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RELATIONS BETWEEN THE CONFEDERATE MATRIX AND THE COMPANION MATRIX OF THE POLYNOMIAL

Yan Lai Liang [§]
Department of Mathematics and Computer Science
Yulin Teachers College
Yulin, Guang-Xi, 537000, P.R. CHINA
and
Department of Applied Mathematics
Dalian University of Technology
Dalian, Liao Ning, 116024, P.R. CHINA
e-mail: yulinsjxlyl@163.com

Abstract: In this paper, we study the relations between the confederate matrix and the companion matrix of the polynomial by defining some linear operators at the polynomial space $C_n[x]$. Similarity formulas of the two matrices are obtained.

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1. Introduction

For the polynomial basis as a basic tool for the study on the theories as matrix and operator theories, polynomial and rational interpolation theories, from the import in 19-th century, there is a mass of study achievements as stated by literatures. In 1980s, S. Barnett and his co-authors used the standard power basis of polynomial and the associated matrices to explain the properties of Bezout matrix, Hankel matrix, and Vandermonde matrix, and the problems concerning inversion and diagonalization of these matrices; and they applied the polyno-

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[§]Correspondence address: Department of Applied Mathematics, Dalian University of Technology, Dalian, Liao Ning, 116024, P.R. CHINA

mial basis to the fields of information system theory and linear control system theory etc. (refer to [1]-[2]). In this paper, we discuss some important matrices by defining some linear operators at the polynomial space $C_n[x]$, and these important matrices are: confederate matrix, companion matrix, and upper displacement matrix. The paper studies the relations among these matrices and gets their similarity formula.

2. Definition and Lemma

Let $C_n[x]$ stand for the n-dimensional linear space of complex coefficient polynomial with degree less than n, and let $P(x) = (1, x, x^2, \cdots, x^{n-1})$ be the standard power basis of $C_n[x]$, and $\{Q_k(x)\}_{k=0}^{n}$ be a sequence of polynomials with degree $Q_k(x) = k(k = 0, 1, \cdots, n)$. Set

$$Q(x) = (Q_0(x), Q_1(x), \cdots, Q_{n-1}(x)). \tag{1}$$

Evidently, $Q(x)$ is a basis of $C_n[x]$, which is called a general polynomial basis.

Definition 1. For a polynomial

$$p(x) = \sum_{k=0}^{n} p_k x^k \ (p_n \neq 0), \tag{2}$$

where p_k in C. The matrix

$$C(p) = \begin{bmatrix} 0 & 0 & \dots & 0 & -\frac{p_0}{p_n} \\ 1 & 0 & \dots & 0 & -\frac{p_1}{p_n} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -\frac{p_{n-1}}{p_n} \end{bmatrix} \tag{3}$$

is called the secondary companion matrix of $p(x)$.

I. Gohberg and al had discussed the properties of the first and secondary Chebyshev polynomial basis

$$T(x) = (T_0(x), T_1(x), \dots, T_{n-1}(x)) \text{ and } U(x) = (U_0(x), U_1(x), \dots, U_{n-1}(x)),$$

which satisfy the recurrence relations following of

$$\begin{array}{lll} T_0(x) = 1, & T_1(x) = x, & T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \ k \geq 2, \\ U_0(x) = 1, & U_1(x) = 2x, & U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x), \ k \geq 2, \end{array} \tag{4}$$

and the relations between some matrices with respect to $T(x)$ and $U(x)$, and got the inversion formula of Chebyshev-Vandermonde matrix. But their studies are limited to the recurrence relations (4) stated above. For detailed discussion please see [3].

Now we will extend the conceptions as the classic companion matrix under the standard power basis to a general polynomial basis with the recurrence relations of any polynomial items, then we will discuss the relations among these matrices.

