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ANALYTICAL SOLUTIONS OF A FRACTIONAL OSCILLATOR BY THE DECOMPOSITION METHOD

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Abstract: The equation of motion of a driven fractional oscillator is obtained from the corresponding equation of motion of a driven harmonic oscillator by replacing the second-order time derivative by a fractional derivative of order α with $1 < \alpha \leq 2$. The fractional derivative is described in the Caputo sense. The application of Adomian decomposition method, developed for differential equations of integer order, is extended to derive analytical solutions of the fractional oscillator. The response characteristics and phase plane representation of the fractional oscillator are studied for several cases.

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1. Introduction

In recent years considerable interest has been shown in the so called fractional calculus, which allows one to consider integration and differentiation of any order, not necessarily integer. Such interest has been stimulated by the applications that this calculus finds in different areas of physics and engineering,

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possibly including fractal phenomena. A review of some applications of fractional calculus in contnuum and statistical mechanics is given by Mainardi [9].

In this paper, we will consider the dynamics of the so-called driven fractional oscillator (DFO). This fractional oscillator is obtained by replacing the second time derivative term in the corresponding harmonic oscillator by a fractional derivative of order α, $1 < \alpha \leq 2$. The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 2$, the fractional system of oscillators reduces to the standard system of simple harmonic oscillators. Some aspects of such a system have been studied previously by Gorenflo and Mainardi [6], Mainardi [8], Podlubny [16], and Narahari et al [12, 13]. A feature common to all these authors is that they used Laplace transform method to obtain analytical solutions for (DFO) system in terms of Mittag-Leffler functions. Futheremore, some numerical applications have been made by Blank [3], who has proposed a method based on collocation using spline functions, and Podlubny [15] using the finite difference method.

The Adomian decomposition method [1, 2] will be applied for computing solutions to the driven fractional oscillators (DFO) system considered in this paper. The method provides the solutions in the form of a power series with easily computed terms. The Adomian decomposition method has many advantages over the classical techniques mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations. For more details in using the decomposition method for similar problems see [17, 18, 7]

The paper is organized as follows. A brief review of the fractional calculus theory is given in Section 2. In Section 3 we use the decomposition method to construct our numerical solutions for (DFO) system. In Section 4 we present some examples to show the efficiency and simplicity of the method.

2. Preliminaries and Notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. We first introduce the Riemann-Liouville definition of fractional derivative operator J^{α}.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_{μ}, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^{p} f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_{μ}^{m} iff $f^{(m)} \in C_{\mu}$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0, \;\; x > 0\,, \\ J^0 f(x) &= f(x). \end{aligned}$$

Properties of the operator J^α can be found in [10], we mention only the following: For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity [4].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \qquad (2.1)$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0, \; f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If $m-1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D_*^\alpha J^\alpha f(x) = f(x),$$

and,

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \;\; x > 0.$$

For real $\alpha > 0$ (later only for $1 < \alpha \leq 2$), consider the fractional differential equation

$$\frac{d^\alpha x}{dt^\alpha} + \omega^\alpha x(t) = f(t), \;\; m-1 < \alpha \leq m, \qquad (2.2)$$

subject to the initial conditions

$$x^k(0) = c_k, \quad k = 0, 1, ..., m-1, \qquad (2.3)$$

where ω is an arbitrary constant and $f(t)$ is a given continuous function. Here m is an integer uniquely defined by $m-1 < \alpha \leq m$, which provides the number of the prescribed initial values $x^k(0) = c_k, \quad k = 0, 1, ..., m-1$. Equation (2.2) is called the fractional oscillation equation for $1 < \alpha \leq 2$ and the fractional relaxation equation and the fractional growing oscillations equation corresponding to $0 < \alpha < 1$ and $2 < \alpha \leq 3$, respectively.

The fractional derivative in equation (2.2) is considered in the Caputo sense. The reason for adopting the Caputo definition is as follows: when we interpret the fractional derivative in equation (2.2) as Caputo fractional derivatives then, with suitable conditions on the forcing function $f(t)$ and with initial values $x^k(0) = c_k, \quad k = 0, 1, ..., m-1$ specified, the equation has a unique solution. If we were to interpret the fractional derivative as Riemann-Liouville fractional derivatives, we would have to specify our initial conditions in terms of fractional integrals and their derivatives. The initial conditions required by the Caputo definition coincide with identifiable physical states, and this leads to the preference, among modellers, for the Caputo definition [5]. For more details on the existence and uniqueness of solutions to the fractional differential equations (2.2), see [16, 14, 11].

