International Journal of Pure and Applied Mathematics
Volume 37 No. 1 2007, 55-64

OSCILLATION CRITERIA FOR SECOND ORDER IMPULSIVE DIFFERENCE EQUATIONS WITH DELAY

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Abstract: In this paper, we are concerned with the oscillation criteria of second order impulsive difference equation with delay. By using the discrete vision of impulsive inequality and mathematical analytical techniques, sufficient conditions of oscillation of impulsive delay difference equations are obtained. Some examples are also inserted to illustrate the effect of impulses.

AMS Subject Classification: 35B05, 47A10
Key Words: oscillation, impulses, second order, delay difference equation

1. Introduction

In recent years, there has been intensively studied on the theory of differential equation (see [1], [3], [4], [7], [6], [2]), as we know the impulsive differential equation are the basic tool to studying evolution processes that are subjected to abrupt changes in the state, for instance, many biological, physical and engineering application exhibit impulsive effect (see [1], [3]). there are several papers devoted to the oscillation of second order impulsive delay difference equation, see [5], while impulsive difference equations sometimes can exactly reflect the real world. Moreover, fewer papers were devoted to the oscillation

Received: February 27, 2007

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criteria of impulsive delay difference equations with first order difference term, it is worth mentioning that magnitude of impulses may greatly alter the state.

In this paper, we research the following second order impulsive delay differential equation

$$\begin{cases} \triangle(r_{n-1}\triangle x(n-1)) + p_{n-1}\triangle x(n-1) + f(n, x(n), x(n-l)) = 0, \\ n \geq n_0, \quad n \neq n_k, \quad k = 1, 2, \dots, \\ \triangle x(n_k+1) = M_k(\triangle x(n_k)), \quad n = n_k,\ k = 1, 2, \dots, \\ x(n_k+1) = N_k(x(n_k)), \quad n = n_k,\ k = 1, 2, \dots, \end{cases} \tag{1.1}$$

where $\triangle$ denotes the forward difference operator, i.e. $\triangle x_n = \triangle x(n) = x(n+1) - x(n)$. N is the natural number set, $l \in N$, $0 \leq n_0 < n_1 < \cdots < n_k < \cdots$, and $\lim_{k\to+\infty} n_k = +\infty$.

For convince, we give some following notations, let

$$N[n_1, n_2] = \{n \mid n \in N, n_1 \leq n \leq n_2\},\ N[n_1, n_2) = \{n \mid n \in N, n_1 \leq n < n_2\},$$

$$N[n_1, \infty) = \{n \mid n \in N, n \geq n_1\},$$

$$A(n-1) = \sum_{s=n_0}^{n-1} \frac{p_{s-1} + \triangle r_{s-1}}{r_s}, \quad B(n-1) = \sum_{s=n_0}^{n-1} \frac{\triangle r_{s-1}}{r_s}$$

By a solution of (1.1) we mean a real valued sequence $\{x(n)\}$ defined on $N[n_0 - l, \infty)$ which satisfies (1.1) for $n \geq n_0$. It is obvious that (1.1) has a unique solution $\{x(n)\}_{n_0-l}^{\infty}$, under the initial conditions:

$$x_i = y_i, \quad i = n_0 - l, \cdots n_0,$$

where $y_i (i = n_0 - l, \cdots n_0)$ are given real constants.

In Section 2, we put some lemmas, some conditions and the definition of oscillation; main results are arranged in Section 3. Some examples are given in Section 4.

2. Some Conditions and Some Lemmas

Definition. A solution of (1.1) is said to be nonoscillatory, if the solution is eventually positive or eventually negative; otherwise, the solution is said to be oscillatory.

