATURE C ISOMETRICALLY IMMERSED IN A RIEMANNIAN SPACE FORM $R^m(c)$

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Abstract: The aim of this paper is to give some characterizations of a Riemannian n-manifold M isometrically immersed in a Riemannian space form $R^m(c)$ of constant sectional curvature c for the equality which involves $\widehat{\delta}$-invariant, the squared mean curvature of the Riemannian submanifold and c.

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1. Introduction

In [3], the author introduces new types of curvature invariants, defining two strings of scalar-valued Riemannian curvature functions $M \longrightarrow \mathbb{R}$, namely $\delta(n_1, \ldots, n_k)$ and $\widehat{\delta}(n_1, \ldots, n_k)$ for every $(n_1, \ldots, n_k)$ satisfying $n_1 < n$, $n_j \geq 2$, $j = 1, \ldots, k$ and $n_1 + \cdots + n_k \leq n$. For these two strings of Riemannian invariants [1, 2, 3], one always has trivally

$$\delta(n_1, \ldots, n_k) \geq \widehat{\delta}(n_1, \ldots, n_k) . \tag{1.1}$$

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It was shown by the author in [3] that, for any isometric immersion of a Riemannian n-manifold M in a Riemannian space form $R^m(c)$ of constant sectional curvature c, $\delta(n_1, \ldots, n_k)$ satisfies the following sharp inequality:

$$\delta(n_1, \ldots, n_k) \leq \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)} H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right) c. \quad (1.2)$$

Combining (1.1) and (1.2), one also has the following inequality:

$$\widehat{\delta}(n_1, \ldots, n_k) \leq \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)} H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right) c. \quad (1.3)$$

In [4], the authors classified Riemannian manifolds which satisfy the equality case of (1.1), i.e. satisfy the condition: $\delta(n_1, \ldots, n_k) = \widehat{\delta}(n_1, \ldots, n_k)$. They also determined all submanifolds of a Riemannian space form $R^m(c)$ which realize the equality in (1.3).

In this article, we obtain some characterizations of a Riemannian n-manifold M isometrically immersed in a Riemannian space form $R^m(c)$ of constant sectional curvature c for the equality which involves $\widehat{\delta}$-invariant, the squared mean curvature of the Riemannian submanifold and c.

2. Preliminaries

Let M be an n-dimensional submanifold of a Riemannian space form $R^m(c)$ of constant sectional curvature c. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $R^m(c)$, respectively. Then the Gauss and Weingarten formulas of M in $R^m(c)$ are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.1)$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (2.2)$$

for vector fields X, Y tangent to M and ξ normal to M, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold in $R^m(c)$. The second fundamental form and the shape operator are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle. \quad (2.3)$$

The mean curvature vector H of the submanifold M is defined by $H = \frac{1}{n}\text{trace}\, h$. A submanifold M is called minimal in $R^m(c)$ if its mean curvature vector vanishes identically. Moreover a submanifold M is said to be totally geodesic at a point $p \in M$ if its second fundamental form vanishes at p.

Denote by R the Riemann curvature tensor of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y;Z,W) = (\langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle)\, c \\ + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \end{aligned} \tag{2.4}$$

for vectors X, Y, Z, W tangent to M.

For a Riemannian n-manifold M, denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM$, $p \in M$. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined in [4] by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.5}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the r-plane section L is defined in [3] to be

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2.6}$$

The scalar curvature of the r-space spanned by $\{e_{i_1}, \ldots, e_{i_r}\}$ is denoted by $\tau_{i_1 \cdots r}$.

For an integer $k \geq 0$, denote by $S(n,k)$ the set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Let $S(n)$ denote the union of all $S(n,k)$, $k \geq 0$. For each k-tuple $(n_1, \ldots, n_k) \in S(n,k)$, Riemannian invariants $S(n_1, \ldots, n_k)(p)$ and $\widehat{S}(n_1, \ldots, n_k)(p)$ are defined respectively by

$$\begin{aligned} S(n_1, \ldots, n_k)(p) &= \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \\ \widehat{S}(n_1, \ldots, n_k)(p) &= \sup\{\tau(L_1) + \cdots + \tau(L_k)\}, \end{aligned} \tag{2.7}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

