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ON SEQUENTIALLY-k-SPACES

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Abstract: The class of sequentially-k-spaces is introduced and investigated.

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1. Introduction

A topological space X is called k-space if X is a Hausdorff space and X is an image of a locally compact space under a quotient mapping. A natural question arises: when a k-space satisfies that its product with every k-spaces is also a k-space? Michael in [4] showed that a k-space has this property iff it is a locally compact space. A similar question, related to the class of quasi-k-spaces, Hausdorff images of locally countably compact spaces under quotient mappings, was answered by M. Sanchis [5].

The study of k-spaces and quasi-k-spaces suggests to define, in a natural way, the class of sequentially-k-spaces, namely, the class of Hausdorff images of locally sequentially compact spaces under quotient mappings.

The aim of this paper is to start the study of sequentially-k-spaces. We will show the basic properties of this class of spaces, in analogy with the classical known results about k-spaces. Moreover, we will ask when a sequentially-k-space satisfies that its product with every sequentially-k-spaces is also a sequentially-k-space.

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2. Sequentially-k-Spaces

All spaces are assumed to be Hausdorff.

Definition 2.1. A topological space X is locally sequentially compact if for every $x \in X$ and any neighborhood U of x there exists a neighborhood V of x such that $\overline{V}$ is sequentially compact and $\overline{V} \subset U$.

Definition 2.2. A topological space X is called a "sequentially-k-space" if X is a Hausdorff image of a locally sequentially compact space under a quotient mapping.

Theorem 2.3. *A Hausdorff space is a sequentially-k-space iff for each $A \subset X, A$ is closed provided that the intersection of A with any sequentially compact subspace K of X is closed in K.*

Proof. Let X be a sequentially-k-space and let $f : Y \to X$ be a quotient mapping of a locally sequentially compact space Y onto X. Suppose that the intersection of a set A with any sequentially compact space K of X is closed in K. Take a point $y \in \overline{f^{-1}(A)}$ and a neighborhood $U \subset Y$ of the point y such that $\overline{U}$ is sequentially compact. By hypothesis the set $A \cap f(\overline{U})$ is closed in the sequentially compact space $f(\overline{U})$ (see Theorem 3.10.32 in [2]). Now if $y \notin f^{-1}(A)$ then $f(y) \notin A \cap f(\overline{U})$ thus there exists an open set T in X containing $f(y)$ such that $T \cap (A \cap f(\overline{U})) = \emptyset$.

It follows that $f^{-1}(T) \cap f^{-1}(A) \cap \overline{U} = \emptyset$, where the set $f^{-1}(T) \cap \overline{U}$ represents a neighborhood of y disjoint from $f^{-1}(A)$. This is a contradiction. Then $y \in f^{-1}(A)$.

Conversely, consider a Hausdorff space X and denote by $\mathcal{SK}(X)$ the family of non-empty sequentially compact subspaces of X. Let $\tilde{X} = \oplus\{K : K \in \mathcal{SK}(X)\}$. The surjective mapping $f : \nabla_{K \in \mathcal{SK}(X)}, i_K : \tilde{X} \to X$, where i_K is the embedding of the subspace K in the space X is continuous (see Proposition 2.1.11 in [2]).

Moreover the space $\tilde{X}$ is a locally sequentially compact space. It is easy to verify that if a set A is closed provided that the intersections of $A \cap K$ is closed in K for all $K \in \mathcal{SK}(X)$, then f is a quotient mapping. □

Corollary 2.4. *A Hausdorff space X is a sequentially-k-space if and only if for each $A \subset X, A$ is open provided that the intersection of A with any sequentially compact subspace K of X is open in K.*

A sequentially-k-space need not be a k-space, and viceversa.

Example 2.5. A k-space which is not a sequentially-k-space: $\beta\mathbb{N}$.

Trivially $\beta\mathbb{N}$ is a k-space. To show that $\beta\mathbb{N}$ is not a sequentially-k-space observe that in $\beta\mathbb{N}$ the convergent sequences are those definitely constant. Therefore the sequentially compact subspaces in $\beta\mathbb{N}$ are the finite subspaces, so $A \cap F$ is closed in F for each finite subset F of $\beta\mathbb{N}$ and every $A \subset \beta\mathbb{N}$.

Example 2.6. A sequentially-k-space which is not a k-space.

The space $X = \omega_2 - \{$ ordinals whose cofinality is $\omega_1\}$ is a sequentially compact space (so, a fortiori, it is a sequentially-k-space) which is not a k-space.