Here, we always assume that the following conditions hold:

(H_1) $uf(n,u,v) > 0, (uv > 0)$ and there exists a nonnegative sequence $\{q_n\}$ and a function ϕ such that

$$\frac{f(n+1,u,v)}{\phi(v)} \geq q_n, \quad v \neq 0,$$

where ϕ satisfies $x\phi(x) > 0 (x \neq 0)$, and

$$\phi(u) - \phi(v) = g(u,v)(u-v), \quad uv \neq 0$$

and $g(u,v)$ is a nonnegative function.

(H_2) $M_k(x), N_k(x)$ are continuous on R, and there exist positive numbers a_k^*, a_k, b_k^*, b_k such that $a_k^* \leq \frac{M_k(x)}{x} \leq a_k$, $b_k^* \leq \frac{N_k(x)}{x} \leq b_k$.

(H_3) $\{r_n\}$ and $\{p_n\}$ is a positive sequence, while $\{r_n\}$ is a decreasing sequence and $p_{n-1} \leq r_{n-1}$.

(H_4)

$$\sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i} \exp(-A(s)) = +\infty\,.$$

(H_4')

$$\sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i} \exp(-B(s)) = +\infty\,.$$

(H_5)

$$\sum_{i=n_0, i\neq n_k}^{+\infty} \prod_{n_0<n_k<i} \frac{1}{a_k} \prod_{n_0<s<i, s\neq n_k} \frac{1}{1-\frac{p_s}{r_s}} q_i = +\infty\,.$$

(H_5')

$$\sum_{i=n_0, i\neq n_k}^{+\infty} \prod_{n_0<n_k<i} \frac{1}{a_k} q_i = +\infty\,.$$

In order to obtain our results, we need the following important lemma which is a discrete version of Theorem 1.4.1 in [6] by Lakshmikantham, Bainov and Simeonov.

Lemma 1. *Assume that:*

$$\begin{cases} \triangle m(n) \leq l_n m(n) + q_n, & n \geq n_0,\ n \neq n_k,\ k = 1,2,\ldots, \\ m(n_k+1) \leq b_k m(n_k) + e_k, & n = n_k, \end{cases} \tag{2.1}$$

where $\{l_n\}$ and $\{q_n\}$ are two real valued sequences and $l_n > -1, b_k, c_k$ are constants and $b_k \geq 0$. Then

$$\triangle m(n) \leq m(n_0) \prod_{n_0<n_k<n} b_k \prod_{n_0<i<n, i\neq n_k, k\in N} (1+l_i) + \sum_{n_0<n_k<n} e_k \prod_{n_k<n_j<n} b_j \times \prod_{n_0<i<n, i\neq n_j, j\in N} (1+l_i) + \sum_{i=n_0, i\neq n_k}^{n-1} \prod_{i<n_k<n} b_k \prod_{i<s<n, s\neq n_k} (1+l_s) q_i, \quad n \geq n_0 .$$

The proof can be followed from mathematical induction and direct analysis and it is omitted.

Lemma 2. *Let $x(n)$ be a solution of* (1.1), *suppose that there exists some $N_1 \geq n_0 (N_1 \in N)$ such that $x(n) > 0$, for $n \geq N_1$. If $(H_1) \sim (H_4)$ are satisfied, then $\triangle x(n_k) \geq 0$ and $\triangle x(n) \geq 0$, for $n \in N(n_k, n_{k+1}]$, where $n_k \geq N_1, k = 1, 2, \ldots$.*

Proof. At first, we prove that $\triangle x(n_k) \geq 0$ for any $n_k \geq N_1$. If it is not true, then there exists some j such that $\triangle x(n_j) < 0, n_j \geq N_1$, then let

$$\triangle x(n_j) \exp(A(n_j)) = -\delta < 0 .$$

From (1.1) and $(H_1) \sim (H_3)$. Especially by (H_3) one can derive $p_{n-1} \leq 2r_n + r_{n-1}$, i.e. $1 - \frac{1}{3}\frac{p_{n-1}+\triangle r_{n-1}}{r_n} \geq 0$, then we have for $n \in N(n_{j+k}, n_{j+k+1}]$:

$$\begin{aligned}
&\triangle(\triangle x(n-1)\exp(A(n-1))) + \frac{1}{r_n} f(n, x(n), x(n-l)) \exp(A(n)) \\
&= (\triangle x(n) - \triangle x(n-1)) \exp(A(n-1)) + \triangle x(n-1)(\exp(A(n)) - \exp(A(n-1))) \\
&\quad + \frac{1}{r_n} f(n, x(n), x(n-l)) \exp(A(n)) \\
&= \triangle^2 x(n-1) \exp(A(n)) + \triangle x(n-1) \exp(A(n))(1 - \exp(-\frac{p_{n-1} + \triangle r_{n-1}}{r_n})) \\
&\quad + \frac{1}{r_n} f(n, x(n), x(n-l)) \exp(A(n)) = \{\triangle^2 x(n-1) + \triangle x(n-1) \\
&\quad \times [1 - \exp(-\frac{p_{n-1} + \triangle r_{n-1}}{r_n})] + \frac{1}{r_n} f(n, x(n), x(n-l))\} \exp(A(n)) \\
&\quad = \{\triangle^2 x(n-1) + \triangle x(n-1) \\
&\quad \times [\frac{p_{n-1} + \triangle r_{n-1}}{r_n} - \frac{1}{2!}(\frac{p_{n-1} + \triangle r_{n-1}}{r_n})^2 (1 - \frac{1}{3}\frac{p_{n-1} + \triangle r_{n-1}}{r_n}) \\
&\quad - \cdots - \frac{1}{2k!}(\frac{p_{n-1} + \triangle r_{n-1}}{r_n})^2 (1 - \frac{1}{2k+1}\frac{p_{n-1} + \triangle r_{n-1}}{r_n}) - \cdots] \\
&\quad + \frac{1}{r_n} f(n, x(n), x(n-l))\} \exp(A(n))
\end{aligned}$$

$$\leq \frac{1}{r_n}[\triangle(r_{n-1}\triangle x(n-1)) + f(n, x(n), x(n-l))]\exp(A(n)) = 0\,,$$

i.e. for $n \in N(n_{j+k}, n_{j+k+1}]$ $(k = 0, 1, 2, \cdots)$.

$$\triangle(\triangle x(n-1)\exp(A(n-1))) \leq -\frac{1}{r_n} f(n, x(n), x(n-l))\exp(A(n)) < 0\,.$$

Hence, $\triangle x(n-1)\exp(A(n-1))$ is monotonically decreasing on $(n_{j+k},\, n_{j+k+1}]$ $(k = 0, 1, 2, \cdots)$. So

$$\begin{aligned}\triangle x(n_{j+1})\exp(A(n_{j+1})) &\leq \triangle x(n_j+1)\exp(A(n_j+1))\\ &= M_j(\triangle x(n_j))\exp(A(n_j))(\exp(A(n_j+1)) - \exp(A(n_j)))\\ &\leq a_j^* \triangle x(n_j))\exp(A(n_j)\,,\end{aligned}$$

and

$$\begin{aligned}\triangle x(n_{j+2})\exp(A(n_{j+2})) &\leq \triangle x(n_{j+1}+1)\exp(A(n_{j+1}+1))\\ &= M_{j+1}(\triangle x(n_{j+1}))\exp(A(n_{j+1}))(\exp(A(n_{j+1}+1)) - \exp(A(n_{j+1})))\\ &\leq a_{j+1}^* \triangle x(n_{j+1}))\exp(A(n_{j+1})\\ &\leq a_j^* a_{j+1}^* \triangle x(n_j))\exp(A(n_j)\,.\end{aligned}$$

By induction, we obtain

$$\triangle x(n)\exp(A(n)) \leq -(a_j^* a_{j+1}^* \cdots a_{j+l}^*)\delta = -\delta \prod_{n_j \leq n_k \leq n} a_k^*\,.$$