3. Analysis of the Numerical Method

In this section, we are concerned with providing good quality algorithm for the solution of a system of driven fractional oscillators (DFO) of the general form:

$$\frac{d^\alpha x}{dt^\alpha} + \omega^\alpha x(t) = f(t), \quad 1 < \alpha \leq 2, \tag{3.1}$$

subject to the initial conditions

$$x(0) = a, \quad \dot{x}(0) = b, \tag{3.2}$$

where ω is the natural frequency and $f(t)$ is the forcing function.

Equations (3.1) can be written in terms of operator from as

$$D_{*t}^\alpha x(t) = f(t) - \omega^\alpha x(t), \tag{3.3}$$

where the fractional differential operator D_{*t}^α is defined as in equation (2.1) denoted by

$$D_{*t}^\alpha = \frac{d^\alpha}{dt^\alpha}.$$

Operating with J^α, the inverse of the operator D_{*t}^α, on both sides of equation (3.3) yields

$$J^{\alpha} D_{*t}^{\alpha} x(t) = J^{\alpha} f(t) - \omega^{\alpha} J^{\alpha} x(t). \tag{3.4}$$

Therefore, it follows that

$$x(t) = \phi(t) + J^{\alpha} f(t) - \omega^{\alpha} J^{\alpha} x(t), \tag{3.5}$$

where

$$\phi(t) = \sum_{k=0}^{m-1} x^{(k)}(0^{+}) \frac{t^k}{k!}.$$

The Adomian decomposition method [1, 2] assumes a series solution for $x(t)$ given by the infinite sum of components

$$x(t) = \sum_{n=0}^{\infty} x_n(t). \tag{3.6}$$

Substituting (3.6) and the initial condition (3.2) into (3.5) and indentifying the zeroth component x_0 by the term arising from the initial condition and from the force function, then we have the following recursive relations

$$x_0 = \phi(t) + J^{\alpha} f(t),$$

$$x_{n+1}(t) = -\omega^{\alpha} J^{\alpha} x_n(t)..... \tag{3.7}$$

The components of $x_n(t), \;\; n \geq 1$ are determined in the following recursive way

$$\begin{aligned} x_1 &= -\omega^{\alpha} J^{\alpha}[x_0] = -\omega^{\alpha}[J^{\alpha}\phi + J^{\alpha} J^{\alpha} f]\,, \\ x_2 &= -\omega^{\alpha} J^{\alpha}[x_1] = (-\omega^{\alpha})^2 [J^{\alpha} J^{\alpha}\phi + J^{\alpha} J^{\alpha} J^{\alpha} f]\,, \\ x_3 &= -\omega^{\alpha} J^{\alpha}[x_2] = (-\omega^{\alpha})^3 [J^{\alpha} J^{\alpha} J^{\alpha}\phi + J^{\alpha} J^{\alpha} J^{\alpha} J^{\alpha} f]\,, \\ &\vdots \end{aligned}$$

As a result, the series solution is given by

$$x(t) = x_0 + x_1 + x_2 + x_3 + \dots\,. \tag{3.8}$$

Define the $\gamma-$term approximation solution as

$$\psi_{\gamma} = \sum_{n=0}^{\gamma-1} x_n(t), \tag{3.9}$$

and the exact solution $x(t)$ is given by

$$x(t) = \lim_{\gamma \to \infty} \psi_{\gamma}\,. \tag{3.10}$$

To demonstrate the effectiveness of the method for solving (DFO), we consider here the following three examples.

4. Numerical Examples

Example 4.1. Consider the following fractional differential equation:

$$\frac{d^\alpha x}{dt^\alpha} + \omega^\alpha x(t) = 0, \quad 1 < \alpha \leq 2, \tag{4.1}$$

subject to the initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{4.2}$$

This equation describes a simple harmonic fractional oscillator where the forcing function in this case is $f(t) = 0$.