The Riemannian invariants $\delta(n_1, \ldots, n_k)(p)$ and $\widehat{\delta}(n_1, \ldots, n_k)(p)$ introduced in [3] are given by

$$\begin{aligned} \delta(n_1, \ldots, n_k)(p) &= \tau(p) - S(n_1, \ldots, n_k)(p), \\ \widehat{\delta}(n_1, \ldots, n_k)(p) &= \tau(p) - \widehat{S}(n_1, \ldots, n_k)(p). \end{aligned} \tag{2.8}$$

Clearly, $\delta(n_1,\ldots,n_k) \geq \widehat{\delta}(n_1,\ldots,n_k)$ for any k-tuple $(n_1,\ldots,n_k) \in S(n,k)$. If a Riemannian manifold M satisfies $\delta(n_1,\ldots,n_k) = \widehat{\delta}(n_1,\ldots,n_k)$ identically, it is called an $S(n_1,\ldots,n_k)$-space. It follows from (2.7) and (2.8) that a Riemannian n-manifold is an $S(n_1,\ldots,n_k)$-space if and only if $\tau(L_1)+\cdots+\tau(L_k)$ is independent of the choice of k mutually orthogonal subspaces $L_1,\ldots,L_k$ which satisfy $\dim L_j = n_j$, $j=1,\ldots,k$ (see [4]).

We recall the following theorems and lemmas for later use.

Theorem 2.1. (see [3]) *Given an n-dimensional submanifold M in a Riemannian space form $R^m(c)$ of constant sectional curvature c, we have*

$$\delta(n_1,\ldots,n_k) \leq \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)}H^2 + \frac{1}{2}\left(n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\right)c, \quad (2.9)$$

for any k-tuple $(n_1,\ldots,n_k) \in S(n)$.

The equality case of inequality (2.9) holds at a point $p \in M$ if and only if, there exists an orthonormal basis $\{e_1,\ldots,e_m\}$ at p, such that the shape operators of M in $R^m(c)$ at p take the following forms:

$$A_r = \begin{bmatrix} A_1^r & \ldots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \ldots & A_k^r & \\ & \mathbf{0} & & \mu_r I \end{bmatrix}, \quad r = n+1,\ldots,m, \quad (2.10)$$

where I is an identity matrix and each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \mu_r, \quad r = n+1,\ldots,m. \quad (2.11)$$

Lemma 2.2. (see [4]) *For a given integer j with $2 \leq j \leq n-2$, if M is an $S(j)$-space, then it is an $S(j+1)$-space.*

Lemma 2.3. (see [4]) *An $S(n-2)$-space is a Riemannian space form.*

Theorem 2.4. (see [4]) *Let M be a Riemannian n-manifold and k an integer ≥ 2.*

(1) *If M is an $S(n_1,\ldots,n_k)$-space, then M is a Riemannian space form unless $n_1 = \cdots = n_k$ and $n_1+\cdots+n_k = n$.*

(2) *M is an $S(n_1,\ldots,n_k)$-space with $n_1 = \cdots = n_k$ and $n_1+\cdots+n_k = n$ if and only if M is a conformally flat space.*

Theorem 2.5. (see [4]) *Let $(n_1,\ldots,n_k)\in S(n)$ and M a Riemannian n-manifold isometrically immersed in a Riemannian space form $R^m(c)$ of constant sectional curvature c. Then M satisfies*

$$\widehat{\delta}(n_1,\ldots,n_k) = \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)}H^2+\frac{1}{2}\left(n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\right)c \quad (2.12)$$

identically if and only if M is a Riemannian space form and the immersion is totally umbilical.

3. The Immersions which Realize the Equality in (1.3)

In this section, we want to give the main results of the paper by the following theorems.

Theorem 3.1. *Let $(n_1,\ldots,n_k)\in S(n)$ and M a Riemannian n-manifold isometrically immersed in a Riemannian space form $R^m(c)$ of constant sectional curvature c. Then M satisfies (2.12) identically unless $n_1=\cdots=n_k$ and $n_1+\cdots+n_k=n$ for $k\geq 1$ if and only if M is a Riemannian space form of constant sectional curvature c and the immersion is minimal.*

Proof. Assume that M is a submanifold in a Riemannian space form $R^m(c)$ which satisfies equality (2.12) identically unless $n_1=\cdots=n_k$ and $n_1+\cdots+n_k=n$ for $k\geq 1$. Then, as it is stated in the proof of Theorem 2.5, by (1.1) and (1.2), it follows that

$$\delta(n_1,\ldots,n_k)=\widehat{\delta}(n_1,\ldots,n_k)$$

and

$$\tau-\{\tau(L_1)+\cdots+\tau(L_k)\} = \frac{n^2\left(n+k-1-\sum n_j\right)}{2\left(n+k-\sum n_j\right)}H^2 +\frac{1}{2}\left(n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\right)c, \quad (3.1)$$

for any mutually orthogonal k plane sections $L_1,\ldots,L_k\subset T_pM$, $p\in M$ with $\dim L_j=n_j$, $j=1,\ldots,k$. So M is an $S(n_1,\ldots,n_k)$-space.