Therefore

$$\triangle x(n) \leq -\delta \prod_{n_j \leq n_k \leq n} a_k^* \exp(-A(n))\,. \tag{2.2}$$

From (2.2) and (1.1) we have

$$\begin{aligned}&\triangle x(n) \leq -\delta \prod_{n_j \leq n_k \leq n} a_k^* \exp(-A(n)), \quad k = 1, 2, \ldots,\\ &x(n_k+1) \leq b_k x(n_k), \quad n = n_k,\ k = 1, 2, \ldots\,.\end{aligned} \tag{2.3}$$

Applying Lemma 1, we have

$$x(n) \leq x(n_j) \prod_{n_j < n_k < n} b_k - \delta \sum_{i=n_j, i \neq n_k}^{n-1} \prod_{s < n_k < n} b_k \prod_{n_j < n_i < s} a_i^* \exp(-A(s))\,, \tag{2.4}$$

$$x(n) \le \prod_{n_j<n_k<n} b_k\{x(n_j) - \delta \sum_{s=n_j, s\neq n_i}^{n-1} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i} \exp(-A(s))\}\,. \tag{2.5}$$

In view of $x(n) > 0$ and (H_4), we can easily find that the right side of (2.5) converges to $-\infty$ as $n \to +\infty$, while the left side of (2.5) is eventually positive, which is a contradiction, therefore $\triangle x(n_k) \ge 0$ for $n_k \ge N_1$. Because $\triangle x(n)\exp(A(n))$ is decreasing for $n \in N(n_k, n_{k+1}]$, and by the condition (H_2), we have $\triangle x(n)\exp(A(n)) \ge \triangle x(n_{k+1})\exp(A(n_{k+1})) \ge 0$, which implies $\triangle x(n) \ge 0$. The proof of this theorem is complete. □

Remark. In the case when $x(n)$ is eventually negative, if (H_4) holds true, then $\triangle x(n_k) \le 0$, and $\triangle x(n) \le 0$ for $n \in (n_k, n_{k+1}]$, where $n_k \ge N_1$.

3. Oscillation Criteria

Theorem 1. *If the conditions* $(H_1) \sim (H_5)$ *are satisfied, then every solution of* (1.1) *oscillates.*

Proof. Let $x(n)$ be a solution of (1.1). If the conclusion is not true, without loss of generality, there exists at least one nonoscillatory solution. We might as well as assume that $x(n) > 0$ for $t \ge n_0$. It follows from Lemma 2. that $\triangle x(n) \ge 0$ for $n \in (n_k, n_{k+1}](k = 1, 2, \cdots)$. Let

$$V(n) = \frac{r_n \triangle x(n)}{\phi(x(n+1-l))}\,. \tag{3.1}$$

Because of conditions (H_1), (H_3), we know that $V(n_k) > 0$ $(k = 1, 2, \dots)$, $V(n) > 0$ for any $n \ge n_0$. Using condition (H_1) and (1.1), we obtain:

$$\begin{aligned}\triangle V(n) &= \frac{r_{n+1}\triangle x(n+1)}{\phi(x(n+2-l))} - \frac{r_n\triangle x(n)}{\phi(x(n+1-l))}\\ &\le \frac{\triangle(r_n\triangle x(n))}{\phi(x(n+1-l))} + \frac{r_n\triangle x(n)}{\phi(x(n+2-l))} - \frac{r_{n-1}\triangle x(n-1)}{\phi(x(n+1-l))}\\ &= \frac{-p_n\triangle x(n)}{\phi(x(n+1-l))} - \frac{f(n+1, x(n+1), x(n+1-l))}{\phi(x(n+1-l))}\\ &\quad - \frac{r_n\triangle x(n)\triangle\phi(x(n+1-l))}{\phi(x(n+1-l))\phi(x(n+2-l))} \le -\frac{p_n}{r_n}V(n) - q_n\,. \end{aligned} \tag{3.2}$$

