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ON THE MULTIPLICATION MAP FOR A LINE BUNDLE ON A CURVE

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Abstract: Let X be a smooth projective curve, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X,L)$ a linear subspace. Here we study the multiplication map $H^0(X,L) \otimes V \to H^0(X,L^{\otimes 2})$.

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Let X be an integral projective curve and $L \in \mathrm{Pic}(X)$. For any integer x such that $0 \le x \le h^0(X,L)$ let $G(x,H^0(X,L))$ denote the Grassmannian of all x dimensional linear subspaces of $H^0(X,L)$. For any $V \in G(x,H^0(X,L))$ let $\mu_{L,V} : H^0(X,L) \otimes V \to H^0(X,L^{\otimes 2})$ denote the multiplication map. Set $\beta(L,V) := \dim(\mathrm{Im}(\mu_{L,V})$. For any integer x such that $0 \le x \le h^0(X,L)$ let $\beta(L,x)$ be the maximal of all integers $\beta(L,V)$ for $V \in G(x,H^0(X,L))$. In this way we obtained a non-decreasing sequence of integers $\{\beta(L,x)\}$, $0 \le x \le h^0(X,L)$. Set $\mu(L) := \mathrm{Im}(\mu_{L,H^0(X,L)})$. To avoid trivialities we always assume $h^0(X,L) \ge 2$. The pair (L,V) is said to have maximal rank if for every integer $t \ge 2$ the symmetric multiplication map $\sigma^t_{L,V} : S^t(V) \to H^0(X,L^{\otimes t})$ has maximal rank, i.e. it is either injective or surjective. We will say that the pair (L,V) has the expected projectively normal closure if for every integer $t \ge 2$ the multiplication map $H^0(X,L^{\otimes t}) \otimes V \to H^0(X,L^{\otimes t+1})$ is surjective, while the multiplication map $\mu_{L,V}$ either is surjective or $\mathrm{Ker}(\mu_{L,V}) = \wedge^2(V)$. Notice

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that for arbitrary L, V we always have $\wedge^2(V) \subseteq \mathrm{Ker}(\mu_{L,V})$.

Theorem 1. *Fix integers d, g, x such that $g \ge 0$, $d - g \ge x \ge 2$, $d \ge g + 3$ and $d \le (-1 + \sqrt{8g+1})/2$. Let X be a general genus g smooth curve and L a general element of $\mathrm{Pic}^d(X)$. A general $V \in G(x, H^0(X, L))$ spans L. For any $V \in G(x, H^0(X, L))$ spanning L the multiplication map $\alpha_t : H^0(X, L^{\times t}) \otimes V \to H^0(X, L^{\times t+1})$ is surjective for all $t \ge 2$, while $\mathrm{Ker}(\mu_{L,V}) = \wedge^2(V)$. Now assume $x \ge 4$ and V general in $G(x, H^0(X, L)$. Then the map induced by V is very ample and the image $C \subset \mathbf{P}^{x-1}$ is a general non-special degree d embedding of a general genus g curve. The pair (L, V) has maximal rank and the expected projectively normal closure*

Question 1. Fix any $m \in \mu(L)$. What is the minimal integer x such that $x \in \mu_{L,V}$ for some $V \in G(x, H^0(X, L))$. Call $\gamma(L, m)$ this integer. Set $\mu(L, x) := \{m \in \mu(L) : \gamma(m) = x\}$. Study this partion of $\mu(L)$ for suitable X, L.

Remark 1. Every element of $\mu(L)$ is a linear combination of at most $h^0(X, L)$ elements of $\mu(L, 1)$. Hence $\mu(L, 1)$ spans $\mu(L)$.

Proposition 1. *Fix an integer x such that $0 < x \le h^0(X, L) - 2$ and take any $V \in G(x, H^0(X, L))$. If $\beta(L, W) = \beta(L, V)$ for a general $W \in G(x + 1, H^0(X, L))$ such that $V \subset H^0(X, L)$, then $\mu_{L,V} = \mu(L)$.*

Proof. By semicontinuity we have $\mu_{L,V} = \mu_{L,W}$ for all $W \in G(x+1, H^0(X, L))$ such that $V \subset H^0(X, L)$. Take as W the linear span of V and all $M \in G(1, H^0(X, L)$ and apply Remark 1. □

From Proposition 1 we immediately get the following result.

Corollary 1. *Assume $x \le h^0(X, L) - 2$ and $\beta(L, x) = \beta(L, x + 1)$. Then $\beta(L, x) = \beta(L, h^0(X, L))$.*

For any linear subspace $V \subset H^0(X, L)$ let $\sigma^2_{L,V} : S^2(X) \to H^0(X, L^{\otimes 2})$ denote the symmetric multiplication map. Notice that $\mathrm{Im}(\sigma^2_{L,V}) \subseteq \mu_{L,V} \subseteq \mu(L)$ and that $\mathrm{Im}(\sigma^2_{L,H^0(X,L)}) = \mu(L)$.

Question 2. Fix any linear subspace $V \subset H^0(X, L)$ such that $\mathrm{Im}(\sigma^2_{L,V}) \ne \mu(L)$. Is $\mathrm{Im}(\sigma^2_{L,V}) \ne \mu_{L,V}$?

