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THE FUZZY SHEAF OF THE GROUPS FORMED BY FUZZY TOPOLOGICAL GROUPS OVER POINTED FUZZY TOPOLOGICAL SPACE

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Abstract: The purpose of this paper to consider both fuzzy homotopy and fuzzy sheaf theory and to construct an algebraic fuzzy sheaf by means of the fuzzy topological group.

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1. Introduction

In his paper [20] of 1965, Zadeh introduced the notion of fuzzy sets and fuzzy set operations. Subsequently, Chang [2], Wong [18], Lowen [16] and others applied some basic consepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. In an analogous application with groups, Rosenfeld [17] formulated the elements of a theory of fuzzy groups. Foster [8] brought together the structure of a fuzzy topological space and that of a fuzzy group to form a combined structure, that of a fuzzy topological group. In the following years, Liang and Hai [12] changed the definition of a fuzzy topological group in

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order to make sure that an ordinary topological group. Also in [12, 14, 13], they characterized fuzzy topological groups, fuzzy normal subgroups, fuzzy quotient groups, the direct group of finite family of fuzzy topological groups. Chon [5], Jha and Singh [10, 11] and other developed fuzzy topological groups consepts. Changyou ve Wang Jin [3, 4], Chuanlin [7] defined consept of fuzzy fundamental group and fuzzy homotopy in fuzzy topological spaces. Besides they studied fuzzy topological and homotopic invariance of fuzzy fundamental groups. Yıldız [19] considered both homotopy and sheaf theory and constructed an algebraic sheaf by means of the topological groups and he gave some algebraic topological characterizations.

In this paper we consider both fuzzy homotopy and fuzzy sheaf theory and construct an algebraic fuzzy sheaf by means of the fuzzy topological group.

2. Preliminaries

Definition 1. Let X be non-empty set and $I = [0,1]$ unit interval, we denoted all functions from X to I by I^X. A Fuzzy set in X is every element of I^X.

Definition 2. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow I$ which associates with each point $x \in X$ and it is denoted by $A = \{(x, \mu_A(x)) : x \in X\} \subset X \times I$.

Definition 3. Let X be non empty set and $\lambda \in (0,1]$. If $P_{x_0}^{\lambda} : X \longrightarrow I$,

$$P_{x_0}^{\lambda}(x) = \begin{cases} \lambda\,, & x = x_0\,, \\ 0\,, & x \neq x_0\,, \end{cases}$$

then $P_{x_0}^{\lambda}$ is called a fuzzy point on X. Here x_0 is called support of $P_{x_0}^{\lambda}$ and λ is called value of $P_{x_0}^{\lambda}$.

Definition 4. A *fuzzy topology* on a set X is a family τ of fuzzy sets in X which satisfies following conditions and the pair (X, τ) is called a fuzzy topological space.

(i) $\emptyset, X \in \tau$.

(ii) If $A, B \in \tau$ then $A \wedge B \in \tau$.

(iii) If $A_j \in \tau$ for all $j \in J$, then $\bigvee_{j \in I} A_j \in \tau$.

Definition 5. Let (X, τ),(Y, τ') are fuzzy topological spaces. A function f of (X, τ) into (Y, τ') is fuzzy continuous iff for each open fuzzy set W in τ' the inverse image $f^{-1}(W)$ is in τ. Conversely, f is fuzzy open iff for each open fuzzy set V in τ the image $f(V)$ is in τ'.

Definition 6. Let (X,τ) fuzzy topological space and a fuzzy set A in a fuzzy topological space (X,τ) is called a *neighbourhood* of a fuzzy point P_x^r iff there exists $V \in \tau$ such that $P_x^r \in V \leq A$.

Definition 7. Let A, B are fuzzy sets and any fuzzy point P_x^r. A fuzzy point P_x^r is said to be quasi-coincident with A, denoted by $P_x^r qA$, iff $r+A(x) > 1$. A fuzzy set A is said to be quasi-coincident with B, denoted by AqB, if there exists $x \in X$ such that $A(x)+B(x) > 1$.

