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WEAK LIFTING MODULES

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Abstract: In this paper we define weak lifting modules in two different ways with respect to general classes of modules, which is a generalization of lifting modules. We give the relations between lifting modules and weak lifting modules and characterize them.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary left R-modules. We use $N \leq M, N \ll M, N \leq_{cc} M$ and $N|M$ to indicate that N is a submodule, small submodule, coclosed submodule and direct summand of M respectively.

Let M be a module. Let N and L be submodules of M. N is called a supplement of L in M if $M = N+L$ and $N\cap L \ll N$. Let $B \leq A \leq M$, following Keskin [5], B is called a coessential submodule of A in M if $A/B \ll M/B$. A submodule A of M is called coclosed if A has no proper coessential submodule. Also, we will call B a coclosure of A in M, if B is a coessential submodule of A and B is coclosed in M.

Let M be a module. M is called a lifting module, if for any submodule N of

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M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. By Mohamed and Müller [6], Proposition 4.8, the module M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand.

In the view of the definitions and Keskin [5], Lemma 1.1, we know that the following statements are equivalent for an amply supplemented module M.

(i) M is a lifting module.

(ii) For every submodule N of M, every coclosure of N in M is a direct summand of M.

(iii) For every submodule N of M, every supplement of N in M is a direct summand of M.

Following Dogruoz and Smith [2], by a class of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphisms. If $\mathcal{X}$ is a class of R-modules and M is an R-module then an $\mathcal{X}$-submodule of M will be a submodule N of M such that N belongs to $\mathcal{X}$.

In Doğruöz and Smith [3], an R-module M is called weak type 1 $\mathcal{X}$-extending if for every $\mathcal{X}$-submodule N of M there exists a complement K of N in M such that $K|M$ and M is called weak type 2 $\mathcal{X}$-extending if for every $\mathcal{X}$-submodule N of M there exists a closure L of N in M such that $L|M$. Dually, we shall say that an R-module M is called weak type 1 $\mathcal{X}$-lifting if for every $\mathcal{X}$-submodule N of M there exists a supplement K of N in M such that $K|M$ and M is called weak type 2 $\mathcal{X}$-lifting if for every $\mathcal{X}$-submodule N of M there exists a coclosure L of N in M such that $L|M$. Clearly, if a module M is lifting, then it is weak type 1 $\mathcal{X}$-lifting and weak type 2 $\mathcal{X}$-lifting. Let $\mathcal{X} = \{0\}$, then all R-modules are weak type 1 $\mathcal{X}$-lifting and weak type 2 $\mathcal{X}$-lifting, but not all R-modules are lifting. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is not lifting (see Mohamed and Müller [6]). In this paper, we give the basic properties of weak type 1 $\mathcal{X}$-lifting modules and weak type 2 $\mathcal{X}$-lifting modules.

Following Ganesan and Vanaja [4], a module M is called a UCC module if every submodule has a unique coclosure in M. For example, semisimple modules, hollow modules are all examples of UCC modules. The $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is not UCC (see Ganesan and Vanaja [4]).

For other definitions we refer to [6], [1], [7].

2. Main Results

Proposition 1. *Let $\mathcal{X}$ be any class of R-modules. Then any type 1 (respectively, type 2) $\mathcal{X}$-lifting module is weak type 1 (weak type 2) $\mathcal{X}$-lifting.*

Proof. It is clear. □

Proposition 2. *Let $\mathcal{X}$ be any class of R-modules. Then any weak type 2 $\mathcal{X}$-lifting R-module is weak type 1 $\mathcal{X}$-lifting.*

Proof. Let M be any weak type 2 $\mathcal{X}$-lifting R-module. Let N be any $\mathcal{X}$-submodule of M. Then there exist submodules K and K' of M such that $M = K \oplus K'$ and $N/K \ll M/K$. Clearly $N = K \oplus (N \cap K')$ and $M = N + K'$. Hence $N \cap K' \ll K'$. It follows that the direct summand K' is a supplement of N in M. Hence M is weak type 1 $\mathcal{X}$-lifting. □

Lemma 1. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be classes of R-module. Then any type 1 (respectively, type 2, weak type 1, weak type 2) $\mathcal{Y}$-lifting module is type 1 (respectively, type 2, weak type 1, weak type 2) $\mathcal{X}$-lifting.*

Proof. Clear. □

Let M be any R-module and $\mathcal{X}$ be any class of R-modules. Define the family $\mathcal{X}(M)$ to be the set of all submodules N of M with $N \in \mathcal{X}$.

