International Journal of Pure and Applied Mathematics
Volume 37 No. 1 2007, 1-12

DISCRETE-TIME QUEUEING MODEL GEOM$^{[X]}$/G/1 WITH MULTIPLE VACATIONS AND SET-UP TIMES

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Abstract: The problem Geom$^{[X]}$/G/1, a queueing model of discrete time with multiple vacations and set-up times with batch arrival is considered. The generating functions of the steady state queue length, waiting time and their stochastic decomposition are derived. The busy period, the whole vacation period, the idle period and the on-line period of model are given.

AMS Subject Classification: 60K25, 68M20
Key Words: queueing model, stochastic decomposition, multiple vacations

1. Introduction

In recent years discrete-time queues have been used extensively to analyze high-speed computer communication networks. Their importance has increased due to the emergence of the broadband integrated service digital networks (B-ISDN), which can provide transfer of video, voice, and data communication through high-speed local area networks (LANs), on-demand video distribution, video telephony communications, etc. The asynchronous transfer mode (ATM) is envisaged as the basic transfer model for implementing B-ISDN. In these systems, all information flow is transmitted in fixed-size units called cells. For more details, see the books by Bruneel and Kim (1993) [1], Woodward (1994)

Received: September 3, 2006

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[18], and Onvural (1994) [8], as well as special issues of various journals, and conference proceedings such as performance evaluation Tran Gia et al (1995) [15], queueing systems Miyazawa and Takagi (1994) [7], and Perros et al (1993) [9].

The stochastic server system making use of the idle period to maintain service facilities or to do some auxiliary work is called vacation queueing. Because of its importance in the computer, communication network and management engineering, this model has been widespread concerned especially in recent years, see for reference papers some researching progress and references given by Doshi (1986) [4] and Tian (1994) [12]. About Geom/G/1 queue model with vacation, most works were concentrated to the study of models of multiple vacations, single vacation and set-up time Takagi [10] and Tian (see [13], [11], [14]). Zhang et al have adopted strategy of multiple adaptive vacations [19]. Ma et al [6], Wei, et al (see [16]-[17]) have considered Geom/G/1 with vacations and set-up time. Takagi [10], Lee [5] and Chaudhy [2] have considered the batch arrival of Geom/G/1. Based on that, we give a model which we will study in this paper. Then we generalize the results of [6]. The models given in [10] and [19] are special cases of this paper.

Then rest of the paper is organized as follows. In the next section we describe the model and give its embedded Markov chain. In Section 3, we discuss some performance measures of model, including stochastic decomposition of steady-state length and waiting time, the whole vacation, busy period, busy period cycle and the on-line period. Two special cases are given in Section 4. Conclusion of the paper is stated in Section 5.

2. Describing Model and Embedded Markov Chain

In Geom$^{[X]}$/G/1 system, we denote by Λ the number of messages that arrive during a single slot. The probability distribution and probability generating function (PGF) of Λ are specified, respectively by

$$\lambda(k) = p(\Lambda = k), k = 0, 1, 2, \cdots ; \quad \Lambda(z) = \sum_{k=0}^{\infty} \lambda(k) z^k.$$

We denote by λ and $\lambda^{(i)}$ the mean and the i-th factorial moment, respectively, of Λ, that is

$$\lambda = E[\Lambda] = \Lambda^{(1)}(1),\ \lambda^{(i)} = E[\Lambda(\Lambda - 1) \cdots (\Lambda - i + 1)] = \Lambda^{(i)}(1),\ i = 2, \cdots .$$

In the Geom$^{[X]}$/G/1 queueing model with multiple vacation and set-up times with exhaustive service, the server begins a vacation if there are no messages in the system at the end of a service. If the server returns from a vacation to find the system nonempty, it starts to work immediately and continues to work until the system becomes empty again. If the server returns from a vacation to find no messages waiting in the queue, it begins another vacation immediately, and repeats vacations in this manner until it finds at least one waiting message upon returning from a vacation. If the server finds at least one waiting message upon returning from a vacation, then the server immediately reboot service system. The length V of each vacation (measured in slots) is assumed to be an integral multiple of a slot duration, and an independent and identically distributed random variable. The PGF for V is denoted by $V(u)$. Thus we have

$$v(l) = p(V = l), \quad l = 1, 2, \cdots ; \quad V(u) = \sum_{l=1}^{\infty} v(l)u^l, \quad |u| \leq 1.$$