Definition 8. Let (X,τ) fuzzy topological space and a fuzzy set A in a fuzzy topological space (X,τ) is called $Q-neighbourhood$ of P_x^r iff there exists $B \in \tau$ such that $P_x^r qB$ and $B \leq A$.

Definition 9. Let X be a group and G a fuzzy set in X with membership function μ_G. Then G is a *fuzzy group* in X iff the following conditions are satisfied;

(i) $\mu_G(xy) \geq Min\{\mu_G(x), \mu_G(y)\}$, $\forall x, y \in X$.

(ii) $\mu_G(x^{-1}) \geq \mu_G(x)$, $\forall x \in X$.

3. Fuzzy Homotopy

Definition 10. (see [6]) Let (X,τ), (Y,τ') be fuzzy topological spaces and (R,τ^*) usually topological space, $(I,\widetilde{\varepsilon})$ fuzzy topological space induced by (I,τ_I^*) topological space and $f,g : (X,\tau) \longrightarrow (Y\ ,\tau')$ are fuzzy continuous functions. If there exists a fuzzy continuous function $F : (X,\tau) \times (I,\widetilde{\varepsilon}) \longrightarrow (Y,\tau')$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$ then F is called *homotopy* from f to g and denoted by $f \overset{F}{\sim} g$.

Definition 11. Let (X,τ), (Y,τ') be fuzzy topological spaces and (R,τ^*) usually topological space, $(I,\widetilde{\varepsilon})$ fuzzy topological space induced by (I,τ_I^*) topological space and $f,g : (X,\tau) \longrightarrow (Y\ ,\tau')$ are fuzzy continuous functions. If there exists a fuzzy continuous function $F : (X,\tau) \times (I,\widetilde{\varepsilon}) \longrightarrow (Y,\tau')$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$ and $F(x,t) = f(x) = g(x)$ for every $x \in X_0 \subseteq X$ then f and g are called *homotopic relative to* X_0 and denoted by $f \overset{F}{\sim} g\ (rel\ X_0)$

Theorem 1. (see [6]) *Let (X,τ), (Y,τ') are fuzzy topological spaces. $\sim (rel\ X_0)$ relation, which is given by Definition 1, is equivalent relation on $FCO((X,\tau),(Y,\tau')) = \{f | f : (X,\tau) \longrightarrow (Y,\tau')$ fuzzy continuous and onto function $\}$.*

Corollary 1. *Equivalent relation, which is given by Theorem 1, divides*

$FCO((X,\tau),(Y,\tau'))$ set into different equivalence clases and the equivalence clases of $f \in FCO((X,\tau),(Y,\tau'))$ function is denoted by $[f]$.

Theorem 2. (see [6]) *Let $(X,\tau),(Y,\tau')$ and (Z,τ'') are fuzzy topological spaces, $f_0, f_1 : (X,\tau) \longrightarrow (Y,\tau')$, $g_0, g_1 : (Y,\tau') \longrightarrow (Z,\tau'')$ be fuzzy continuous functions. If $f_0 \overset{F}{\sim} f_1\ (rel X_0)$, $g_0 \overset{G}{\sim} g_1(rel\ X_0)$, then $g_0 f_0 \sim g_1 f_1 (rel\ X_0)$.*

Definition 12. (see [6]) Let (X,τ), (Y,τ') are fuzzy topological spaces. If there exists $f' : (Y,\tau') \longrightarrow (X,\tau)$ fuzzy continuous function, which satisfies the following conditions:

(i) $ff' \sim i_Y$,

(ii) $f'f \sim i_X$,

then $f : (X,\tau) \longrightarrow (Y,\tau')$ fuzzy continuous function is called *fuzzy homotopy equivalence*. If there is a homotopy equivalence function between (X,τ) and (Y,τ'), then these spaces are called fuzzy homotopy equivalence or these are same homotopy type and denoted by $(X,\tau) \simeq (Y,\tau')$.