Define

$$\mathcal{X}^{ce}(M) = \{A \leq M | \exists B \in \mathcal{X}, B/A \ll M/A\}.$$

Note that $\mathcal{X}(M) \subseteq \mathcal{X}^{ce}(M)$.

Proposition 3. *For any class $\mathcal{X}$ of R-modules, an R-module M is weak type 1 $\mathcal{X}$-lifting if and only if M is weak type 1 $\mathcal{X}^{ce}$-lifting.*

Proof. Since $\mathcal{X}(M) \subseteq \mathcal{X}^{ce}(M)$, the sufficiency follows by Lemma 1. Conversely, let N be an $\mathcal{X}^{ce}$-submodule of M. There exists a submodule $L \in \mathcal{X}$ such that $L/N \ll M/N$. Since M is weak type 1 $\mathcal{X}$-lifting, there exists a supplement K of L in M such that $K|M$. Thus $M = K + L$ and $K \cap L \ll K$. Since $L/N \ll M/N$, $M = K + N$. Note that $K \cap N \ll K$. It follows that K is a supplement of N in M. Thus M is weak type 1 $\mathcal{X}^{ce}$-lifting. □

Lemma 2. *Let $\mathcal{X}$ be any class of R-modules and let M be an R-module. Then M is type 2 $\mathcal{X}^{ce}$-lifting if and only if M is weak type 2 $\mathcal{X}^{ce}$-lifting.*

Proof. The necessity is clear by Proposition 1. Conversely, let $N \in \mathcal{X}^{ce}(M)$ and K be a coclosure of N in M. Then $N/K \ll M/K$ and $K \leq_{cc} M$ and there exists an $\mathcal{X}$-submodule N_1 of N such that $N_1/N \ll M/N$. Thus $N_1/K \ll M/K$ by Ganesan and Vanaja [4], Lemma 2.5. Thus $K \in \mathcal{X}^{ce}$. By assumption there exists a direct summand L of M such that $K/L \ll M/L$. Therefore $K = L$ is a direct summand of M and M is type 2 $\mathcal{X}^{ce}$-lifting. □

Lemma 3. *Let $\mathcal{X}$ be any class of R-modules and let M is an amply supplemented UCC module. Then M is type 1 $\mathcal{X}$-lifting if and only if M is type 1 $\mathcal{X}^{ce}$-lifting.*

Proof. The sufficiency follows by Lemma 1. Now suppose that M is type 1 $\mathcal{X}$-lifting and N is an $\mathcal{X}^{ce}$-submodule of M. There exists an $\mathcal{X}$-submodule L such that $L/N \ll M/N$. Let K be a supplement of N in M. Since M is an amply supplemented UCC module, $(L\cap K)/(N\cap K) \ll M/(N\cap K)$ by Ganesan and Vanaja [4], Proposition 3.10. Note that $N\cap K \ll M$. Hence $L\cap K \ll M$ and so $L\cap K \ll K$ by Keskin [5], Lemma 1.1. Clearly $M = L + K$. Therefore K is a supplement of L in M. Thus $K|M$. Hence M is type 1 $\mathcal{X}^{ce}$-lifting. □

Lemma 4. *For any class $\mathcal{X}$ of R-module, an R-module M is type 2 $\mathcal{X}$-lifting if and only if M is type 2 $\mathcal{X}^{ce}$-lifting.*

Proof. The sufficiency follows by Lemma 1. Suppose that N is an $\mathcal{X}^{ce}$-submodule of M. There exists an $\mathcal{X}$-submodule L such that $L/N \ll M/N$. Let K be a coclosure of N in M. Then $N/K \ll M/K$ and $K \leq_{cc} M$. By Ganesan and Vanaja [4], Lemma 2.5, $L/K \ll M/K$. This means that K is a coclosure of L in M. Since M is type 2 $\mathcal{X}$-lifting, $K|M$. Hence M is type 2 $\mathcal{X}^{ce}$-lifting. □

Theorem 1. *For any class $\mathcal{X}$ of R-module, an R-module M is type 2 $\mathcal{X}$-lifting if and only if M is weak type 2 $\mathcal{X}^{ce}$-lifting.*

Proof. By Lemma 2 and Lemma 4. □

Note that if $\mathcal{X}$ is closed under coessential submodules, then $\mathcal{X}(M) = \mathcal{X}^{ce}(M)$.