For the classical Geom$^{[X]}$/G/1 queueing, we introduce above vacation strategy. Arrival interval time T obey parameter $1 - \lambda(0)$ $(0 < \lambda(0) < 1)$ of geometric distribution. If we denote by X the service time (measured in slots) of each message. Each service is started and completed at exact slot boundaries. The probability distribution and the PGF of X are specified, respectively by

$$b(j) = p(X = j),\ j = 1, 2, \cdots ; \quad B(u) = \sum_{j=1}^{\infty} b(j)u^j,\ |u| \leq 1.$$

The mean and the i-th moment of the service time distribution are denoted by b and $\lambda^{(i)}, i = 2, \cdots$, respectively

$$b = E[X] = B^{(1)}(1), \quad b^{(i)} = E[X^{(i)}].$$

If we denote by Y the set-up time (measured in slots) of each time of the system. Each service is started and completed at exact slot boundaries. The probability distribution and the PGF of Y are specified, respectively by

$$u(j) = p(Y = j),\ j = 1, 2, \cdots ; \text{ and } U(z) = \sum_{j=1}^{\infty} u(j)z^j, |z| \leq 1.$$

We assume throughout this paper that arrival interval, service time, vacation time and set-up time are mutual independence and obey first-come and first-served (FCFS) service discipline (in the late arrival model).

Let L_n be the number of customers in the present system (queue size) immediately after the service completion of the n-th customer, where $n = 0, 1, \cdots$. If A_n denotes the number of messages that arrive during the service completion of the n-th message, and α denotes the number of messages at the beginning of the busy period in this system. Then $\{L_n, n \geq 1\}$ is an embedded chain of discrete time queue process. It is obvious

$$L_{n+1} = \begin{cases} L_n - 1 + A_{n+1}, & L_n \geq 1, \\ \alpha + A_{n+1} - 1, & L_n = 0. \end{cases}$$

By Foster rule, we can prove that the Markov chain $\{L_n, n \geq 1\}$ is positive recurrence if and only if $\rho = \lambda b < 1$.

Let $C_j^{(v)}$ and $C_j^{(u)}$ denote probability distribution that arrive j messages in a vacation and setup time, respectively. Since the number of messages that arrive in each complete service time is independent and mutual distribution. Then we have

$$k_j = p(A = j), j \geq 0; \quad A(z) = \sum_{j=0}^{\infty} k_j z^j = B[\Lambda(z)].$$

From the above definitions, considering the condition that there are any messages or not in the vacation, we have

$$b_j = p(\alpha + A - 1 = j) = \sum_{i=1}^{j} C_i^{(v)} \sum_{r=0}^{j+1-i} C_r^{(u)} k_{j+1-i-r}, \quad j \geq 0.$$

3. Analysis of Performance Measures of System

If $\rho < 1$, system is positive recurrence, we denote

$$\beta = E[U] + \frac{E[V]}{1 - v(\lambda(0))}.$$

Theorem 1. *If $\rho < 1$, the steady-state queue length Π can be decomposed into the sum of two stochastic variables, i.e., $\Pi = \Pi_0 + \Pi_d$, where Π_0 denotes the steady-state length of classical Geom$^{[X]}$/G/1 model whose PGF has been given in [10]. Then*

$$\Pi_d(z) = \frac{1 - \frac{1}{1-v(\lambda(0))} U[\Lambda(z)][V[\Lambda(z)] - v(\lambda(0))]}{\beta(1 - \Lambda(z))} \tag{1}$$

is the PGF of additional length Π_d.

Proof. The steady-state distribution $\{\pi_k, k \geq 0\}$ satisfies $\Pi\tilde{P} = \Pi$,

$$\pi_j = \pi_0 b_j + \sum_{i=1}^{j+1} \pi_i k_{j+1-i}, \quad j \geq 0,$$

where $\Pi = (\pi_0, \pi_1, \cdots)$. Taking PGF of above equation, we have

$$\Pi(z) \quad = \pi_0 R(z) + \sum_{j=0}^{\infty}\sum_{i=1}^{j+1} \pi_i k_{j+1-i} z^j = \pi_0 R(z) + \frac{1}{z} A(z)[\Pi(z) - \pi_0], \qquad (2)$$

where

$$R(z) = \sum_{j=0}^{\infty} z^j b_j = \frac{1}{z}\{v(\lambda(0))\frac{\Lambda(z) - \lambda(0)}{1 - \lambda(0)} U[\Lambda(z)] + (V[\Lambda(z)] - v(\lambda(0)))\} B[\Lambda(z)].$$