Theorem 3. *The same homotopy type equivalence, given by Definition 12, is a equivalence relation on fuzzy topological spaces.*

Theorem 4. *If (X,τ) and (Y,τ') are fuzzy topological equivalence spaces, then (X,τ) and (Y,τ') are fuzzy homotopy equivalence spaces. Conversely this theorem always is not true.*

4. Topological and Fuzzy Topological Groups

Let $X \neq \emptyset$ and $(X,*)$ be a group. If $A, B \in I^X$ and $C, D \subseteq X$, then $A.B \in I^X$, $A^{-1} \in I^X$, $C * D \subseteq X$ and $C^{-1} \subseteq X$ are defined as following type;

$$(A.B)(x) = Sup\{\min\{A(x_1), A(x_2)\}, x_1 * x_2 = x\}\,,\ A^{-1}(x) = A(x^{-1})\,,$$

$$C * D = \{c * d : c \in C, d \in D\}\,,\ C^{-1} = \{c^{-1} : c \in C\}\,.$$

Definition 13. (see [5]) Let $X \neq \emptyset$ be a set, $(X,*)$ be a group and (X,τ) be fuzzy topological space. A triad $(X,*,\tau)$ is called a *fuzzy topological group* if satisfies the following conditions:

(i) For all $x, y \in X$ and any fuzzy open Q-neigbourhood W of the fuzzy point P^r_{x*y} there are fuzzy open Q-neigbourhoods U and V of P^r_x and P^r_y, respectively such that $U.V \subseteq W$.

(ii) For all $x \in X$ and any fuzzy open Q-neigbourhoods V' of $P^r_{x^{-1}}$ there exists a fuzzy open Q-neigbourhoods U of P^r_x such that $U^{-1} \subseteq V'$.

Theorem 5. (see [9]) *Let (Y,τ') be a fuzzy topological space, $(Z,*'',\tau'')$ fuzzy topological group, and f, g elements of $FC(Y,Z)$ which is the set of all* *fuzzy continuous functions from* Y *to* Z. *İf* $f, g \in FC(Y,Z)$, *then* $f \odot g : Y \longrightarrow Z$ *and* $f^{-1} : Y \longrightarrow Z$

$$(f \odot g)(y) = f(y) *'' g(y)\,, \quad f^{-1}(y) = (f(y))^{-1}\,,$$

for every $y \in Y$, *are fuzzy continuous.*

Definition 14. (see [9]) Let X be a set, $r \in [0,1]$ and $r^* \in I^X$. For all $x \in X$ defined by $r^*(x) = r$ which is fuzzy set and τ subset of I^X. If τ satisfies following conditions, then τ called a *fully stratified fuzzy topological space:*

(i) $r^* \in \tau$.

(ii) If $A, B \in \tau$ *then* $A \wedge B \in \tau$.

(iii) If $A_j \in \tau$ *for all* $j \in I$ *için için* $\underset{j \in I}{\vee} A_j \in \tau$.

Theorem 6. (see [9]) *Let* (Y, τ') *be a fully stratified fuzzy topological space,* $(Z, *'', \tau'')$ *fuzzy topological group and "e" the identity element of the group* $(Z, *'')$. *Then the function* e' *from the fully stratified fuzzy topological space* Y *into the fuzzy topological space* Z *with the type;*

$$e'(y) = e$$

for every $y \in Y$, *is fuzzy continuous.*

Proof. Let $U \in \tau''$. Then $((e')^{-1}(U))(y) = U(e'(y)) = U(e)$ for every $y \in Y$. This means that the fuzzy set $(e')^{-1}(U)$ is constant. Therefore, since the fuzzy space (Y, τ') is fully stratified, we have $(e')^{-1}(U) \in \tau''$. Thus, the function e' is fuzzy continuous. □

Theorem 7. (see [9]) *Let* (Y, τ') *be fully stratified fuzzy topological space and* $(Z, *'', \tau'')$ *fuzzy topological group. Then* $(FC(Y,Z), \odot)$ *is a group.*

Theorem 8. (see [9]) *Let* (Y, τ') *be fully stratified fuzzy topological space and* $(Z, *'', \tau'')$ *fuzzy topological group. If* $(Z, *'')$ *is Abelian group, then* $(FC(Y,Z), \odot)$ *group is Abelian too.*

5. Fuzzy Sheaf

Let (X,τ) fuzzy topological space as base set. Then, it can be constituted the pointed fuzzy topological space (X, P_m^r) with the same type of homotopy for any fuzzy point $P_m^r \in I^X$.