Sequential spaces (see [3]) are examples of sequentially-k-spaces.

Theorem 2.7. *Every sequential Hausdorff space and, in particular, every first countable space is a sequentially-k-space.*

Proof. Let X be a sequential Hausdorff space. This means that $A \subset X$ is closed iff for each sequence S including its limit, $A \cap S$ is closed in S.

Suppose that $A \cap K$ is closed in K for each sequentially compact $K \subset X$. Then $A \cap S$ is closed in S, where S is a sequence with its limit. So, since X is sequential, A is closed in X. □

Theorem 2.8. *A mapping f of a sequentially-k-space X to a topological space Y is continuous iff for every sequentially compact $K \subset X$ the restriction $f_{/K} : K \to Y$ is continuous.*

Proof. It suffices to show that the continuity of all restrictions $f_{/K}$ implies the continuity of f. Take a closed set $A \subset Y$; for every sequentially compact $K \subset X$ we have that $f^{-1}(A) \cap K = (f_{/K})^{-1}(A)$ is closed in K, therefore, by Theorem 2.3, $f^{-1}(A)$ is closed, hence f is continuous. □

Theorem 2.9. *A continuous mapping $f : X \to Y$ of a topological space X to a sequentially-k-space Y is closed (open) if for every sequentially compact subspace $K \subset X$ the restriction $f_K : f^{-1}(K) \to K$ is closed (open).*

Proof. By Proposition 1.4 in [2] we have that if $f : X \to Y$ is closed (open) then $f_L : f^{-1}(L) \to L$ is closed (open) for each subspace $L \subset Y$. Therefore it suffices to show that if all restrictions f_K are closed (open) then f is closed (open). First, suppose that $f^{-1}(K) \to K$ is closed (open) for every sequentially compact subspace $K \subset Y$ and consider a closed (open) set $A \subset X$. The equalities

$$f(A) \cap K = f(A \cap f^{-1}(K)) = f_K(A \cap f^{-1}(K))$$

and the fact that f_K is closed implies that $f(A) \cap K$ is closed (open) in k for every sequentially compact subspace $K \subset Y$. Since Y is a sequentially-k-space, it follows that $f(A)$ is closed (open). □

From the definition of a sequentially-k-space we have:

Theorem 2.10. *If there exists a quotient mapping $f : X \to Y$ of a sequentially-k-space X onto a Hausdorff space Y, then Y is a sequentially-k-space.*

3. On Products of Sequentially-k-Spaces

Observe that a regular sequential space X is locally sequentially compact if and only if for every $x \in X$ there exists a neighborhood U of the point x such that $\overline{U}$ is a countably compact subspace of X.

Therefore we have:

Theorem 3.1. *Let X a regular sequential Hausdorff space. The following are equivalent:*

1. X is locally compact,

2. $X \times Y$ is a sequentially-k-space, for each sequentially-k-space Y.

Proof. Sequential compactness and countable compactness are equivalent in the class of sequential spaces (see Theorem 3.10.31 in [2]). Therefore for sequential regular Hausdorff spaces Theorem 2.4 in [5] works again. □

An analogous of the Whitehead Theorem (see [6]) is the following:

Theorem 3.2. *For every locally sequentially compact space X and any quotient mapping $g : Y \to Z$, where Y is a sequential space, the Cartesian product $f : id_X \times g : X \times Y \to X \times Z$ is a quotient mapping.*

Proof. If X is locally sequentially compact then X is a countably compact space, therefore the result follows applying Theorem 4.1. in [4]. □

It follows the result.

Theorem 3.3. *If X is a locally sequentially compact space then $X \times Y$ is a sequential space for every sequential space Y.*

Proof. Let $g : X' \to Y$ a quotient mapping with domain a (locally compact) metrizable space X'. The product $f = i_X \times g$ is a quotient mapping by Theorem 3.3. therefore $X \times Y$ is the quotient of the space $X \times X'$, which is a sequential space by a result of Boehme (see [1]). It follows that $X \times Y$ is a sequential space. □

It comes natural the following question: if $X \times Y$ is sequential for every sequential space Y, then is X a locally sequentially compact space ?

The following result gives an answer to this question.

Theorem 3.4. *The following properties of a regular sequential space X are equivalent:*

1. X is locally sequentially compact,

2. $X \times Y$ is sequential for every sequential space Y.

Proof. It suffices to recall that in the class of regular sequential spaces the local sequential compactness is equivalent to the local countable compactness and apply Theorem 4.2. in [4]. □

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