International Journal of Pure and Applied Mathematics
Volume 37 No. 1 2007, 51-53

THE MONODROMY GROUP OF THE HYPERPLANE SECTION OF THE JOIN OF d GENERIC POINTS IN A PROJECTIVE SPACE

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Abstract: Fix a hyperplane $H \subset \mathbf{P}^n$. For all general $(P_1, \dots, P_d) \in (\mathbf{P}^n)^d$ let X_d be the union of all lines $\langle \{P_i, P_j\} \rangle$, $1 \le i < j \le d$. Hence $X_d \cap H$ is the union of $\binom{d}{2}$ points. Moving the points $P_1, \dots, P_d$ we get a permutation group $G_{d,n}$ on $\{1, \dots, \binom{d}{2}\}$: the Galois (or monodromy) group of the join of d generic points of $\mathbf{P}^n$. We study this primitive (but not 2-transitive) permutation group.

AMS Subject Classification: 14N05
Key Words: monodromy group, generic Galois group, hyperplane section, secant variety

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Fix integers $d \ge 4$ and $n \ge 3$. Let $P_1 \dots, P_d \in \mathbf{P}^n$ be d distinct points such that the secant variety $S(\{P_1, \dots, P_s\})$ of it (i.e. the set of all lines $\langle \{P_i, P_j\} \rangle$ with $1 \le i \le j$) is formed by $d(d-1)/2$ distinct lines. Equivalently, assume that no plane contains at least 4 of the points $P_1, \dots, P_d$. $S(\{P_1, \dots, P_s\})$ is a reduced curve and hence one may consider the Galois (or monodromy) group of its generic hyperplane section in the classical sense explained in [1], Chapter 3. However, it is very easy to check that this group is the trivial group. Here

Received: February 27, 2007

we will consider a different monodromy group. We fix a hyperplane $H \subset \mathbf{P}^n$ and move the points $P_1 \dots, P_d \in \mathbf{P}^n$ in a Zariski open subset of the symmetric product of d copies of $\mathbf{P}^n$. Then take the intersection $S(\{P_1, \dots, P_s\}) \cap H$. As explained in the introduction of [2] we get a group $G_{d,n}$ and we will say that $G_{d,n}$ is the Galois (or monodromy) group of a hyperplane section of the join of a generic set of d points of $\mathbf{P}^n$. Set $X_{d,n} := S(\{P_1, \dots, P_s\})$ when $\{P_1 \dots, P_d\}$ is generic in the symmetric product of d copies of $\mathbf{P}^n$. Set $T := X_{d,n} \cap H$. Hence $G_{d,n}$ is a subgroup of the full permutation group $S_{d(d-1)/2}$. We will write $[P_i, P_j]$, $1 < i < j \leq d$ for the element $\langle \{P_i, P_j\} \rangle \cap H$ on which $G_{d,n}$ acts.

It is easy to check that $G_{d,n}$ is 1-transitive. Here we will show that it is not 2-transitive. Fix $[P, P']$ and two different elements $[Q, Q']$ and $[A, A']$. Let E be the stabilizer of $[P, P']$ in $G_{d,n}$. Since $G_{d,n}$ is transitive, E has index $d(d-1)/2$ in $G_{d,n}$. Notice that $3 \leq \sharp(\{P, P', Q, Q'\}) \leq 4$ and $3 \leq \sharp(\{P, P', A, A'\}) \leq 4$. There are $2d-4$ pairs $[B, B']$ such that $\sharp(\{P, P', B, B'\}) = 3$. There are $(d-2)(d-3)/2$ pairs $[B, B']$ such that $\sharp(\{P, P', B, B'\}) = 4$. We easily get that E has two orbits on $T \backslash \{[P, P']\}$: one orbit with cardinality $2d-4$ corresponding to all $[Q, Q']$'s such that $\sharp(\{P, P', Q, Q'\}) = 3$ and one orbit with cardinality $(d-2)(d-3)/2$ corresponding to all $[Q, Q']$'s such that $\sharp(\{P, P', Q, Q'\}) = 4$. Since $(d-2)(d-3)/2 \neq 2d-4$, $G_{d,n}$ is not a primitive transformation group in the sense of [3]. Now assume $d \geq 6$. Let F (resp. F') the stabilizer in $G_{d,n}$ of the points $[P_1, P_2]$ and $[P_1, P_3]$ (resp. $[P_1, P_2]$ and $[P_3, P_4]$. F (resp. F') has index $2d-4$ (resp. $(d-2)(d-3)/2$ in E. The action of F on $T \backslash \{[P_1, P_2], [P_1, P_3]\}$ has 3 orbits which may be described in the folowing way:

(a) $\{[P_1, P_j]\}$, $4 \leq j \leq d$; this orbit has cardinality $d-3$;

(b) $\{P_2, P_j], [P_3, P_j]\}$, $4 \leq j \leq d$; this orbits has cardinality $2d-6$;

(c) $\{[P_i, P_j]\}$, $4 \leq i < j \leq d$; this orbit has cardinality $\binom{d-3}{2}$.

The action of F' on $T \backslash \{[P_1, P_2], [P_3, P_4]\}$ has 3 orbits which may be described in the folowing way:

(i) $\{[P_1, P_j], [P_2, P_j], [P_3, P_j], [P_4, P_j]\}$, $5 \leq j \leq d$; this orbit has cardinality $4d-16$;

(ii) $\{P_1, P_3], [P_2, P_3], [P_1, P_4], [P_2, P_4]\}$; this orbits has cardinality 4;

(iii) $\{[P_i, P_j]\}$, $5 \leq i < j \leq d$; this orbit has cardinality $\binom{d-4}{2}$.

And so on. Notice that the action of F' on the orbit (iii) is just isomorphic to the action of $G_{d-4,n}$ on $X_{d,n} \cap H$. Thus we get the folowing result.

Theorem 1. *$\sharp(G_{d,n}) = [d(d-1)(d-2)(d-3)/4] \cdot \sharp(G_{d-4,n})$ for all $d \geq 6$.*

Now we consider higher order secant varieties of $\{P_1, \ldots, P_s\} \subset \mathbf{P}^n$. We assume $d \geq n+1$ and that the points $P_1, \ldots, P_d$ are il linearly general position. We fix an integer t such that $1 \leq t-1 \leq n-1$. Fix a codimension t linear subspace $H_t \subset \mathbf{P}^n$. Let $S^t(\{P_1, \ldots, P_s\})$ denote the union of all t-linear subspaces of $\mathbf{P}^n$ spanned by $t+1$ of the points $P_1, \ldots, P_d$. Hence $S^1(\{P_1, \ldots, P_s\}) = S(\{P_1, \ldots, P_s\})$. Notice that $H_t \cap S^t(\{P_1, \ldots, P_s\})$ is finite and $\sharp(H_t \cap S^t(\{P_1, \ldots, P_s\}) = \binom{d}{t+1}$ for general $P_1, \ldots, P_d$. Set $X_{d,n,t} := S^t(\{P_1, \ldots, P_s\})$ for generic $\{P_1, \ldots, P_d\}$. Moving the points $P_1, \ldots, P_d$ we get a permutation group $G_{d,n,t} \subseteq S_{\binom{d}{t+1}}$ acting on the finite set $X_{d,n,t} \cap H_t$. It is easy to check that $G_{d,n,t}$ is transitive.

We work over an algebraically closed field $\mathbb{K}$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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