Furthermore $(Z, *'', \tau'')$ fuzzy topological group with base fuzzy point $P_e^r \in I^Z$. Then, it is denoted by $(Z, *'', P_e^r)$.

If $(Z, *'', P_e^r)$ is any fuzzy topological group, then the fuzzy set of fuzzy homotopy class of homotopy maps preserving the base fuzzy points from (X, P_m^r) to $(Z, *'', P_e^r)$, with respect to the fuzzy continuous base point, will be denoted by

$$\begin{aligned} H_m &= [(X, P_m^r), (Z, *'', P_e^r)] = \{[f]_{P_m^r} | f : (X, P_m^r) \longrightarrow (Z, *'', P_e^r), \\ f(P_m^r) &= P_{f(m)}^r = P_e^r,\ f \text{ fuzzy continuous}\}. \end{aligned}$$

In addition to, membership function of H_m is $\mu_{H_m} : H_m \longrightarrow [0, 1]$,

$$\mu_{H_m}([f]_{p_m^r(x)}) = \begin{cases} r, & m = x, \\ 0, & m \neq x. \end{cases}$$

Proposition 1. *Let $f, g \in FC((X, P_m^r), (Z, *'', P_e^r))$, then the fuzzy function $(f \odot g)(x) = f(x) *'' g(x)$ preserves the base point.*

Proof. We have

$$f(P_m^r) = P_{f(m)}^r = P_e^r, \quad g(P_m^r) = P_{g(m)}^r = P_e^r.$$

for every $f, g \in FC((X, P_m^r), (Z, *'', P_e^r))$. In this case we can write,

$$(f \odot g)(P_m^r) = f(P_m^r) *'' g(P_m^r) = P_e^r *'' P_e^r = P_e^r. \qquad \square$$

Corollary 2. *We can write*

$$[f]_{P_m^r}.[g]_{P_m^r} = [f \odot g]_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$$

for every $[f]_{P_m^r}, [g]_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$.

Theorem 9. *If (X, P_m^r) pointed fuzzy topological space, $(Z, *'', P_e^r)$ pointed fuzzy topological group, then $([(X, P_m^r), (Z, *'', P_e^r)], \cdot]$ is a group. In addition to that such a groups are also different from each other.*

Proof. (i) Associative property. For every $[f]_{P_m^r}, [g]_{P_m^r}, [h]_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$:

$$\begin{aligned} [f]_{P_m^r} \cdot ([g]_{P_m^r} \cdot [h]_{P_m^r}) &= [f]_{P_m^r} \cdot [g \odot h]_{P_m^r} = [f \odot (g \odot h)]_{P_m^r} \\ &= [(f \odot g) \odot h]_{P_m^r} = [f \odot g]_{P_m^r} \cdot [h]_{P_m^r} = ([f]_{P_m^r} \cdot [g]_{P_m^r}) \cdot [h]_{P_m^r}. \end{aligned}$$

This means associative property is satisfied.

(ii) Identity element. For every $[f]_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$

$$[f]_{P_m^r} \cdot [e']_{P_m^r} = [f \odot e']_{P_m^r} = [f]_{P_m^r} \text{ ve } [e']_{P_m^r} \cdot [f]_{P_m^r} = [e' \odot f]_{P_m^r} = [f]_{P_m^r}.$$

Thus identity element is $[e']_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$.