It follows from conditions $(H_2), (H_3)$ and (1.1) that

$$V(n_k+1) = \frac{r_{n_k+1}\triangle x(n_k+1)}{\phi(x(n_k+2-l))} \le a_k V(n_k)\,, \tag{3.3}$$

i.e.

$$\begin{cases} \triangle V(n) \le -\dfrac{p_n}{r_n} V(n) - q_n, & k = 1, 2, \dots, \\ V(n_k + 1) \le a_k V(n_k), & n = n_k,\ k = 1, 2, \dots . \end{cases} \tag{3.4}$$

Applying Lemma 1 again, we obtain

$$V(n) \le \prod_{n_0 < n_k < n} a_k \prod_{n_0 < i < n, i \ne n_k} (1 - \frac{p_i}{r_i}) \{ V(n_0) - \sum_{i=n_0, i \ne n_k}^{n-1} \prod_{n_0 < n_k < i} \frac{1}{a_k} \prod_{n_0 < s < i, s \ne n_k} \frac{1}{1 - \frac{p_s}{r_s}} q_i \} \qquad (n \ge n_0)\,. \tag{3.5}$$

In view of condition (H_5), (3.5) and $V(t) \ge 0$, we get a contradiction, as $t \to +\infty$. Hence, every solution of (1.1) oscillates. So we prove this theorem. □

According to the proof of Theorem 1, if $p_n = 0$, we can easily have the following corollaries.

Corollary 1. *If the conditions* $(H_1) \sim (H_3), (H_4')$ *and* (H_5') *are satisfied, then every solution of* (1.1) *oscillates.*

Corollary 2. *If the conditions* $(H_1) \sim (H_3)$ *and* (H_4') *are satisfied and there exists a positive integer* k_0 *for* $k \ge k_0$ *such that* $a_k \le 1$, *and*

$$\sum_{i=n_0, i \ne n_k}^{+\infty} q_i = +\infty \tag{3.6}$$

holds, then every solution of (1.1) *oscillates.*

Corollary 3. *If the conditions* $(H_1) \sim (H_3), (H_4')$ *are satisfied and there exists a positive integer* k_0 *and* λ *such that*

$$\frac{1}{a_k} \ge (\frac{n_k + 1}{n_k})^\lambda, \quad k \ge k_0\,, \tag{3.7}$$

and

$$\sum_{n \ne n_k, k \in N}^{+\infty} n^\lambda p_n = +\infty\,, \tag{3.8}$$

then every solution of (1.1) *oscillates.*

Proof. Without loss of generality, let $k_0 = 1$. Then we have

$$\sum_{i=n_0, i \ne n_k, k \in N}^{n} p_i \prod_{n_0 \le n_l \le i} \frac{1}{a_l} = \sum_{i=n_0}^{n_1 - 1} p_i + \frac{1}{a_1} \sum_{i=n_1+1}^{n_2 - 1} p_i + \cdots + \frac{1}{a_1 a_2 \cdots a_j} \sum_{i=n_{j+1}}^{n_{j+1} - 1} p_i$$

$$\geq \frac{1}{a_1}\sum_{s=n_1+1}^{n_2-1} p_s + \cdots + \frac{1}{a_1a_2\cdots a_n}\sum_{s=n_{l+1}}^{n} p_s$$

$$\geq \frac{1}{n_1^{\lambda}}[\sum_{s=n_1+1}^{n_2-1} n_2^{\lambda} p_s + \cdots + \frac{1}{a_1a_2\cdots a_n}\sum_{s=n_{l+1}}^{n} n_{l+1}^{\lambda} p_s]$$

$$\geq \frac{1}{n_1^{\lambda}}[\sum_{s=n_1+1}^{n_2-1} s^{\lambda} p_s + \cdots + \frac{1}{a_1a_2\cdots a_n}\sum_{s=n_{l+1}}^{n} s_{l+1}^{\lambda} p_s] \geq \frac{1}{n_1^{\lambda}}\sum_{s=n_1+1, s\neq n_k, \in N}^{n} s^{\lambda} p_s\,,$$

for $n \in N(n_{l+1}, n_{l+2})$. Let $n \to +\infty$, we can obtain that $(H_5^{'})$ holds. It follows from Corollary 1 that every solution of (1.1) oscillates. □