Substituting equation (4.1) and the initial conditions (4.2) into equation (3.7), yields the following recursive relations:

$$\begin{aligned} x_0 &= 1, \\ x_{n+1} &= -J^\alpha[\omega^\alpha x_n(t)]. \end{aligned}$$

Using the above recursive relationship, the first few terms of the decomposition series are given by

$$x_1 = -\frac{(\omega t)^\alpha}{\Gamma(\alpha+1)}, \quad x_2 = \frac{(\omega t)^{2\alpha}}{\Gamma(2\alpha+1)}, \quad x_3 = -\frac{(\omega t)^{3\alpha}}{\Gamma(3\alpha+1)}, \cdots$$

Substituting $x_0, x_1, x_2, ...$ into (3.8) gives the solution $x(t)$ in a series form solution by

$$x(t) = 1 - \frac{(\omega t)^\alpha}{\Gamma(\alpha+1)} + \frac{(\omega t)^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{(\omega t)^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(\omega t)^{4\alpha}}{\Gamma(4\alpha+1)} - .... \tag{4.3}$$

Setting $\alpha = 2$ in equation (4.3), the solution of a simple harmonic oscillator is obtained and given by $x(t) = \cos(\omega t)$. In this case, the motion is periodic, the total energy is a constant of motion and the plane phase diagram is a closed curve, namely an ellipse.

To graph the phase plane diagram of equation (4.3), we define the total energy of the fractional oscillator as

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mp^2,$$

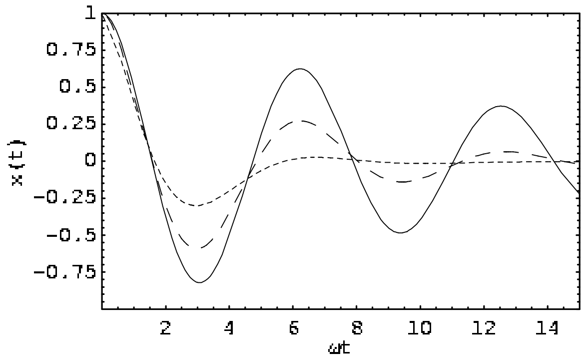

Figure 1: Response function of equation (4.3) for different values of α: (——) $\alpha = 1.9$, (—) $\alpha = 1.75$, (...) $\alpha = 1.5$

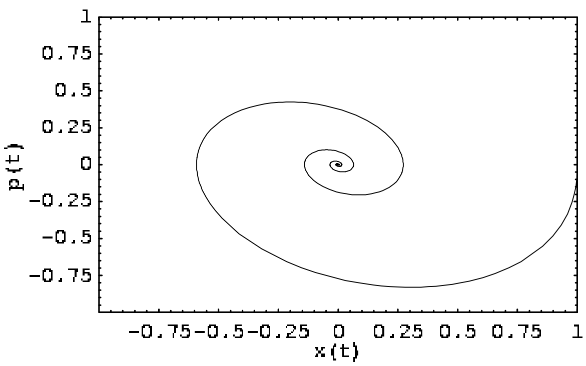

Figure 2: Phase plane diagram of equation (4.3) for $\alpha = 1.75$

where p is the generalized momentum of the fractional oscillator defined by

$$p = \Big(\frac{d^{\alpha/2}x}{dt^{\alpha/2}}\Big).$$

Figure 1 shows the evolution results for $\alpha = 1.5, \quad 1.75$ and 1.9. Comparison between these results shows how the displacement of the fractional oscillator varies as a function of time and how this time variation depends on the parameter α. It can be seen that the behavior of the driven fractional oscillator is very similar to the behavior of the damped harmonic oscillator, where the motion is still oscillatory, but the total energy decrease and the phase plane diagram is no longer a closed curve, but a logarithmic spiral.

Figures 2 and 3 show the phase plane trajectories for $\alpha = 1.75$ and $\alpha = 1.9$, respectively. A similar behavior to the damped oscillator is also observed in

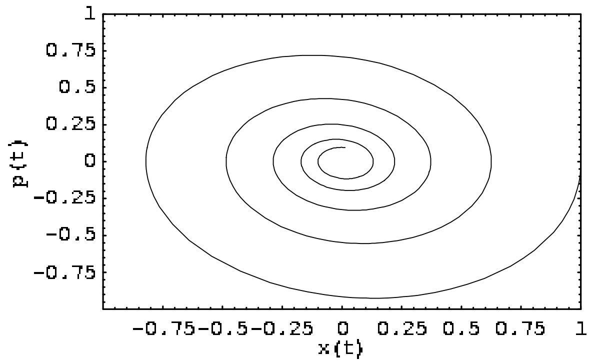

Figure 3: Phase plane diagram of equation (4.3) for $\alpha = 1.9$

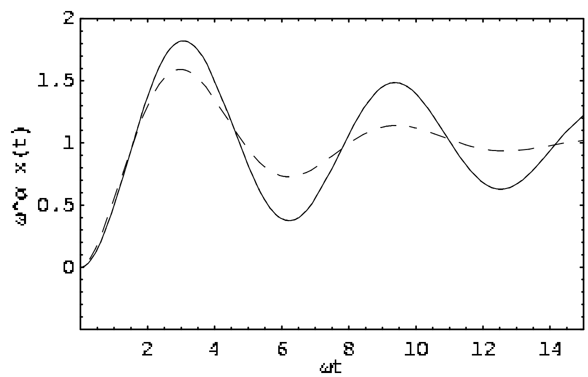

Figure 4: Response function of equation (4.6) for different values of α: (——)$\alpha = 1.9$, $(- - -)\alpha = 1.75$

the two plane diagrams.