(iii) Inverse element. For every $[f]_{P_m^r}, [f^{-1}]_{P_m^r} \in [(X, P_m^r), (Z, *'', P_e^r)]$;

$$\begin{aligned} [f]_{P_m^r} \cdot [f^{-1}]_{P_m^r} = [f \odot f^{-1}]_{P_m^r} = [e']_{P_m^r} \text{ and } [f^{-1}]_{P_m^r} \cdot [f]_{P_m^r} \\ = [f^{-1} \odot f]_{P_m^r} = [e']_{P_m^r}. \end{aligned}$$

Thus we can write $[f^{-1}]_{P^r_m} = [f]^{-1}_{P^r_m}$. □

Theorem 10. *Let (X, P^r_m) pointed fuzzy topological space, $(Z, *'', P^r_e)$ pointed fuzzy topological group. If $(Z, *'')$ group is Abelian, then $(H_m, \cdot)$ group is Abelian.*

Proof. For $[f]_{P^r_m}, [g]_{P^r_m} \in H_m$, if $(Z, *'')$ group is Abelian, then $(FS(Y, Z), \odot)$ group is Abelian (Theorem 8). Therefore

$$[f]_{P^r_m} \cdot [g]_{P^r_m} = [f \odot g]_{P^r_m} = [g \odot f]_{P^r_m} = [g]_{P^r_m} \cdot [f]_{P^r_m}$$

then $(H_m, \cdot)$ groups is Abelian. □

Theorem 11. *Let (X, P^r_m) pointed fuzzy topological space, $(Z, *'', P^r_e)$ pointed fuzzy topological group, then $H_m = [(X, P^r_m), (Z, *'', P^r_e)]$ is fuzzy group on X.*

Proof. (i) For every $[f]_{P^r_m}, [g]_{P^r_m} \in H_m = [(X, P^r_m), (Z, *'', P^r_e)]$

$$\mu_{H_m}([f]_{p^r_m}.[g]_{P^r_m}) = r = \min\{\mu_{H_m}([f]_{p^r_m}), \mu_{H_m}([g]_{p^r_m})\}\,.$$

(ii) For every $[f]_{P^r_m}$ there exists $[f^{-1}]_{P^r_m} \in [(X, P^r_m), (Z, *'', P^r_e)]$ (Theorem 9) and

$$\mu_{H_m}([f]^{-1}_{p^r_m}) = \mu_{H_m}([f^{-1}]_{p^r_m}) = r = \mu_{H_m}([f]_{p^r_m}).$$ □

Let us denote by $H(X)$ the disjoint union of the fuzzy groups $[(X, P^r_m), (Z, *'', P^r_e)] = H_m$ obtained for each $P^r_m \in I^X$ (or $m \in X$), (X, P^r_m) pointed fuzzy topological spaces, i.e.

$$H(X) = \bigvee_{\substack{P^r_m \in I^X \\ m \in X}} [(X, P^r_m), (Z, *'', P^r_e)] = \bigvee_{m \in X} H_m\,.$$

Thus $H(X)$ is a fuzzy set over I^X. Let us now define a function $\psi : H(X) \longrightarrow I^X$ as

$$[f]_{P^r_m} \in H(X) \Longrightarrow \exists m \in X \ni \; [f]_{P^r_m} \in H_m \le H(X)$$
$$\Longrightarrow \psi([f]_{P^r_m}) = P^r_m \in I^X\,, \quad m \in X.$$

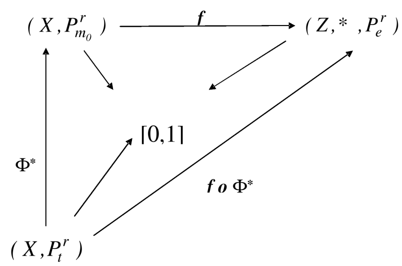
For $m_0 \in X$ arbitrary fixed point, $W = W(P^r_{m_0})$ is open Q-neigbourhood of $P^r_{m_0}$ in I^X. Now we can define a function $s : W \longrightarrow H(X)$ as follows:

If $P^r_{m_0} \in I^X$ for $m_0 \in X$, then there exists H_{m_0} group in $H(X)$. Let $[f]_{P^r_{m_0}}$ be the homotopy class in the group H_{m_0}.

