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INITIAL-BOUNDARY VALUE PROBLEM FOR A CLASS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: For a class of integro-differential equation with nonlinear damped and memory terms arising from the models of nonlinear viscoelasticity, we consider the existence of a weak solution to the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial u}{\partial t}) + f(u) + h(\frac{\partial u}{\partial t}) + \int_0^t g(t-\tau)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u(\tau)d\tau = 0, \quad (x,y;t) \in \Omega \times (0,\infty),$$

where Ω is bounded domain of R^2, and $f,\ g$ are power like functions. By virtue of the Galerkin method combined with the a priori estimate, it is proved that under rather mild conditions on nonlinear terms and initial data the above-mentionded problem admits a weak solution.

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1. Introduction

This paper is devoted to the study of existence of weak solution u of the nonlinear damped plate equation by Galerkin method and by integral estimate

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$$\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial u}{\partial t}) + \int_0^t g(t-\tau)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u(\tau)d\tau$$

$$+f(u) + h(\frac{\partial u}{\partial t}) = 0 \quad (x,y;t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad (x,y) \in \Omega\,, \tag{1.2}$$

where $\Omega(= (0,1) \times (0,1))$ is bounded domain of R^2. If Ω has a nonempty boundary Γ, then it will be assumed regular only if we are interested in $u = \frac{\partial u}{\partial v} = 0$ on Γ (here $\frac{\partial u}{\partial v}$ represents the normal derivative). Otherwise, no regularity is required upon Γ. $u(x,t)$ represents displacement, and, g represents relaxation function. Integral term in model of function includes the influence of memory term.

For the sake of readability we assume that

$$f(s) = \gamma |s|^\xi s\,, \quad h(s) = \beta |s|^\rho s\,, \tag{1.3}$$

where ξ, ρ, γ and β are positive constants.

Next we make a few remarks about earlyer works for some problems related to (1.1). When $g = 0$ and Ω is a bounded domain of R^n with smooth boundary, problem (1.1) was studied by several authors. It is worth mentioning a paper by Woinowsky-Krieger [7] which arises in the dynamic buckling of a hinged extensible beam subject to an axial force, see also Dickey [3] and Eisley [4]. When $f(s) \neq 0, h(s) \neq 0, g \neq 0$ and Ω is a bounded domain with smooth boundary, [1], [2], [5] proved, in the framework of nonlinear viscoelasticity, the exponential decay by assuming that the kernel of the memory decays exponentially.

2. Notation and Statement of Results

Let us introduce some notations that will be used throughout this work. In the standard $L^2(\Omega)$ space we write $\|u\|_p$ is the norm in $L^P(\Omega)(1 \leq p \leq \infty)$, and

$$(u,v) = \int_\Omega u(x)v(x)dx, \quad |u|^2 = \int_\Omega |u(x)|^2 dx\,.$$

We define the subspace of $H^1(\Omega)$, and denoted by $H_0^1(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^1(\Omega)$. Especially,

$$(((u_1,v_1),(u_2,v_2))) = \iint\limits_\Omega (u_1u_2 + v_1v_2)dxdy, \ \|(u,v)\|^2 = \iint\limits_\Omega (u^2+v^2)dxdy.$$

Based on the above properties we present the hypotheses for the main results.

Assumptions on the Kernel. We assume that $g : R^+ \to R^+$ is bounded C^1 function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(\tau) d\tau = l > 0\,, \tag{2.1}$$

and such that there exist positive constants ξ_1 and ξ_2 satisfying

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad t > 0\,. \tag{2.2}$$

Theorem 1. *Let the initial data $\{u_0, u_1\}$ belong to $H_0^1(\Omega) \times L^2(\Omega)$, and assume that assumptions (2.1)-(2.2) hold. Then, problem (1.1)-(1.3) possesses a weak solution u in the class*

$$u \in L^\infty([0,\infty); H_0^1(\Omega)), \quad \frac{\partial u}{\partial t} \in L^\infty([0,\infty); L^2(\Omega))\,.$$

3. Existence of Weak Solution

Proof. Step 1. Let $\{\omega_\nu\}$ be a basis in $H_0^1(\Omega)$, and let us consider V_m the space generated by $\omega_1, \cdots, \omega_m$. Let

$$u_m(t) = \sum_{j=1}^m \delta_{jm}(t)\omega_j$$

be the solution of the approximate Cauchy problem

$$\begin{aligned}
(\frac{\partial^2 u_m}{\partial t^2}, \nu) &+ (((\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y}), (\frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y}))) + (((\frac{\partial^2 u_m}{\partial t \partial x}, \frac{\partial^2 u_m}{\partial t \partial y}), (\frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y}))) \\
&- \int_0^t g(t-\tau)(((\frac{\partial u_m}{\partial x}(\tau), \frac{\partial u_m}{\partial y}(\tau)), (\frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y})))d\tau \\
&+ \gamma(|u_m|^\xi u_m, \nu) + \beta(|\frac{\partial u_m}{\partial t}|^\rho \frac{\partial u_m}{\partial t}, \nu) = 0\,, \quad \nu \in H_0^1(\Omega)\,, \qquad (3.1)
\end{aligned}$$

$$u_m(0) = u_{0m} \to u_0 \quad in\ H_0^1(\Omega), \quad \frac{\partial u_m}{\partial t}(0) = u_{1m} \to u_1 \text{ in } L^2(\Omega)\,. \tag{3.2}$$

By standard methods in differential equations, we prove the existence of solution to the approximate problem on some interval $[0, t_m)$ and this solution can be extended to the closed interval [0, T] by using the estimate below.