Firstly, let the polynomial sequence $\{Q_k(x)\}_{k=0}^n$ satisfy the recurrence relations following of

$$\begin{aligned}&Q_0(x) = \alpha_0, \quad Q_1(x) = \alpha_1 x Q_0(x) - a_{01}Q_0(x),\\ &Q_k(x) = \alpha_k x Q_{k-1}(x) - a_{k-1,k}Q_{k-1}(x) - \cdots - a_{1,k}Q_1(x) - a_{0,k}Q_0(x),\\ &2 \le k \le n,\end{aligned} \tag{5}$$

where all coefficients $a_{i,k}$ and $\alpha_k(\neq 0)$ are determined beforehand.

Definition 2. For a polynomial $p(x) = \sum\limits_{k=0}^{n} p_k x^k = \sum\limits_{k=0}^{n} \widetilde{p}_k Q_k(x)$ ($p_n \neq 0, \widetilde{p}_n \neq 0$), we define its confederate matrix as

$$C_Q(p) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \dots & \dots & (\frac{a_{0n}}{\alpha_n} - \frac{1}{\alpha_n}.\frac{\widetilde{p}_0}{\widetilde{p}_n}) \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \dots & \dots & (\frac{a_{1n}}{\alpha_n} - \frac{1}{\alpha_n}.\frac{\widetilde{p}_1}{\widetilde{p}_n}) \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \dots & \dots & (\frac{a_{2n}}{\alpha_n} - \frac{1}{\alpha_n}.\frac{\widetilde{p}_2}{\widetilde{p}_n} \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{\alpha_{n-1}} & (\frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n}.\frac{\widetilde{p}_{n-1}}{\widetilde{p}_n}) \end{bmatrix} \tag{6}$$

with respect to the general basis $Q(x)$ (see [4]).

Consider another polynomial sequence $\{Q_k(x)\}_{k=0}^n$ which satisfies the recurrence relations following of

$$\begin{gathered} Q_0(x) = \delta_0, \quad Q_1(x) = \delta_1 Q_0(x) - w_{12} x Q_0(x), \\ Q_k(x) = \delta_k Q_0(x) + x \sum_{i=1}^{k} w_{i,k+1} Q_{i-1}(x), \quad 2 \le k \le n, \end{gathered} \tag{7}$$

where all coefficients $w_{i,k+1}$ and δ_k are determined beforehand, and $w_{i,k+1} \neq 0$.

We import the matrix relating only to $Q(x)$

$$W_Q = \begin{bmatrix} 0 & w_{12} & w_{13} & \dots & w_{1n} \\ 0 & 0 & w_{23} & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & w_{n-2,n} \\ \vdots & \vdots & \ddots & \ddots & w_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (8)$$

When $Q(x)$ is the standard power basis $(1, x, x^2, \dots, x^{n-1})$, W_Q will degenerate to the following so-called upper displacement matrix

$$Z_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (9)$$

The confederate matrix $C_Q(p)$ and matrix W_Q of a polynomial play an important role in the displacement structure of the generalized Bezout matrix, Vandermonde matrix with a general polynomial basis (see [6]). Below we will discuss the relations among the matrices mentioned above. Thus we give the following lemmas firstly.

Lemma 1. *For the polynomial $p(x)$ with the given form as (2), we define the linear operator σ at $C_n[x]$*

$$\sigma(f) = xf(x) - \frac{f^{n-1}(0)}{(n-1)!} \cdot \frac{p(x)}{p_n}, \quad f(x) = \sum_{k=0}^{n-1} f_k x^k \in C_n[x]. \quad (10)$$

Then the matrix of σ under the standard power basis $P(x) = (1, x, x^2, \cdots, x^{n-1})$ of $C_n[x]$ is $C(p)$ in (3).