If P^r_t is any fuzzy point in W, then (X, P^r_t) and $(X, P^r_{m_0})$ are having the same homotopy type. Therefore, there is a homotopy equivalence function,

$$(X, P^r_t) \xrightarrow{\Phi_*} (X, P^r_{m_0})\,.$$

Hence



From diagram functions f and Φ_* is fuzzy continuous and base-point preserving, $fo\ \Phi_*$ composition is fuzzy continuous, too. In addition this composition preserves the based point, i.e.

$$(fo\Phi_*)(P_t^r) = f(\Phi_*(P_t^r)) = f(P_{m_0}^r) = P_{f(m_0)}^r = P_e^r\ .$$

$[h]_{P_t^r} \in H_t$ is a homotopy class of function $fo\Phi_* = h$. In that case we can define s function as follows

$$s : W \longrightarrow H(X),\ s \longrightarrow s(P_t^r) = [h]_{P_t^r} \text{ for every } P_t^r \in W\ .$$

In this way s is well-defined. Therefore for all $P_t^r \in W$, $(\psi os)(P_t^r) = \psi(s(P_t^r)) = \psi([h]_{P_t^r}) = P_t^r$,, then $\psi os = I_W$. Hence we can writes$(W) = \bigcup_{P_t^r \in W} [h]_{P_t^r}$.

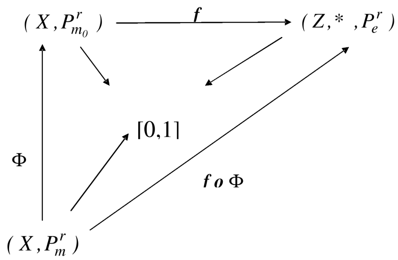
Remark 1. Let us denote by $\Gamma(W, H(X))$ the set of s functions which are defined like above.

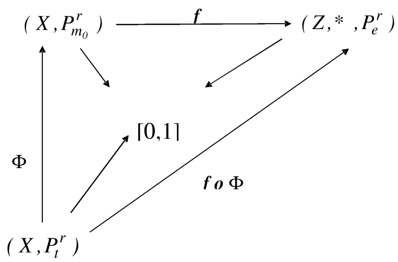
If we can define $s(W)$ as an open fuzzy set, then it can be easily shown that the $\beta = \{s(W) : W = W(P_m^r) \leq I^X,\ m \in X,\ s \in \Gamma(W, H(X))\}$ family is fuzzy topology-base on $H(X)$. Thus $H(X)$ is a fuzzy topological space.

Now we can show that $\psi : H(X) \longrightarrow I^X$ function is local topological. If $[h]_{P_m^r} \in H(X)$ and $m \in X$, then $\psi([h]_{P_m^r}) = P_m^r$. Therefore, there is a function $s : W \longrightarrow H(X)$ such that $s(P_t^r) = [h]_{P_t^r}$, $P_t^r \in W = W(P_{m_0}^r)$. Now let us assume that $s(W) = U([h]_{P_m^r})$ and $\psi|_U = \psi^*$

(i) The function $\psi^* = \psi|_U : U \longrightarrow W$ is injective. Because for any $[g]_{P_m^r}, [h]_{P_t^r} \in U = s(W)$ there are the fuzzy point P_m^r, P_t^r respectively in W such that

$$s(P_m^r) = [g]_{P_m^r} = [fo\Phi]_{P_m^r}\ , \quad s(P_t^r) = [h]_{P_t^r} = [fo\Phi']_{P_t^r}$$





If $\psi^*([g]_{P^r_m}) = \psi^*([h]_{P^r_t})$, then

$$\begin{aligned}\psi^*(s(P^r_m)) &= \psi^*(s(P^r_t)) \Longrightarrow (\psi^*([fo\Phi]_{P^r_m}) = \psi^*([fo\Phi']_{P^r_t}) \\ &\Longrightarrow P^r_m = P^r_t \text{ and } m = t, m, t \in X\,.\end{aligned}$$