Substituting $R(z)$ and $A(z)$ into (2), we get

$$\Pi(z) = \pi_0 B[\Lambda(z)] \times \{\frac{1 - v(\lambda(0))\frac{\Lambda(z)-\lambda(0)}{1-\lambda(0)} U[\Lambda(z)] - (V[\Lambda(z)] - v(\lambda(0)))}{B[\Lambda(z)] - z}\}. \quad (3)$$

According to the normalization condition $L_v(1) = 1$ and the $L'Hospital$ rule, we get $\pi_0 = (1-\rho)\beta^{-1}$. Substituting it into (3), we have

$$\begin{aligned} \Pi(z) \quad &= \frac{(1-\rho)(1-z)B[\Lambda(z)]}{B[\Lambda(z)] - z} \\ &\times\{\frac{1 - v(\lambda(0))\frac{\Lambda(z)-\lambda(0)}{1-\lambda(0)} U[\Lambda(z)] - (V[\Lambda(z)] - v(\lambda(0)))}{\beta(1 - \Lambda(z))}\} \qquad (4) \\ &= \Pi_0(z) \times \Pi_d(z). \end{aligned}$$

Thus, from (4), we obtain

$$E(\Pi_d) = \frac{\frac{2\lambda E[U]E[V]}{1-v(\lambda(0))} + \lambda E[U(U-1)] + \lambda E[V(V-1)]}{2\beta},$$

$$E(\Pi_v) = \rho + \frac{\lambda^2 b^{(2)} - \lambda\rho + \lambda^{(2)} b}{2(1-\rho)} + E(\Pi_d).$$

The proof is completed. □

Theorem 2. *If $\rho < 1$, the steady-state's waiting-time W_v can be decomposed into the sum of two stochastic variables, i.e., $W_v = W_0 + W_d$, where W denotes the steady-state's waiting-time of classical Geom$^{[X]}$/G/1 model whose PGF has been given in [10]. Then, one has that*

$$W_d(z) = \frac{1 - \frac{1}{1-v(\lambda(0))}U(u)[V(u) - v(\lambda(0))]}{\beta(1-u)}, \tag{6}$$

it is the PGF of additional delay-time W_d.

Proof. Assuming that a group of messages that arrive in the same slot constitute a supermessage in a Geom/G/1 system. That is, the PGF $\Lambda_g(z)$ and the mean λ_g for the number of supermessages that arrive in a slot in this Geom/G/1 system are respectively given by

$$\Lambda_g(z) = \lambda(0) + [1 - \lambda(0)]z, \quad \lambda_g = 1 - \lambda(0). \tag{7}$$

The PGF $B_g(u)$ for the service time of a supermessage is given by

$$B_g(u) = \frac{\Lambda[B(u)] - \lambda(0)}{1 - \lambda(0)}. \tag{8}$$

From (4) the PGF for the number of supermessages present in the corresponding Geom$^{[X]}$/G/1 system at the end of the service to a supermessage is given by

$$\begin{aligned}\Pi_g(z) &= \frac{(1-\rho)(1-z)B_g[\Lambda_g(z)]}{B_g[\Lambda_g(z)] - z} \\ &\times \{\frac{1 - \frac{1}{1-v(\lambda(0))}U[\Lambda_g(z)](V[\Lambda_g(z)] - v(\lambda(0)))}{\beta(1-\Lambda_g(z))}\}.\end{aligned} \tag{9}$$

In the Geom/G/1 system of supermessages with FCFS discipline, the number of supermessages present in the system at the end of the service to a supermessage equals the number of supermessages that arrive in the time interval during which that supermessage was in the system. Therefore, of $W_g(u)$ denotes the PGF for the waiting time of a supermessage, we have the relation

$$\begin{aligned}\Pi_g(z) &= W_g(\Lambda_g(z))B_g[\Lambda_g(z)] \\ &= \frac{(1-\rho)(1-z)B_g[\Lambda_g(z)]}{B_g[\Lambda_g(z)] - z}\{\frac{1 - \frac{U[\Lambda_g(z)](V[\Lambda_g(z)] - v(\lambda(0)))}{1-v(\lambda(0))}}{\beta(1-\Lambda_g(z))}\}.\end{aligned} \tag{10}$$

From (9) and (10), we get

$$W_g(u) = \frac{(1-\rho)(1-u)}{\Lambda(B(u))-u} \times \{\frac{1-\frac{1}{1-v(\lambda(0))}U(u)[V(u)-v(\lambda(0))]}{\beta(1-u)}\}$$
$$= \frac{(1-\rho)(1-u)}{\Lambda(B(u))-u} \times W_d(u). \quad (11)$$

where we have used (7) and (8). The first factor on the r.h.s. of (11) is the PGF for the waiting time of a supermessage in the system without vacations (see [10], p. 17) and the second factor is PGF $W_g(u)$ for the residual vacation time of the system.