Let $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_m\}$ be an orthonormal basis at a point p such that L_j is spanned by $e_{n_1+\cdots+n_{j-1}+1},\ldots,e_{n_1+\cdots+n_j}$ for $j=1,\ldots,k$ and e_{n+1} is in the direction of the mean curvature vector. From (3.1) and Theorem 2.1, we know that the shape operators of M at p take (2.10) and (2.11) forms.

Let take $k = 1$ and $2 \le n_1 \le n-2$. Because M satisfies equality (2.12) identically. Then from Lemma 2.2 and Lemma 2.3, M is a Riemannian space form. On the other hand, statement (1) of Theorem 2.4 also implies that M is a Riemannian space form. In this case, from (3.1), it is found that

$$K = \frac{\left(n+k-1-\sum n_j\right)\left(n+k-\sum n_j\right)\mu_{n+1}^2}{n\left(n-1\right)-\sum\limits_{j=1}^{k} n_j\left(n_j-1\right)} + c \tag{3.2}$$

and

$$\sum_{j=1}^{k} \tau\left(L_j\right) = \sum_{j=1}^{k} \frac{n_j\left(n_j-1\right)}{2} \frac{\left(n+k-1-\sum n_j\right)\left(n+k-\sum n_j\right)\mu_{n+1}^2}{n\left(n-1\right)-\sum\limits_{j=1}^{k} n_j\left(n_j-1\right)} + \sum_{j=1}^{k} \frac{n_j\left(n_j-1\right)}{2} c. \tag{3.3}$$

Furthermore, (2.6) and Gauss' equation yield

$$\sum_{j=1}^{k} \tau\left(L_j\right) = \sum_{j=1}^{k} \frac{n_j\left(n_j-1\right)}{2} c + \sum_{r=n+1}^{m} \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\alpha_j} h^r_{\beta_j\beta_j} - \left(h^r_{\alpha_j\beta_j}\right)^2\right), \quad \alpha_j, \beta_j \in \Delta_j, \quad j = 1, \ldots, k, \tag{3.4}$$

where $\Delta_1 = \{1, \ldots, n_1\}, \ldots, \triangle_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}$. Let take $h^{n+1}_{ii} = a_i$, for $1 \le i \le n$. Thus from (3.4) and Theorem 2.1, it follows that

$$\begin{aligned}\sum_{j=1}^{k} \tau\left(L_j\right) &= \sum_{j=1}^{k} \frac{n_j\left(n_j-1\right)}{2} c \\ &\quad + \frac{1}{2}\left(\sum_{j=1}^{k} \left(n_j-1\right) \sum_{\alpha_j \in \Delta_j} \left(a_{\alpha_j}\right)^2 - \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(a_{\alpha_j} - a_{\beta_j}\right)^2\right) \\ &\quad + \frac{1}{2} \sum_{r=n+2}^{m} \left(\sum_{j=1}^{k} \left(n_j-1\right) \sum_{\alpha_j \in \Delta_j} \left(h^r_{\alpha_j\alpha_j}\right)^2 - \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\alpha_j} - h^r_{\beta_j\beta_j}\right)^2\right)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{r=n+1}^{m} \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\beta_j}\right)^2 \\
&= \sum_{j=1}^{k} \frac{n_j\left(n_j-1\right)}{2} c + \frac{1}{2} \sum_{j=1}^{k} \frac{n_j-1}{n_j} \mu^2_{n+1} - \frac{1}{2} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{\alpha_j<\beta_j} \left(a_{\alpha_j} - a_{\beta_j}\right)^2 \\
&- \frac{1}{2} \sum_{r=n+2}^{m} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\alpha_j} - h^r_{\beta_j\beta_j}\right)^2 - \sum_{r=n+1}^{m} \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\beta_j}\right)^2 . \qquad (3.5)
\end{aligned}
$$