Since (X, P^r_m) with (X, P^r_t) is same homotopy type, we can write the following equality;

$$\Phi \sim \Phi' \Longrightarrow fo\Phi \sim fo\Phi' \Longrightarrow [fo\Phi]_{P^r_m} = [fo\Phi']_{P^r_t} \Longrightarrow [g]_{P^r_m} = [h]_{P^r_t}\,.$$

(ii) The function $\psi^* = \psi|_U : U \longrightarrow W$ is fuzzy continuous. In fact, if $[h]_{P^r_m} \in U = s(W)$, then $\psi^*([h]_{P^r_m}) = P^r_m \in W$ and $V = V(P^r_m) \subset W$ is a Q-neigbourhood of P^r_m. Then $s(V) \subset U = s(W)$ is a Q-neigbourhood of $[h]_{P^r_m}$ and $\psi^*(s(V)) = V \subset W$, ψ^* is fuzzy continuous.

(iii) $(\psi^*)^{-1} = (\psi|_U)^{-1} = s : W \longrightarrow U = s(W)$ is fuzzy continuous. In fact, if P^r_m is any fuzzy point of W, $s(P^r_m) = [h]_{P^r_m} \in U'$ and $U' = U'([h]_{P^r_m}) \subset U$ is a Q-neigbourhood of $[h]_{P^r_m}$, then $(\psi|_U)(U') \subset W$ is Q-neigbourhood of P^r_m in W and $s(\psi|_U)(U') = U' \subset U$. Therefore $(\psi^*)^{-1}$ is fuzzy continuous.

Hence, the following theorem can be given.

Theorem 12. *Let $(Z, *'', \tau'')$ be fuzzy topological group and let (X, P^r_m) be pointed fuzzy topological space which have same homotopy type. If for every $P^r_m \in I^X$,*

$$H(X) = \bigvee_{\substack{P^r_m \in I^X \\ m \in X}} [(X, P^r_m), (Z, *'', P^r_e)] = \bigvee_{m \in X} H_m$$

and $\psi : H(X) \longrightarrow I^X$ *is function such that* $\psi([h]_{P_m^r}) = P_m^r$ *for every* $[h]_{P_m^r} \in H(X)$*, then there exist a fuzzy topology on* $H(X)$ *such that* ψ *is a locally topological mapping with respect to this fuzzy topology. Thus the pair* $(H(X), \psi)$ *is fuzzy sheaf on* I^X*.*

Definition 15. The fuzzy sheaf $(H(X), \psi)$ given by the previous theorem is called fuzzy sheaf of the fuzzy groups formed by $(Z, *'', P_e^r)$ pointed fuzzy topological group over I^X and (X, P_m^r) pointed fuzzy topological spaces.

Definition 16. The group $[(X, P_m^r), (Z, *'', P_e^r)] = \psi^{-1}(P_m^r)$ is called the stalk of the fuzzy sheaf on I^X and is denoted by $H(X)_{P_m^r}$ for every $m \in X$.

If $P_m^r \in I^X$ is an arbitrary fixed fuzzy point, then there is $W = W(P_m^r)$ an open Q-neigbourhood of P_m^r and function $s : W \longrightarrow H(X)$ such that s is fuzzy continuous and $\psi os = I_W$. Hence the function s is called a section of $H(X)$ over W. Let us denote all of the sections of $H(X)$ over W, by $\Gamma(W, H(X))$.

Theorem 13. *Let the set of all sections of* $H(X)$ *over* W *is* $\Gamma(W, H(X))$*. We can define*

$$(s_1 \circledast s_2)(P_m^r) = s_1(P_m^r).S_2(P_m^r) = [h]_{P_m^r}.[g]_{P_m^r} = [h \odot g]_{P_m^r}$$

for any $s_1, s_2 \in \Gamma(W, H(X))$*. Then* $(\Gamma(W, H(X)), \circledast)$ *is a group.*

Proof. Firstly we show the operation $\circledast$ is well defined on $\Gamma(W, H(X))$.