The results of our computations, for different values of α, are in perfect agreement with the analytical solutions obtained by [8, 12, 13] using Laplace transformand Mittag-Leffler functions.

Example 4.2. We next consider the (FDO) system of the form:

$$\frac{d^\alpha x}{dt^\alpha} + \omega^\alpha x(t) = f(t), \ \ 1 < \alpha \leq 2, \tag{4.4}$$

subject to the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0, \tag{4.5}$$

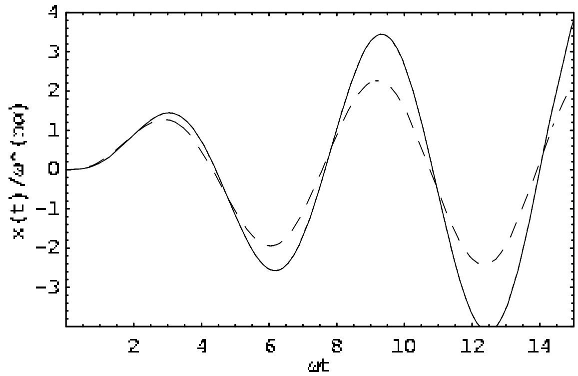

Figure 5: Response function of equation (4.9) for different values of α: (——)$\alpha = 1.9$, $(- - -)\alpha = 1.75$

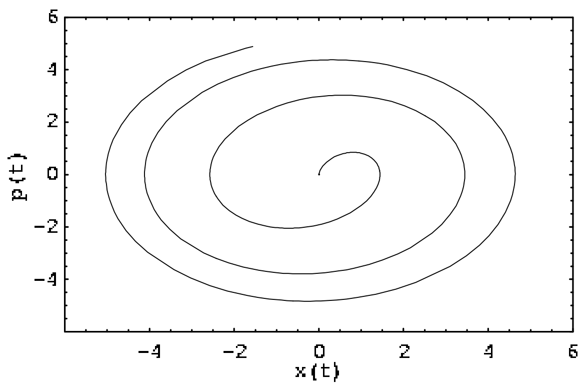

Figure 6: Phase plane diagram of equation (4.9) for $\alpha = 1.9$

where the forcing function in this case is the step function given by

$$f(t) = \begin{cases} A, & \text{for } t > 0\,, \\ 0, & \text{for } t < 0\,. \end{cases}$$

Substituting equation (4.4) and the initial conditions (4.5) into equation (3.7), yields the following recursive relations:

$$x_0 = a + J^\alpha A = a + A\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad x_{n+1} = -J^\alpha[\omega^\alpha x_n(t)].$$

Using the above recursive relationship, the first few terms of the decomposition series are given by

$$x_1 = -(\omega)^{\alpha}\Big[\frac{at^{\alpha}}{\Gamma(\alpha+1)} + \frac{At^{2\alpha}}{\Gamma(2\alpha+1)}\Big],$$
$$x_2 = (\omega)^{2\alpha}\Big[\frac{at^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{At^{3\alpha}}{\Gamma(3\alpha+1)}\Big],$$
$$x_3 = -(\omega)^{3\alpha}\Big[\frac{at^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{At^{4\alpha}}{\Gamma(4\alpha+1)}\Big],$$
$$\vdots$$

Substituting $x_0, x_1, x_2, ...$ into (3.8) gives the solution $x(t)$ in a series form solution by

$$x(t) = \sum_{n=0}^{\infty}(-\omega)^{n\alpha}\Big[\frac{at^{n\alpha}}{\Gamma(n\alpha+1)} + \frac{At^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\Big] \qquad (4.6)$$

To examine the effect of the step function $f(t) = A$, we choose $x(0) = a = 0$ and $A = 1$. Figure 4 shows the response function for $\alpha = 1.75$ and 1.9. The results here are very similar to the results of the previous example: The behavior of the driven fractional oscillator (DFO) for the step function is similar to the behavior of the damped oscillator.