Hence, from the equality of (3.3) and (3.5), we can write that

$$
\begin{aligned}
&\left[\sum_{j=1}^{k} \frac{\left(n_j-1\right)}{2} \left(n_j \frac{\left(n+k-1-\sum n_j\right)\left(n+k-\sum n_j\right)}{n\left(n-1\right) - \sum\limits_{j=1}^{k} n_j\left(n_j-1\right)} - \frac{1}{n_j} \right)\right] \mu^2_{n+1} \\
&= -\frac{1}{2} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{\alpha_j<\beta_j} \left(a_{\alpha_j} - a_{\beta_j}\right)^2 - \frac{1}{2} \sum_{r=n+2}^{m} \sum_{j=1}^{k} \frac{1}{n_j} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\alpha_j} - h^r_{\beta_j\beta_j}\right)^2 \\
&\qquad - \sum_{r=n+1}^{m} \sum_{j=1}^{k} \sum_{\alpha_j<\beta_j} \left(h^r_{\alpha_j\beta_j}\right)^2 . \qquad (3.6)
\end{aligned}
$$

Taking into account (3.6), unless $n_1 = \cdots = n_k$ and $n_1 + \cdots + n_k = n$ for $k \geq 1$, it is obvious that $\mu_{n+1} = 0$ because of

$$
\sum_{j=1}^{k} \frac{\left(n_j-1\right)}{2} \left(n_j \frac{\left(n+k-1-\sum n_j\right)\left(n+k-\sum n_j\right)}{n\left(n-1\right) - \sum\limits_{j=1}^{k} n_j\left(n_j-1\right)} - \frac{1}{n_j} \right) > 0 \, .
$$

By this way from (3.6), it follows that

$$
\begin{gathered}
a_{\alpha_j} = a_{\beta_j};\ \alpha_j, \beta_j \in \Delta_j,\ j = 1, \ldots, k, \\
h^r_{\alpha_j\alpha_j} = h^r_{\beta_j\beta_j};\ \alpha_j, \beta_j \in \Delta_j,\ j = 1, \ldots, k,\ r = n+2, \ldots, m, \\
h^r_{\alpha_j\beta_j} = 0;\ \alpha_j, \beta_j \in \Delta_j,\ j = 1, \ldots, k,\ r = n+1, \ldots, m.
\end{gathered} \qquad (3.7)
$$

As a result it can be easily seen from (3.7) and Theorem 2.1 that $a_{\alpha_j} = a_{\beta_j} = 0$; $\alpha_j, \beta_j \in \Delta_j$, $j = 1, \ldots, k$ and $h^r_{\alpha_j\alpha_j} = h^r_{\beta_j\beta_j} = 0$; $\alpha_j, \beta_j \in \Delta_j$, $j = 1, \ldots, k$, $r = n+2, \ldots, m$. Moreover from the proof of Theorem 2.1, it is found that $h^r_{\alpha\beta} =$

0; $(\alpha, \beta) \notin \Delta^2$, $\alpha \neq \beta$, $r = n+1, \ldots, m$, where $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \cdots \cup (\Delta_k \times \Delta_k)$ and $a_{\alpha\alpha} = \mu_{n+1} = 0$; $\alpha \notin \Delta$, where $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$ and again $h^r_{\alpha\alpha} = 0$; $\alpha \notin \Delta$, $r = n+2, \ldots, m$. Hence M is a totally geodesic submanifold in $R^m(c)$. As a result M is minimal in $R^m(c)$. In addition from (3.2), M is a Riemannian space form of constant sectional curvature c. The converse is clear. □

By the similar considerations we can easily prove the following theorem.

Theorem 3.2. *Let $(n_1, \ldots, n_k) \in S(n)$ and M a Riemannian n-manifold isometrically immersed in a Riemannian space form $R^m(c)$ of constant sectional curvature c. Then M satisfies (2.12) identically and $\mu_{n+1} = 0$ if and only if M is a Riemannian space form of constant sectional curvature c and the immersion is minimal.*

Now, assume M is an n-dimensional $(n > 2)$ submanifold of a Riemannian manifold $R^m(c)$ of constant sectional curvature c.

Corollary 3.3. *Let N^2 be a 2-dimensional submanifold of an Euclidean $(m-n+2)$-space E^{m-n+2} and the product submanifold $M = N^2 \times E^{n-2}$ be isometrically immersed in an Euclidean m-space E^m. Then M satisfies $\tau - \sup K = \frac{n^2(n-2)}{2(n-1)} H^2$ identically if and only if M is a Riemannian space form of zero sectional curvature and the immersion is minimal.*

Corollary 3.3 follows from Theorem 3.1 easily.

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