The waiting time W_v of an arbitrary message consists of two independent components. One is the waiting time W_g of a supermessage to which the arbitrary message belongs the other, denoted by J, is the sum of the service times for those messages within the same supermessage that are served before the arbitrary message. Note that these components are independent. If $J(u)$ denotes the PGF for J, then the PGF $W(u)$ for the waiting time of an arbitrary message in the original Geom$^{[X]}$ G/1 system is given by

$$W_v(u) = W_g(u)J(u). \quad (12)$$

In order to get $J(u)$, we note that the number of messages within the supermessage that are served before the arbitrary message is equivalent to the forward recurrence time in a discrete-time renewal time is given by the number of messages included in the supermessage (see [3], Section 2.1 for more details about $J(u)$' argument). Hence we get

$$J(u) = \frac{1-\Lambda[B(u)]}{\lambda(1-B(u))}. \quad (13)$$

From (11), (12) and (13), we have

$$W_v(u) = \{\frac{1-\frac{1}{1-v(\lambda(0))}U(u)[V(u)-v(\lambda(0))]}{\beta(1-u)}\}$$
$$\times \frac{(1-\rho)(1-u)(1-\Lambda[B(u)])}{\lambda(1-B(u))(\Lambda[B(u)]-u)} = W_d(u)W_0(u). \quad (14)$$

From (14), we have

$$E(W_d) = \frac{\frac{2E[U]E[V]}{1-v(\lambda(0))} + E[U(U-1)] + E[V(V-1)]}{2\beta},$$
$$E(W_v) = \frac{\lambda^2 b^{(2)} - \lambda\rho + \lambda^{(2)} b}{2\lambda(1-\rho)} + E(W_d).$$

The proof is completed. □

Comparing Theorem 1 with Theorem 2, we have that $E[L_d] = \lambda[W_d]$, i.e., the well-known Little's formula also holds.

We denote by B_v a busy period whose length (measured in slots), it is defined as a time interval that is started at the end of vacation and terminated at the beginning of the next vacation. If there are no messages in the present system at the end of a vacation, the following busy period is zero. Let Q_b denote messages numbers which busy period starts in the system, then we have

$$P(Q_b = j) = \sum_{i=1}^{j} C_i^{(v)} C_{j-i}^{(u)}, \quad j \geq 1. \tag{15}$$

This is the probability distribution of Q_b.

Because arrival internal distribution is memoryless, when $Q_b = k$, the busy-length is $\tilde{B}^{(k)}$, namely $B_v = \widetilde{B}^{(k)}, k \geq 1$. And $\tilde{B}$ denotes busy period in the classical Geom$^{[X]}$/G/1 model, whose the mean is $E[\tilde{B}] = b(1-\rho)^{(-1)}$ (see [10], p. 38). Thus we obtain

$$\begin{aligned} B_v(z) = E(z^{B_v}) = \sum_{j=1}^{\infty} P(Q_b = j)[\widetilde{B}(z)]^j \\ = \frac{U(\Lambda[\widetilde{B}(z)])}{1 - v(\lambda(0))} V(\Lambda[\widetilde{B}(z)]) - v(\lambda(0)), \end{aligned} \tag{16}$$

where $\widetilde{B}(z)$ denotes the PGF of busy period in the classical Geom$^{[X]}$/G/1 model; from (16), we have

$$E(B_v) = \lambda\beta E(\widetilde{B}) = \frac{\rho\beta}{1-\rho}.$$

$E(B_v)$ denotes the mean of B_v.