Proof. For any $g(x) \in C_n[x]$, when deg $g(x) < n-1$, since $g^{n-1}(0) = 0$, we have $\sigma(g) = xg(x)$. Thus:

$$\begin{aligned} &\sigma(1) = x, \quad \sigma(x^k) = x^{k+1} \quad (k = 1, 2, \cdots, n-2), \\ &\sigma(x^{n-1}) = x^n - \frac{(n-1)!}{(n-1)!} \cdot \frac{p_0 + p_1 x + \cdots + p_{n-1}x^{n-1} + p_n x^n}{p_n} \\ &\qquad = -\frac{p_0}{p_n} - \frac{p_1}{p_n}x - \cdots - \frac{p_{n-1}}{p_n}x^{n-1}. \end{aligned}$$

Therefore, the matrix of σ under the standard power basis of $C_n[x]$ is the $C(p)$ in (3). □

Lemma 2. *For $f(x)=\sum_{k=0}^{n-1} f_k x^k \in C_n[x]$, we define τ following of*

$$\tau(f)=\frac{f(x)-f(0)}{x}. \tag{11}$$

Then τ is a linear operator at $C_n[x]$, and the matrix of τ under the standard power basis $P(x)=(1,x,x^2,\cdots,x^{n-1})$ of $C_n[x]$ is the upper displacement matrix Z_0 in (9).

Proof. It is easy to prove that τ is a linear operator at $C_n[x]$. From

$$\tau(1)=0,\tau(x^k)=x^{k-1}(k=1,2,\cdots,n-1),$$

we may get this conclusion. □

3. Main Results

Theorem 1. *Suppose the sequence of polynomial $\{Q_k(x)\}_{k=0}^{n}$ satisfies the recurrence structure* (5)*, for a given polynomial*

$$p(x)=\sum_{k=0}^{n} p_k x^k=\sum_{k=0}^{n} \widetilde{p}_k Q_k(x) \quad (p_n\neq 0,\widetilde{p}_n\neq 0).$$

Its confederate matrix $C_Q(p)$ is similar with the secondary companion matrix $C(p)$, and they have the following relation

$$C_Q(p)=M^{-1}C(p)M\,, \tag{12}$$

where M denote the transition matrix from the standard power basis $P(x)$ to the general polynomial basis $Q(x)$, and it is an upper triangular matrix.

Proof. By considering the linear operator σ in Lemma 1, we note that the coefficients of the recurrence relations (5) and the coefficients of the polynomial $p(x)$ have the relation

$$p_n=\widetilde{p}_n\alpha_1\alpha_2\cdots\alpha_n.$$

Thus we get

$$\sigma(f)=xf(x)-\frac{f^{(n-1)}(0)}{(n-1)!}\cdot\frac{p(x)}{\widetilde{p}_n\alpha_1\alpha_2\cdots\alpha_n}, f(x)=\sum_{k=0}^{n-1} f_k x^k\in C_n[x].$$

From (5) we may get

$$\alpha_0 = Q_0(x), xQ_{k-1}(x) = \frac{1}{\alpha_k}Q_k(x) + \frac{1}{\alpha_k}\sum_{i=0}^{k-1} a_{i,k}Q_i(x)(k = 1, 2, \cdots, n)\,.$$

Since $\deg Q_k(x) = k(k = 0, 1, \cdots, n-1)$, so

$$\sigma(Q_0(x)) = xQ_0(x) = \frac{1}{\alpha_1}Q_1(x) + \frac{a_{01}}{\alpha_1}Q_0(x),$$

when $k = 1, 2, \cdots, n-1$, we have

$$\sigma(Q_{k-1}(x)) = xQ_{k-1}(x) = \frac{1}{\alpha_k}Q_k(x) + \frac{1}{\alpha_k}\sum_{i=0}^{k-1} a_{i,k}Q_i(x),$$

and since the first coefficient of $Q_{n-1}(x)$ is $\alpha_0\alpha_1\cdots\alpha_{n-1}$, we may get

$$\sigma(Q_{n-1}(x)) = xQ_{n-1}(x) - \frac{1}{\alpha_n}\cdot\frac{1}{\widetilde{p}_n}p_n(x)$$

$$= (\frac{a_{0n}}{\alpha_n} - \frac{1}{\alpha_n}\cdot\frac{\widetilde{p}_0}{\widetilde{p}_n})Q_0(x) + \cdots + (\frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n}\cdot\frac{\widetilde{p}_{n-1}}{\widetilde{p}_n})\cdot Q_{n-1}(x).$$