Corollary. *Let $\mathcal{X}(M)$ be a class of R-modules which is closed under coessential submodules. Then M is type 2 $\mathcal{X}$-lifting if and only if M is weak type 2 $\mathcal{X}$-lifting.*

Proof. By Theorem 1. □

Proposition 4. *Let $\mathcal{X}$ be a class of R-modules. Then any weak type 2 $\mathcal{X}$-lifting UCC module is type 2 $\mathcal{X}$-lifting.*

Proof. Clear. □

Proposition 5. *Let $\mathcal{X}$ be a class of all R-modules. Then an amply supplemented R-module M is weak type 2 $\mathcal{X}$-lifting if and only if M is lifting.*

Proof. Suppose first that M is weak type 2 $\mathcal{X}$-lifting. Let K be any coclosed submodule of M. By hypothesis, there exists a direct summand L of M such that $K/L \ll M/L$ and hence $K = L$. Thus M is a lifting module. The converse is obvious. □

Theorem 2. *Let $\mathcal{X}$ be a class of semisimple R-modules. Let M_1 be a weak type 2 $\mathcal{X}$-lifting module and let M_2 be a semisimple module. Then $M = M_1 \oplus M_2$ is weak type 2 $\mathcal{X}$-lifting.*

Proof. Let S be any semisimple submodule of M. Note first that $S + M_1 = M_1 \oplus ((S + M_1) \cap M_2)$. Since M_2 is semisimple, the module $(S + M_1) \cap M_2$ is a direct summand of M_2. Thus $S + M_1$ is a direct summand of M. Now there exists a submodule S' of S such that $S = (S\cap M_1)\oplus S'$. Since M_1 is a weak type 2 $\mathcal{X}$-lifting, it follows that there exists a direct summand K of M_1 such that $(S\cap M_1)/K \ll M_1/K$. But $S+M_1 = (S\cap M_1)+S'+M_1 = M_1\oplus S'$. Thus $K\oplus S'$ is a direct summand of $S+M_1$ and so $K\oplus S'|M$. Clearly $(S\cap M_1)/K \ll M/K$.

By Ganesan and Vanaja [4], Lemma 2.5, $((S\cap M_1)\oplus S')/(K\oplus S') \ll M/(K\oplus S')$ and so $S/(K \oplus S') \ll M/(K \oplus S')$. Thus M is weak type 2 $\mathcal{X}$-lifting. □

Theorem 3. *Let $\mathcal{X}$ be a class of semisimple R-modules. Let M_1 be a weak type 2 $\mathcal{X}$-lifting module and let M_2 be a projective weak type 2 $\mathcal{X}$-lifting module. Then $M = M_1 \oplus M_2$ is weak type 2 $\mathcal{X}$-lifting.*

Proof. Let S be any semisimple submodule of M. Then there exists a submodule S' of S such that $S = (S\cap M_2)\oplus S'$. Note that $S'\cap M_2 = 0$, and hence $(M_2\oplus S')/S' \cong M_2$, an projective module. Let $f : M/S' \to (M_2+S')/S'$, $f(m_1+m_2 + S') = m_2 + S', m_1 \in M_1, m_2 \in M_2$. Then f is an epimorphism and so $(M_2 + S')/S'$ is a direct summand of M/S'. Hence $M/S' = ((M_2 \oplus S')/S') \oplus (M'/S'),\ M' \le M$. Note that $M = M_2 \oplus M'$. Thus $M' \cong M/M_2 \cong M_1$. It follows that M' is weak type 2 $\mathcal{X}$-lifting. There exists a direct summand K' of M' such that $S'/K' \ll M'/K'$. Since M_2 is weak type 2 $\mathcal{X}$-lifting, there exists a direct summand K of M_2 such that $(S \cap M_2)/K \ll M_2/K$. By Ganesan and Vanaja [4], Lemma 2.5, $((S\cap M_2)\oplus S')/(K\oplus K') \ll M/(K\oplus K')$ and so $S/(K \oplus K') \ll M/(K \oplus K')$. Clearly $K \oplus K'$ is a direct summand of $M_2 \oplus M' = M$. Thus M is weak type 2 $\mathcal{X}$-lifting. □

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