We denote by V_G a whole vacation whose length (measured in slots). It is defined as a time interval that starts at the beginning of vacations which service period of system has completed and terminates at the end of the vacations, and the server immediately enter set-up time period or idle period. Let V_G denote that it consists of successive J vacations, Then

$$J_j = p(J = j) = p(V^{(j-1)} < T \leq V^{(j)}), \quad p(J \geq j) = [v(\bar{p})]^{j-1}, \ \ j \geq 1,$$

$$\sum_{j=1}^{\infty} P(J \geq j) z^j = \frac{z(1 - J(z))}{1 - z} \quad \Longrightarrow \quad J(z) = 1 - \frac{1 - z}{1 - zv(\lambda(0))}.$$

Therefore, we have

$$V_G(z) = E[z^{V_G}] = \sum_{j=1}^{\infty} E[z^{V_G}|J=j]p(J=j)$$
$$= J(V(z)) = 1 - \frac{1-V(z)}{1-v(\lambda(0))V(z)}.$$

Therefore, we have too,

$$E(V_G) = \frac{E(V)}{1-v(\lambda(0))}.$$

We denote by R a busy period cycle whose length (measured in slots). It is defined as a time interval starting at the end of busy period and terminating at the beginning of the next busy period. Then one busy period cycle consists of the set-up period, the busy period and the vacation period. Thus, we get

$$E(R) = E(V) + E(B_v) + E(U) = \frac{\beta}{(1-\rho)}.$$

Now, under steady-state condition, let p_B, p_V and p_U denote the probability that all the system enter the busy period, the set-up period and the vacation period in an arbitrary slot, respectively. Applying Renewal Reward Theorem, we obtain

$$p_B = \rho; \quad p_V = \frac{E[V](1-\rho)}{\beta(1-v(\lambda(0)))}; \quad p_{\mathrm{U}} = \frac{(1-\rho)E[U]}{\beta}.$$

Vacation can be regarded as the time server leave the system, but in the usual idle period, server still need to stay at post to receive the newly arriving messages. We denote by T_s on-line period whose length (measured in slots). It is defined as a time interval that is started at the end of one vacation and terminated at the beginning of the next vacation. Then the on-line period is consists of the set-up period and the busy period. Let Q_v denote the number of the messages when the set-up time period terminates in the system (its probability distribution is given by equation (15)). Thus, we obtain PGF of the on-line period T_s

$$T_s(z) = E[z^{T_s}] = E[z^U] \sum_{j=1}^{\infty} [\tilde{B}(z)]^j \sum_{i=1}^{j} C_i^{(v)} C_{j-i}^{(u)}$$
$$= \frac{U(z)}{1-v(\lambda(0))} U[\Lambda(\tilde{B}(z))]\{V[\Lambda(\tilde{B}(z))] - v(\lambda(0))\}, \quad (17)$$

where

$$E[z^U] = \sum_{j=0}^{\infty} u_j z^j = U(z).$$

From (17), we obtain

$$E[T_s] = T_s(z)'|_{z=1} = E[U] + \beta \frac{E[S]}{1-\rho} = E[U] + E[B]. \tag{18}$$

4. Two Special Cases

Case 1. The Geom$^{[X]}$/G/1 queue with multiple vacation.

When the set-up period is zero in the Geom$^{[X]}$/G/1 queue with multiple vacation and set-up times, the system becomes Geom$^{[X]}$/G/1 queue with multiple vacation. Therefore, we have

$$L_d(z) \quad = \frac{1 - V[\Lambda(z)]}{E[V](1-\Lambda(z))}, \quad W_d(u) = \frac{1-V(u)}{E[V](1-u)}.$$

Their detailed proofs can be seen in [10].

Case 2. The Geom$^{[X]}$/G/1 queue with set-up times.

When the vacation period is zero in the Geom$^{[X]}$/G/1 queue with single vacation and set-up times, the system becomes Geom$^{[X]}$/G/1 queue with set-up times. Therefore, we have

$$L_d(z) = \frac{1-\lambda(0) - [\Lambda(z) - \lambda(0)]U[\Lambda(z)]}{\beta(1-\Lambda(z))},$$
$$W_d(u) = \frac{1-\lambda(0) - (u-\lambda(0))U(u)}{\beta(1-u)},$$

where

$$\beta = 1 + (1-\lambda(0))E[U].$$

Their detailed proofs can be seen in [10].

5. Conclusion

By conditional probability and generating function, we derive the analytic expressions of steady-state queue length and waiting time distribution and their expectation. At the same time, we give the expression of additional length and delayed waiting time, and the analytic expression of busy period and on-line period. We also prove their stochastic decomposition results. So the results due to Takagi [10] are generalized. This model can be applied to all kinds of up-to-date technical fields, such as computer communication network, flexibility manufacture system (FMS), asynchronously transfer mode (ATM), electronic commerce (EC) and supply chain, etc.

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