Example 4.3. In this example we choose the forcing function to be the sinusoidal function $f(t) = \sin(\omega t)$ and write equation (3.1) as

$$\frac{d^{\alpha}x}{dt^{\alpha}} + \omega^{\alpha}x(t) = \sin(\omega t), \quad 1 < \alpha \leq 2, \qquad (4.7)$$

subject to the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \qquad (4.8)$$

As in the previous example, substituting equation (4.7) and the initial conditions (4.8) into equation (3.7), to obtain the following recursive relations:

$$x_0 = J^{\alpha}[sin(\omega t],$$
$$x_{n+1} = -J^{\alpha}[\omega^{\alpha}x_n(t)].$$

Since the complicated excitation term $f(t)$ can cause difficult fractional integrations and proliferation of terms, we can express $f(t)$ in Taylor series at $x_0 = 0$. Therefore, we replace $f(t)$ by

$$f(t) = \omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \frac{(\omega t)^7}{7!} + ...$$

Hence, the zeroth component is given by

$$x_0 = \frac{(\omega t)^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{(\omega t)^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{(\omega t)^{\alpha+5}}{\Gamma(\alpha+6)} - ....$$

It is easy now to determine the remaining terms of the decomposition series as follows:

$$\begin{aligned} x_1 &= -(\omega)^{\alpha}\Big[\frac{(\omega t)^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{(\omega t)^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{(\omega t)^{2\alpha+5}}{\Gamma(2\alpha+6)} - ....\Big], \\ x_2 &= (\omega)^{2\alpha}\Big[\frac{(\omega t)^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{(\omega t)^{3\alpha+3}}{\Gamma(3\alpha+4)} + \frac{(\omega t)^{3\alpha+5}}{\Gamma(3\alpha+6)} - ....\Big], \\ x_3 &= -(\omega)^{3\alpha}\Big[\frac{(\omega t)^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{(\omega t)^{4\alpha+3}}{\Gamma(4\alpha+4)} + \frac{(\omega t)^{4\alpha+5}}{\Gamma(4\alpha+6)} - ....\Big], \\ &\vdots \end{aligned}$$

and so on, in this manner the other components of the decomposition series can easily obtained of which the n^{th} term has the form

$$\begin{aligned} x_n = (-\omega)^{n\alpha}\Big[\frac{(\omega t)^{(n+1)\alpha+1}}{\Gamma((n+1)\alpha+2)} - \frac{(\omega t)^{(n+1)\alpha+3}}{\Gamma((n+1)\alpha+4)} + \frac{(\omega t)^{(n+1)\alpha+5}}{\Gamma((n+1)\alpha+6)} - ....\Big], \\ = (-\omega)^{n\alpha}\sum_{j=0}^{\infty}(-1)^j \frac{(\omega t)^{2j+1+(n+1)\alpha}}{\Gamma(2j+2+(n+1)\alpha)}. \end{aligned}$$

Substituting $x_0, x_1, x_2, ...$ into (3.9) gives the solution $x(t)$ in a series form solution by

$$x(t) = \sum_{n=0}^{\infty}(-\omega)^{n\alpha}\Big[\sum_{j=0}^{\infty}(-1)^j \frac{(\omega t)^{2j+1+(n+1)\alpha}}{\Gamma(2j+2+(n+1)\alpha)}\Big] \tag{4.9}$$

Figure 5 shows the response function for the sinusoidal function for $\alpha = 1.75$ and 1.9. Figure 6 shows the phase plane trajectory for the sinusoidal function for $\alpha = 1.9$. The results of our method for different values of α $(1 < \alpha \leq 2)$ and the sinusoidal function are in perfect agreement with the analytical solution obtain by Narahari et al [13] using the Laplace transform and Mittag-Leffler functions.

5. Conclusions

In this paper, the Adomian decomposition method proved to be of great interest for solving a system of driven fractional oscillators (DFO). The method provides the solution as infinite series of functions with easily computable components. All the examples show that the results of the present method are in excellent agreement with those obtained by Laplace transform and Mittag-Leffler functions.

The response functions of different force functions are obtained for different values of α $(1 < \alpha \leq 2)$. The numerical results showed that the behavior of the (DFO) is similar to the behavior of the damped harmonic oscillator. It may be concluded that the displacement functions are able to describe processes intermediate between exponential decay $(\alpha = 1)$ and pure sinusiodal oscillation $(\alpha = 2)$.

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