4. Examples

Example 1. Consider the following impulsive delay difference equation

$$\begin{cases} (\triangle^2 x(n-1)) + \dfrac{2\ln n - \ln(n^2-1)}{\ln(n-l)} x(n-l) = 0, \\ n \neq 3k,\ k = 1, 2, \ldots, \\ \triangle x(n_k+1) = \dfrac{k}{k+1}\triangle x(n_k), \quad n = 3k, \end{cases} \tag{4.1}$$

Comparing with (1.1), we can see that $n_k = 3k, r(n) = 1, p_n = 0, q_n = \frac{2\ln n - \ln(n^2-1)}{\ln(n-l)}$, $a_k = a_k^* = \frac{k}{k+1}, b_k = b_k^* = 1$. Obviously, conditions $(H_1) \sim (H_3)$ are satisfied and

$$\sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i}\exp(-B(s)) = \sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i}$$

$$= \sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{i}{i+1} = \frac{j}{j+1} + \frac{j}{j+1}\frac{j+1}{j+2} + \cdots + \frac{j}{j+1}\frac{j+1}{j+2}\cdots$$

$$= \frac{j}{j+1} + \frac{j}{j+2} + \cdots + \frac{j}{j+l}\cdots \geq 1 + 1 + \cdots + 1 + \cdots = +\infty\,.$$

Let $k_0 = 1$, $\lambda = 1$. Then

$$\frac{1}{a_k} = \frac{k+1}{k} = \frac{n_{k+1}}{n_k}\,,$$

and

$$\sum_{n\neq n_k}^{+\infty} n^{\lambda} q_n = \sum_{n\neq n_k}^{+\infty} n q_n = +\infty\,.$$

By Corollary 3, we know that every solution of (4.1) is oscillatory, while equation $(\triangle^2 x(n-1)) + \frac{2\ln n - \ln(n^2-1)}{\ln(n-l)} x(n-l) = 0$ has a nonoscillatory solution $x(n) = \ln n$.

This example illustrates that impulses play an important role.

Example 2. Consider the following impulsive delay difference equation

$$\begin{cases} (\triangle^2 x(n-1)) + \frac{1}{n} x(n-l) = 0, \\ n \neq 3k,\ k = 1, 2, \dots, \\ \triangle x(n_k+1) = \frac{k}{k+1} \triangle x(n_k), \quad n = 3k, \\ x(n_k+1) = \frac{k}{k+1} x(n_k), \quad n = 3k, \end{cases} \tag{4.2}$$

Comparing with (1.1), we can see that $n_k = 3k, r(n) = 1, p_n = 0, q(n) = \frac{1}{n}$, $a_k = a_k^* = \frac{k}{k+1}, b_k = b_k^* = \frac{k}{k+1}$. Obviously, conditions $(H_1) \sim (H_3)$ are satisfied and

$$\sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i} \exp(-B(s)) = \sum_{s=n_j, s\neq n_i}^{+\infty} \prod_{n_j<n_i<s} \frac{a_i^*}{b_i}$$
$$= 1 + 1 + \cdots + 1 + \cdots = +\infty\,,$$

and

$$\sum_{i=n_0, i\neq n_k}^{+\infty} q_i \geq \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = +\infty\,.$$

By Corollary 2, we know that every solution of (4.2) is oscillatory.

Acknowledgments

Supported by the doctoral fund of Guizhou College of Finance and Economics in P.R. China.

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