So, the matrix of σ under the general basis $Q(x) = (Q_0(x), Q_1(x), \cdots, Q_{n-1}(x))$ of $C_n[x]$ is the $C_Q(p)$ in formula (6).We know $C_Q(p)$ is similar with $C(p)$ from Lemma 1, and

$$C_Q(p) = M^{-1}C(p)M,$$

where M is the transition matrix from the standard power basis $P(x)$ to the general polynomial basis $Q(x)$. □

Especially, when $Q(x)$ takes the standard power basis $(1, x, x^2, \cdots, x^{n-1})$, $C_Q(p)$ will degenerate to the secondary companion matrix $C(p)$ of $P(x)$.

Theorem 2. *Suppose the polynomial sequence* $\{Q_k(x)\}_{k=0}^n$ *satisfies the recurrence formula* (7), *then the matrix* W_Q *in the formula* (8) *is similar with* Z_0 *in* (9), *and they have the following relation*

$$W_Q = M^{-1}Z_0M \tag{13}$$

of the above, the meaning of M *is the same as the Theorem 1.*

Proof. Consider the linear operator τ in Lemma 2. For any $g(x) \in C_n[x]$, if $\deg g < n-1$, then $\tau(xg) = g$, and from the recurrence formula (7) and the definition of τ we know that $\tau(Q_0(x)) = 0$,

$$\tau(Q_k(x)) = \sum_{i=1}^{k} w_{i,k+1}Q_{i-1}(x) \quad (k = 1, 2, \cdots, n-1).$$

Thus, the matrix of τ under the general basis $Q_(x)$ of $C_n[x]$ is the matrix W_Q in the formula (8). From Lemma 2 we get that W_Q and Z_0 are similar, and have

$$W_Q = M^{-1} Z_0 M. \qquad \square$$

Furthermore, by the similarity properties of the matrices, we may get the following deductions.

Corollary 1. *Suppose $h(x)$ be a complex coefficient polynomial, then the matrix $h(C_Q(p))$ and $h(C(p))$, $h(W_Q)$ and $h(Z_0)$ are similar, respectively.*

Corollary 2. *Suppose the meaning of $\{Q_k(x)\}_{k=0}^{n}$ is the same as that in Theorem 1, and suppose the polynomial sequence $\{R_k(x)\}_{k=0}^{n}$, $\deg R_k(x) = k$ $(k = 0, 1, \cdots, n)$ satisfies the recurrence relations similar to formula (5), let $C_Q(p)$ and $C_R(p)$ be the confederate matrices of $p(x)$ with respect to the general polynomial basis $Q_(x)$ and $R(x)$, respectively, then $C_Q(p)$ and $C_R(p)$ are similar, and they have the following relation*

$$C_Q(p) = T^{-1} C_R(p) T,$$

where T denote the transition matrix from the general polynomial basis $Q(x)$ to $R(x)$.

References

[1] S. Barnett, *Polynomials and Linear Control System*, Dekker, New York (1983).

[2] S. Barnett, A Bezoutian matrix for Chebyshev polynomials, In: *Applications of Matrix Theory*, Oxford University Press, New York (1989).

[3] I. Gohberg, V. Olshevsky, Fast inversion of Chebyshev-Vandermonde matrices, *Numer. Math.*, **67**, No. 1 (1994), 71-92.

[4] T. Kailath, V. Olshevsky, Displacement structure approach to polynomial Vandermonde and related matrices, *Linear Algebra Appl.*, **261** (1997), 49-90.

[5] J. Maroulas, S. Barnett, Polynomials with respect to a general basic (I): Theory, *J. Math. Anal. Appl.*, **72** (1979), 177.

[6] Z.H. Yang, Polynomial Bezoutian matrix with respect to a general basic, *Linear Algebra and its Application*, **331** (2001), 165-179.