Let $h' \in [h]_{P_m^r}$ and $g' \in [g]_{P_m^r}$, then $h' \sim h$ and $g' \sim g$. This implies $[h]_{P_m^r} = [h']_{P_m^r}$ and $[g]_{P_m^r} = [g']_{P_m^r}$. Therefore we get $[h']_{P_m^r}.[g']_{P_m^r} = [h' \odot g']_{P_m^r} = [h \odot g]_{P_m^r}$.

1. Associative Property. Let $P_m^r \in W$ be a fuzzy point. Then

$$\begin{aligned}((s_1 \circledast s_2) \circledast s_3)(P_m^r) &= (s_1 \circledast s_2)(P_m^r).S_3(P_m^r) = (s_1(P_m^r).S_2(P_m^r)).S_3(P_m^r)\\ &= ([h]_{P_m^r}.[g]_{P_m^r}).[f]_{P_m^r} = ([h \odot g]_{P_m^r}).[f]_{P_m^r}\\ &= [(h \odot g) \odot f]_{P_m^r} = [h \odot (g \odot f)]_{P_m^r} = [h]_{P_m^r}.[g \odot f]_{P_m^r}\\ &= s_1(P_m^r).([g]_{P_m^r}.[f]_{P_m^r}) = s_1(P_m^r).(s_2(P_m^r).S_3(P_m^r)) = (s_1 \circledast (s_2 \circledast s_3))(P_m^r),\end{aligned}$$

for all $s_1, s_2, s_3 \in \Gamma(W, H(X))$. Therefore $(s_1 \circledast s_2) \circledast s_3 = s_1 \circledast (s_2 \circledast s_3)$ for all $s_1, s_2, s_3 \in \Gamma(W, H(X))$.

2. Identity Element. For all $s_1 \in \Gamma(W, H(X))$ and $P_m^r \in W$, If we choose $I \in \Gamma(W, H(X))$, where $I(P_m^r) = [e']_{P_m^r}$. Then

$$(I \circledast s_1)(P_m^r) = I(P_m^r).S_1(P_m^r) = [e']_{P_m^r}.[h]_{P_m^r} = [e' \odot h]_{P_m^r} = [h]_{P_m^r} = s_1(P_m^r),$$
$$(s_1 \circledast I)(P_m^r) = s_1(P_m^r).I(P_m^r) = [h]_{P_m^r}.[e']_{P_m^r} = [h \odot e']_{P_m^r} = [h]_{P_m^r} = s_1(P_m^r).$$

Thus $I \in \Gamma(W, H(X))$ is identity element.

3. Inverse Element. Let $s_1 \in \Gamma(W, H(X))$ be an arbitrary section $s_1^{-1} \in$

$\Gamma(W, H(X))$ such that $s_1^{-1}(P_m^r) = [h^{-1}]_{P_m^r}$. Therefore

$$(s_1 \circledast s_1^{-1})(P_m^r) = s_1(P_m^r).S_1^{-1}(P_m^r) = [h]_{P_m^r}.[h^{-1}]_{P_m^r} = [h \odot h^{-1}]_{P_m^r} = [e']_{P_m^r} = I(P_m^r)\,,$$

$$(s_1^{-1} \circledast s_1)(P_m^r) = s_1^{-1}(P_m^r).S_1(P_m^r) = [h^{-1}]_{P_m^r}.[h]_{P_m^r} = [h^{-1} \odot h]_{P_m^r} = [e']_{P_m^r} = I(P_m^r)\,.$$

On the other hand, the any inverse element of $s \in \Gamma(W, H(X))$, namely, $s^{-1} \in \Gamma(W, H(X))$ which is obtained by means the homotopy inverse of, $(Z, *'', P_e^r)$ is pointed fuzzy topological group. □

Hence $\Gamma(V, H(X))$ is a group. Thus the operation $\circledast : H(X) \times H(X) \longrightarrow H(X)$ is continuous. Hence $(H(X), \psi)$ is an algebraic sheaf.

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