Step 2. The Estimate. Setting $\nu = \frac{\partial u_m}{\partial t}(t)$ in (3.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}\{\frac{1}{2}|\frac{\partial u_m}{\partial t}|^2 + \frac{1}{2}|u_m|^2 + \frac{1}{2}\|(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})\|^2 + \frac{\gamma}{\xi+2}|u_m|^{\xi+2}\} \\
&\qquad + \|(\frac{\partial^2 u_m}{\partial t\partial x}, \frac{\partial^2 u_m}{\partial t\partial y})\|^2 + \beta|\frac{\partial u_m}{\partial t}|^{\rho+2} \\
&= \frac{d}{dt}\int_0^t g(t-\tau)(((\frac{\partial u_m}{\partial x}(\tau), \frac{\partial u_m}{\partial y}(\tau)), (\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})))d\tau \\
&\quad - \int_0^t g'(t-\tau)(((\frac{\partial u_m}{\partial x}(\tau), \frac{\partial u_m}{\partial y}(\tau)), (\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})))d\tau \\
&\quad + (u_m, \frac{\partial u_m}{\partial t}) - g(0)\|(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})\|^2 . \qquad (3.3)
\end{aligned}$$

Integrating (3.3) over (0, t), taking (3.2) and (2.1) in account and using the inequality $ab \le \frac{1}{4\eta}a^2 + \eta b^2$, where η is an arbitrary positive number, we deduce

$$\begin{aligned}
&\frac{1}{2}|\frac{\partial u_m}{\partial t}|^2 + \frac{1}{2}|u_m|^2 + \frac{1}{2}\|(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})\|^2 + \frac{\gamma}{\xi+2}|u_m|^{\xi+2} \\
&\quad + \int_0^t \|(\frac{\partial^2 u_m}{\partial t\partial x}(s), \frac{\partial^2 u_m}{\partial t\partial y}(s))\|^2 ds + \beta\int_0^t |\frac{\partial u_m}{\partial t}(s)|^{\rho+2} ds \\
&\quad \le k_1 + \frac{1}{2}\int_0^t |u_m(s)|^2 ds + \frac{1}{2}\int_0^t |\frac{\partial u_m}{\partial t}(s)|^2 ds \\
&\quad + \frac{1}{2}\int_0^t \|(\frac{\partial u_m}{\partial x}(s), \frac{\partial u_m}{\partial y}(s))\|^2 ds + \eta\|(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})\|^2 \\
&\quad + \frac{\xi_1^2}{2}\|g\|^2_{L^1(0,\infty)}\int_0^t \|(\frac{\partial u_m}{\partial x}(s), \frac{\partial u_m}{\partial y}(s))\|^2 ds \\
&\quad + \frac{1}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\int_0^t \|(\frac{\partial u_m}{\partial x}(s), \frac{\partial u_m}{\partial y}(s))\|^2 ds ,
\end{aligned}$$

where k_1 is a positive constant. Employing Gronwall's Lemma and choosing $\eta > 0$ sufficiently small, from the last inequality we obtain the estimate

$$\begin{aligned}
&|\frac{\partial u_m}{\partial t}|^2 + |u_m|^2 + \|(\frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y})\|^2 + |u_m|^{\xi+2} \\
&\quad + \int_0^t \|(\frac{\partial^2 u_m}{\partial t\partial x}(s), \frac{\partial^2 u_m}{\partial t\partial y}(s))\|^2 ds + \beta\int_0^t |\frac{\partial u_m}{\partial t}(s)|^{\rho+2} ds \le L_1 , \qquad (3.4)
\end{aligned}$$

where L_1 is a positive constant independent of $m \in N$ and $t \in [0,T]$.

Step 3. From the above estimate we infer

$$||\frac{\partial u_m}{\partial t}|^\rho \frac{\partial u_m}{\partial t}|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \iint_\Omega \frac{\partial u_m}{\partial t}|^{2(\rho+1)} dxdydt \leq C\,, \tag{3.5}$$

where C is a positive constant independent of $m \in N$. Consequently, on the one hand, we can extract a subsequence (u_μ) of (u_m). And, we can deduce that

$$u_\mu \to u \text{ weak}^* \text{ in } L^\infty(0,T;H_0^1(\Omega))\,, \tag{3.6}$$

$$\frac{\partial u_\mu}{\partial t} \to \frac{\partial u}{\partial t} \text{ weak}^* \text{ in} L^\infty(0,T;L^2(\Omega))\,. \tag{3.7}$$

On the other hand, we can also extract a subsequence (u_ν) of (u_μ) such that, by the Aubin-Lions Theorem, we have

$$\frac{\partial u_\nu}{\partial t} \to \frac{\partial u}{\partial t} \text{ strongly in } L^2(0,T;L^2(\Omega))\,.$$

Then

$$|\frac{\partial u_\nu}{\partial t}|^\rho \frac{\partial u_\nu}{\partial t} \to |\frac{\partial u}{\partial t}|^\rho \frac{\partial u}{\partial t} \text{ a.e. in } \Omega \times (0,T)\,. \tag{3.8}$$

From $(3.5), (3.8)$ and thanks to Lions Lemma, we deduce

$$|\frac{\partial u_\nu}{\partial t}|^\rho \frac{\partial u_\nu}{\partial t} \to |\frac{\partial u}{\partial t}|^\rho \frac{\partial u}{\partial t} \text{ weak}^* \text{ in } L^2(0,T;L^2(\Omega)) \tag{3.9}$$

Analogously we prove that

$$|u_\nu|^\xi u_\nu \to |u|^\xi u \text{ weak}^* \text{ in } L^2(0,T;L^2(\Omega))\,. \tag{3.10}$$

So, using standard arguments we can pass to the limit in (3.1) in order to obtain the weak solution. □

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