We may prove the following partial result for general X, general L and $V \in G(x, H^0(X, L))$.

Proposition 2. *Fix integers d, g, n, x such that $4 \le x \le d - g$ and $\binom{x+1}{2} \le 2d - g$. Let X be a general curve of genus g, L a general element of $\mathrm{Pic}^d(X)$*

and V a general element of $G(x, H^0(X,L))$. Then $\dim(\mathrm{Im}(\sigma^2_{L,V})) = \binom{x+1}{2}$ *and* $\mathrm{Im}(\sigma^2_{L,V}) \subsetneqq \mathrm{Im}(\mu_{L,V})$.

Proof. Since $d \ge g+3$ the generality of L implies $h^1(X,L) = 0$, that L is very ample and that $u_L(X) \subset \mathbf{P}^{d-g}$ is a general non-special curve with degree d and genus g. The natural map $\sigma^2_{L,H^0(X,L)}$ has maximal rank, i.e. it is either injective or surjective (see [2]). Hence $\mu(L) = H^0(X, L^{\otimes 2}) = H^0(X, L^{\otimes 2})$ (and hence $\dim(\mu(L)) = 2d+1-g$ if $\binom{d-g+2}{2} \ge 2d+1-g$, while $\dim(\mu(L)) = \binom{d-g+2}{2}$ if $\binom{d-g+2}{2} \le 2d+1-g$. Since $\binom{x+1}{2} < 2d+1-g$ and V is general, $\sigma^2_{L,V}$ is injective and $\mathrm{Im}(\sigma^2_{L,V}) \ne \mu(L)$. By semicontinuity it is sufficient to find $W \in G(x, H^0(X,L))$ such that $\mathrm{Im}(\sigma^2_{L,W}) = \binom{x+1}{2} < \beta(L,x)$. Fix a general $S \subset X$ such that $\sharp(S) = d-g-x$ and set $M := L(-S)$. By the generality of S and L we have $h^0(X,L) = x \ge 4$ and M may be seen as a general element of Pic^{x+g-1}. Hence M is very ample and it gives an embedding $h_M : X \to \mathbf{P}^{x-1}$. The multiplication by S induces an inclusion $H^0(X,M) \to H^0(X,L)$ and we take as W the image of this inclusion. Hence $W \in G(x, H^0(X,L))$. The generality of X and M and [1] (case $x = 4$) or [2] (case $x \ge 5$) imply that $\sigma^2_{L,W}$ is injective. Notice that each point of S appears with multiplicity 2 in the base locus of the linear system $|\sigma^2_{L,W}| \subset |L^{\otimes 2}|$. The very ampleness of L and M implies that each point of S appears with multiplicity 1 in the base locus of the linear subsystem of $|L^{\otimes 2}|$ induced by $\mu_{L,W}$. Thus $\mathrm{Im}(\sigma^2_{L,W}) \ne \mathrm{Im}(\mu_{L,W})$. □

Proof of Theorem 1. Since $d \ge g+3$, L is very ample and non-special. Let $h_L : X \to \mathbf{P}^{d-g}$ the embedding associated to the complete linear system $|L|$. The generality of the pair (X,L) implies that $h_L(X)$ has maximal rank (see [1]). Since $d \le (-1+\sqrt{8g+1})/2$, this implies that $h_L(X)$ is contained in no quadric, i.e. $\mathrm{Ker}(\mu_{L,H^0(X,L)}) = \wedge^2(H^0(X,L))$. Since $h^1(X, L^{\otimes t-1}) = 0$ for all $t \ge 2$, the base point free pencil trick gives the surjectivity of the multiplication map $H^0(X, L^{\otimes t}) \otimes V \to H^0(X, L^{\otimes t+1})$ for all integers $t \ge 2$ and all V spanning L. Take a basis e_i, $1 \le i \le d+1-g$ of $H^0(X,L)$ such that e_i, $1 \le i \le x$, is a basis of V. Use the Kronecker basis $e_i \otimes e_j$ of $H^0(X,L) \otimes H^0(X,L)$. With this basis every element α of $H^0(X,L) \otimes H^0(X,L)$ is given by a square matrix $\widetilde{\alpha} = (a_{i,j})$, $1 \le i \le d+1-g$, $1 \le j \le d+1-g$. $\alpha \in H^0(X,L) \otimes V$ if and only if $a_{i,j} = 0$ for all $j > x$. $\alpha \in \wedge^2(H^0(X,L))$ if and only if $\widetilde{\alpha}$ is antysymmetric. Thus $\wedge^2(H^0(X,L)) \cap H^0(X,L) \otimes V$ may be identified with $\wedge^2(V)$. Since $\mathrm{Ker}(\mu_{L,H^0(X,L)}) = \wedge^2(H^0(X,L))$, $\mathrm{Ker}(\mu_{L,V}) = \wedge^2(V)$. Now assume that $x \ge 4$ and that V is general in $G(x, H^0(X,L))$. The pair (L,V) has maximal rank by [1] (case $x = 4$) and [2] (case $x \ge 5$). □

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